Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2022, Volume 28, Number 3, 211–222
DOI: 10.7546/nifs.2022.28.3.211-222

Proceedings of the 25th Jubilee Edition
of the International Conference on
Intuitionistic Fuzzy Sets (ICIFS'2022)
9–10 September 2022, Sofia, Bulgaria

On the intuitionistic fuzzy modal feeble topological structures

Krassimir T. Atanassov [1,2]

[1] Dept. of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering,
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria,
[2] Intelligent Systems Laboratory,
"Prof. Dr Assen Zlatarov" University
8010 Burgas, Bulgaria
e-mail: krat@bas.bg

Received: 16 June 2022 **Revised:** 22 July 2022 **Accepted:** 24 July 2022

Abstract: In the paper, for the first time, ideas for intuitionistic fuzzy modal feeble topological structures (of two types) are introduced, and some of their properties are discussed. These topologies are based on the intuitionistic fuzzy operation @, intuitionistic fuzzy operator $\mathcal{W}$, on the two intuitionistic fuzzy modal operators $\Box, \Diamond$ and of simplest intuitionistic fuzzy extended modal operator D_α.

Keywords: Intuitionistic fuzzy operation, Intuitionistic fuzzy operator, Intuitionistic fuzzy set, Intuitionistic fuzzy topology.
2020 Mathematics Subject Classification: 03E72.

Dedicated to the 75th anniversary
of Prof. Janusz Kacprzyk!

1 Introduction

In the present paper, we combine ideas and definitions from the areas of (general) topology (see, e.g., [8, 11]), of (standard) modal logic (see, e.g., [7, 9, 10]) and of intuitionistic fuzziness (see, e.g., [3–5]), and we introduce the concepts of *Intuitionistic Fuzzy Feeble Topological Structures of*

First and Second Type (IFFTS1 and IFFTS2) and *Intuitionistic Fuzzy Modal Feeble Topological Structures of First and Second Types (IFMFTS1 and IFMFTS2).* It is important to mention that introducing the term "feeble" for some types of topological structures, we aim to make a distinction with term *"weak"* which is used in topology.

The paper is organized as follows. Initially, short remarks over Intuitionistic Fuzzy Sets (IFSs) are given in Section 2. In the next sections, new topology related objects are introduced and some of their basic properties are discussed. In the Conclusion, new directions of the development of the present ideas are formulated.

2 Short remarks on intuitionistic fuzzy sets

Let a set E be fixed. An IFS A in E is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let for every $x \in E$, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$. Therefore, the function π determines the degree of uncertainty of the element x.

Obviously, for every ordinary fuzzy set $\pi_A(x) = 0$ for each $x \in E$ and these sets have the form: $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}$.

Let everywhere below, the universe E be given. One of the geometrical interpretations of the IFSs uses the intuitionistic fuzzy interpretational triangle on Figure 1.

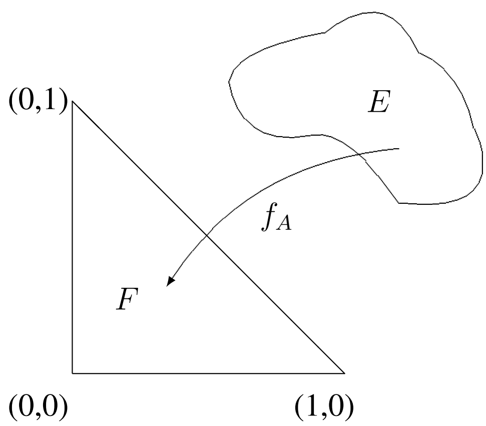

Figure 1. Geormetrical interpretation with the intuitionistic fuzzy interpretational triangle

For every two IFSs A and B many relations and operations have been defined (see, e.g. [1, 3, 4]). The most important of them are the following:

$$
\begin{array}{rcl}
A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) \geq \nu_B(x)); \\
A \supset B & \text{iff} & B \subset A; \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \;\&\; \nu_A(x) = \nu_B(x)); \\
\neg A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A @ B & = & \{\langle x, \dfrac{\mu_A(x) + \mu_B(x)}{2}, \dfrac{\nu_A(x) + \nu_B(x)}{2}\rangle | x \in E\}.
\end{array}
$$

Here, we give definitions of only the first two modal operators (see, e.g., [3, 4]) that are intuitionistic fuzzy interpretations of the classical modal logic operators (see, e.g., [7, 9, 10]):

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\};$$

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

Let

$$
\begin{aligned}
O^* &= \{\langle x, 0, 1\rangle | x \in E\}, \\
U^* &= \{\langle x, 0, 0\rangle | x \in E\}, \\
E^* &= \{\langle x, 1, 0\rangle | x \in E\}.
\end{aligned}
$$

Let for each set X

$$\mathcal{P}(X) = \{Y | Y \subseteq X\}.$$

Let for each set E, $FS(E)$ and $IFS(E)$ be the sets of all FSs and IFSs, respectively, with universe E. Then we see that

$$\mathcal{P}(E^*) = \{A | A \subseteq E\},$$

where

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \subseteq E^*.$$

Therefore, $\mathcal{P}(E^*)$ coincides with $IFS(E)$. On the other hand side, $FS(E)$ coincides with the set

$$\{A | A \subseteq E \;\&\; A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}\} = \{A | A \subseteq E \;\&\; A = \Box A\}\}.$$

3 Definition of a First type of Intuitionistic fuzzy feeble topological structure with a self-dual operation and operator

As a basis of the present research we use the definition from [11], where the conditions that characterize the topological structure $\langle X, \mathcal{O}, \bullet\rangle$ are

C1 $\mathcal{O}(A \bullet B) = \mathcal{O}(A) \bullet \mathcal{O}(B)$,

C2 $A \subseteq \mathcal{O}(A)$,

C3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

C4 $\mathcal{O}(O^*) = O^*$,

where X is a set of subsets, $A, B \in X$, $\mathcal{O} : X \to X$ and $\bullet : X \times X \to X$.

Below, we will call the object $\langle X, \mathcal{O}, \bullet \rangle$ a Feeble Topological Structure (FTS), when some of the four conditions is not valid or if it is valid only partially. The reason for this is that, below we will use a special operation $\bullet$ and in this and the next two sections, a special operator $\mathcal{O}$ that are self-dual and this fact will be included in the names of the introduced objects. In sections 6, 7 and 8 we will use the same operation $\bullet$ and other (standard) IF topological operators.

By analogy with [11], and extending the definitions from there, we will call a first type of an Intuitionistic Fuzzy FTS with a Self-Dual Operation and operator (IFFTSDO1) the object

$$\langle \mathcal{P}(E^*), \mathcal{O}, \bullet \rangle,$$

where E is a fixed universe, $\mathcal{O} : IFS(E^*) \to IFS(E^*)$ is a (self-dual) operator, $\bullet : IFS(E^*) \times IFS(E^*) \to IFS(E^*)$ is a (self-dual) operation, and for every two IFSs $A, B \in \mathcal{P}(E^*)$:

C1 $\mathcal{O}(A \bullet B) = \mathcal{O}(A) \bullet \mathcal{O}(B)$,

C3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

C4 $\mathcal{O}(O^*) = O^*$,

D1 $\neg\mathcal{O}(\neg A) = \mathcal{O}A$,

D2 $\neg(\neg A \bullet \neg B) = A \bullet B$,

D3 $\mathcal{O}(E^*) = E^*$.

As it is seen, in the new definition condition C2 is omitted and this is exactly the reason for the term *"feeble"* in the name of the new object. In the definition we add three new conditions (D1, D2 and D3) that are related to the self-duality of the operator and operation.

In [6], Adrian Ban and the author introduced *"weight-center operator"* over a given IFS A by:

$$\mathcal{W}(A) = \left\{ \left\langle x, \frac{\sum\limits_{y \in E} \mu_A(y)}{\text{card}(E)}, \frac{\sum\limits_{y \in E} \nu_A(y)}{\text{card}(E)} \right\rangle | x \in E \right\},$$

where $\text{card}(E)$ is the number of the elements of a finite set E (see also [4]). For the continuous case, the "summation" may be replaced by integrals over E.

Theorem 1. $\langle \mathcal{P}(E^*), \mathcal{W}, @ \rangle$ is an IFFTSDO1 with the (self-dual) operator "$\mathcal{W}$" and the (self-dual) operation "@".

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Then we check sequentially that

$$\begin{aligned}
\mathcal{W}(A@B) &= \mathcal{W}(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} @ \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\
&= \mathcal{W}(\left\{ \left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2} \right\rangle | x \in E \right\}) \\
&= \left\{ \left\langle x, \frac{\sum\limits_{y \in E} \frac{\mu_A(y)+\mu_B(y)}{2}}{\text{card}(E)}, \frac{\sum\limits_{y \in E} \frac{\nu_A(y)+\nu_B(y)}{2}}{\text{card}(E)} \right\rangle | x \in E \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \left\langle x, \frac{\frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)} + \frac{\sum\limits_{y\in E}\mu_B(y)}{\mathrm{card}(E)}}{2}, \frac{\frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)} + \frac{\sum\limits_{y\in E}\nu_B(y)}{\mathrm{card}(E)}}{2} \right\rangle |x \in E \right\} \\
&= \left\{ \left\langle x, \frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)} \right\rangle |x \in E \right\} @ \left\{ \left\langle x, \frac{\sum\limits_{y\in E}\mu_B(y)}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E}\nu_B(y)}{\mathrm{card}(E)} \right\rangle |x \in E \right\} \\
&= \mathcal{W}(A) @ \mathcal{W}(B);
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}(\mathcal{W}(A)) &= \mathcal{W}\left(\left\{ \left\langle x, \frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)} \right\rangle |x \in E \right\} \right) \\
&= \left\{ \left\langle x, \frac{\sum\limits_{y\in E}\frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)}}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E}\frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)}}{\mathrm{card}(E)} \right\rangle |x \in E \right\} \\
&= \left(\left\{ \left\langle x, \frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)} \right\rangle |x \in E \right\} \right) = \mathcal{W}(A);
\end{aligned}
$$

$$
\mathcal{W}(O^*) = \left\{ \left\langle x, \frac{\sum\limits_{y\in E} 0}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E} 1}{\mathrm{card}(E)} \right\rangle |x \in E \right\} = O^*;
$$

$$
\begin{aligned}
\neg\mathcal{W}(\neg A) &= \neg\mathcal{W}(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}) \\
&= \neg \left\{ \left\langle x, \frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)} \right\rangle |x \in E \right\} \\
&= \left\{ \left\langle x, \frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)} \right\rangle |x \in E \right\} = \mathcal{W}(A);
\end{aligned}
$$

$$
\begin{aligned}
\neg(\neg A @ \neg B) &= \neg(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\} @ \{\langle x, \nu_B(x), \mu_B(x)\rangle | x \in E\}) \\
&= \neg \left\{ \left\langle x, \frac{\nu_A(x) + \nu_B(x)}{2}, \frac{\mu_A(x) + \mu_B(x)}{2} \right\rangle |x \in E \right\} \\
&= \left\{ \left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2} \right\rangle |x \in E \right\} = A @ B;
\end{aligned}
$$

$$
\mathcal{W}(E^*) = \left\{ \left\langle x, \frac{\sum\limits_{y\in E} 1}{\mathrm{card}(E)}, \frac{\sum\limits_{y\in E} 0}{\mathrm{card}(E)} \right\rangle |x \in E \right\} = E^*.
$$

Therefore, Theorem 1 is valid. □

4 Definition of a First type of Intuitionistic fuzzy modal feeble topological structure with a self-dual operation and operator, and a standard modal operator

We will call *First Type of an Intuitionistic Fuzzy Modal Feeble Topological Structure with a Self-Dual Operation and operator (IFMFTSDO1)* the object

$$\langle \mathcal{P}(E^*), \mathcal{O}, \bullet, \circ \rangle,$$

where E is a fixed universe, $\mathcal{O} : IFS(E^*) \to IFS(E^*)$ is a (self-dual) operator, $\bullet : IFS(E^*) \times IFS(E^*) \to IFS(E^*)$ is a (self-dual) operation, $\circ : IFS(E^*) \to IFS(E^*)$ is a (standard) modal operator, and for every two IFSs $A, B \in \mathcal{P}(E^*)$:

C1 $\mathcal{O}(A \bullet B) = \mathcal{O}(A) \bullet \mathcal{O}(B)$,

C3 $\mathcal{O}(\mathcal{O}(A)) = \mathcal{O}(A)$,

C4 $\mathcal{O}(O^*) = O^*$,

D1 $\neg\mathcal{O}(\neg A) = \mathcal{O}A$,

D2 $\neg(\neg A \bullet \neg B) = A \bullet B$,

D3 $\mathcal{O}(E^*) = E^*$,

D4 $\circ\mathcal{O}(A) = \mathcal{O}(\circ A)$,

D5 $\circ(A \bullet B) = \circ A \bullet \circ B$,

D6 $\circ O^* = O^*$,

D7 $\circ E^* = E^*$.

Theorem 2. $\langle \mathcal{P}(E^*), \mathcal{W}, @, \Box \rangle$ is an IFMFTSDO1 with the (self-dual) operator $\mathcal{W}$ and the (self-dual) operation @, and with the (standard) modal operator $\Box$.

Proof. The checks of the first six conditions from the definition of an IFMFTSDO1 coincide with the checks of the conditions from the definition for IFTFSDO1 (see Theorem 1). So, we will check sequentially the remaining four conditions.

Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Then we check that:

$$\begin{aligned}
\Box\,\mathcal{W}(A) &= \Box\,\mathcal{W}(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}) \\
&= \Box \left\{ \left\langle x, \frac{\sum\limits_{y\in E} \mu_A(y)}{\text{card}(E)}, \frac{\sum\limits_{y\in E} \nu_A(y)}{\text{card}(E)} \right\rangle | x \in E \right\} \\
&= \left\{ \left\langle x, \frac{\sum\limits_{y\in E} \mu_A(y)}{\text{card}(E)}, 1 - \frac{\sum\limits_{y\in E} \mu_A(y)}{\text{card}(E)} \right\rangle | x \in E \right\} \\
&= \mathcal{W}(\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}) = \mathcal{W}(\Box\, A);
\end{aligned}$$

$$\begin{aligned}
\Box(A@B) &= \Box(\{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in E\}@\{\langle x,\mu_B(x),\nu_B(x)\rangle|x\in E\})\\
&= \Box\left\{\left\langle x,\frac{\mu_A(x)+\mu_B(x)}{2},\frac{\nu_A(x)+\nu_B(x)}{2}\right\rangle|x\in E\right\}\\
&= \{\langle x,\frac{\mu_A(x)+\mu_B(x)}{2},1-\frac{\mu_A(x)+\mu_B(x)}{2}\rangle|x\in E\}\\
&= \{\langle x,\frac{\mu_A(x)+\mu_B(x)}{2},\frac{(1-\mu_A(x))+(1-\mu_B(x))}{2}\rangle|x\in E\}\\
&= \{\langle x,\mu_A(x),1-\mu_A(x)\rangle|x\in E\}@\{\langle x,\mu_B(x),1-\mu_B(x)\rangle|x\in E\})=\Box A@\Box B;
\end{aligned}$$

$$\Box O^* = \{\langle x,0,1-0\rangle|x\in E\}=O^*;$$

$$\Box E^* = \{\langle x,1,1-1\rangle|x\in E\}=E^*.$$

Therefore, Theorem 2 is valid. □

Theorem 3. $\langle\mathcal{P}(E^*),\mathcal{W},@,\Diamond\rangle$ is an IFMFTSDO1 with the (self-dual) operator $\mathcal{W}$ and the (self-dual) operation @, and with the (standard) modal operator $\Diamond$.

Proof. The first six conditions from the definition of an IFMFTSDO1 again coincide with the conditions from the definition of an IFTFSDO1. So, we will again check sequentially the remaining four conditions.

Let the IFSs $A,B\in\mathcal{P}(E^*)$ be given. Then we check that:

$$\begin{aligned}
\Diamond\mathcal{W}(A) &= \Diamond\mathcal{W}(\{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in E\})\\
&= \Diamond\left\{\left\langle x,\frac{\sum\limits_{y\in E}\mu_A(y)}{\mathrm{card}(E)},\frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)}\right\rangle|x\in E\right\}\\
&= \left\{\left\langle x,1-\frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)},\frac{\sum\limits_{y\in E}\nu_A(y)}{\mathrm{card}(E)}\right\rangle|x\in E\right\}\\
&= \mathcal{W}(\{\langle x,1-\nu_A(x),\nu_A(x)\rangle|x\in E\})=\mathcal{W}(\Diamond A);
\end{aligned}$$

$$\begin{aligned}
\Diamond(A@B) &= \Diamond(\{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in E\}@\{\langle x,\mu_B(x),\nu_B(x)\rangle|x\in E\})\\
&= \Diamond\{\langle x,\frac{\mu_A(x)+\mu_B(x)}{2},\frac{\nu_A(x)+\nu_B(x)}{2}\rangle|x\in E\}\\
&= \{\langle x,1-\frac{\nu_A(x)+\nu_B(x)}{2},\frac{\mu_A(x)+\mu_B(x)}{2}\rangle|x\in E\}\\
&= \{\langle x,\frac{(1-\mu_A(x))+(1-\mu_B(x))}{2},\frac{\nu_A(x)+\nu_B(x)}{2}\rangle|x\in E\}\\
&= \{\langle x,1-\nu_A(x),\nu_A(x)\rangle|x\in E\}@\{\langle x,1-\nu_B(x),\nu_B(x)\rangle|x\in E\})=\Diamond A@\Diamond B;
\end{aligned}$$

$$\Diamond O^* = \{\langle x,1-1,1\rangle|x\in E\}=O^*;$$

$$\Diamond E^* = \{\langle x,1-0,0\rangle|x\in E\}=E^*.$$

This completes the proof. □

5 Definition of a First type of Intuitionistic fuzzy extended modal feeble topological structure with a self-dual operation and operator, and an extended self-dual modal operator

The first extended modal operator (see [2–4]) has the form:

$$\begin{aligned} D_\alpha(A) &= \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)|x \in E\} \\ &= \{\langle x, \mu_A(x) + \alpha(1-\mu_A(x)-\nu_A(x)), \nu_A(x) + (1-\alpha)(1-\mu_A(x)-\nu_A(x))|x \in E\}, \end{aligned}$$

where $\alpha \in [0, 1]$.

It is extended modal operator, because for each IFS A:

$$\begin{aligned} D_0(A) &= \Box A, \\ D_1(A) &= \Diamond A. \end{aligned}$$

Operator D_α is self-dual, because for each IFS A:

$$\begin{aligned} \neg D_\alpha(\neg A) &= \neg\{\langle x, \nu_A(x) + \alpha(1-\mu_A(x)-\nu_A(x)), \mu_A(x) + (1-\alpha)(1-\mu_A(x)-\nu_A(x))|x \in E\} \\ &= \{\langle x, \mu_A(x) + (1-\alpha)(1-\mu_A(x)-\nu_A(x)), \nu_A(x) + \alpha(1-\mu_A(x)-\nu_A(x))|x \in E\} \\ &= D_{1-\alpha}(A). \end{aligned}$$

Now, we define the first type of an Intuitionistic Fuzzy Extended Modal Feeble Topological Structure with a self-dual operation and operator, and with an extended self-dual modal operator (IFEMFTSDO1) as the object $\langle \mathcal{P}(E^*), \mathcal{O}, \bullet, \circ\rangle$ that satisfies the conditions C1, C3, C4, D1, ..., D7 and the new one D8:

$$\neg \circ (\neg A) = \circ(A),$$

where E is a fixed universe, $\mathcal{O} : IFS(E^*) \to IFS(E^*)$ is a (self-dual) operator, $\bullet : IFS(E^*) \times IFS(E^*) \to IFS(E^*)$ is a (self-dual) operation, $\circ : IFS(E^*) \to IFS(E^*)$ is a (self-dual) extended modal operator. In this case the following theorem is valid.

Theorem 4. $\langle \mathcal{P}(E^*), \mathcal{W}, @, D_\alpha\rangle$ is an IFEMFTSDO1 with the (self-dual) operator $\mathcal{W}$, the (self-dual) operation @, and with the (self-dual) extended modal operator D_α for each $\alpha \in [0, 1]$.

Proof. The first six conditions from the definition of an IFMFTSDO1 again coincide with the conditions from the definition of an IFTFSDO1. So, we will again check sequentially the remaining four conditions.

Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Then we check sequentially that:

$$\begin{aligned} D_\alpha(\mathcal{W}(A)) &= D_\alpha(\mathcal{W}(\{\langle x, \mu_A(x), \nu_A(x)\rangle|x \in E\})) \\ &= D_\alpha\left(\left\{\left\langle x, \frac{\sum\limits_{y\in E}\mu_A(y)}{\text{card}(E)}, \frac{\sum\limits_{y\in E}\nu_A(y)}{\text{card}(E)}\right\rangle|x \in E\right\}\right) \end{aligned}$$

$$= \left\{ \left\langle x, \frac{\sum\limits_{y\in E} \mu_A(y)}{\mathrm{card}(E)} + \alpha \left(1 - \frac{\sum\limits_{y\in E} \mu_A(y)}{\mathrm{card}(E)} - \frac{\sum\limits_{y\in E} \nu_A(y)}{\mathrm{card}(E)}\right), \right.\right.$$

$$\left.\left. \frac{\sum\limits_{y\in E} \nu_A(y)}{\mathrm{card}(E)} + (1-\alpha) \left(1 - \frac{\sum\limits_{y\in E} \mu_A(y)}{\mathrm{card}(E)} - \frac{\sum\limits_{y\in E} \nu_A(y)}{\mathrm{card}(E)}\right) \right\rangle | x \in E \right\}$$

$$= \left\{ \left\langle x, \frac{\sum\limits_{y\in E} (\mu_A(y) + \alpha \left(1 - \sum\limits_{y\in E} \mu_A(y) - \sum\limits_{y\in E} \nu_A(y)\right)}{\mathrm{card}(E)}, \right.\right.$$

$$\left.\left. \frac{\sum\limits_{y\in E} \nu_A(y) + (1-\alpha) \left(1 - \sum\limits_{y\in E} \mu_A(y) - \sum\limits_{y\in E} \nu_A(y)\right)}{\mathrm{card}(E)} \right\rangle | x \in E \right\}$$

$$= \mathcal{W}(\{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \nu_A(x)) + (1-\alpha)(1 - \mu_A(x) - \nu_A(x)) | x \in E\})$$
$$= \mathcal{W}(D_\alpha(A));$$

$$D_\alpha(A@B) = D_\alpha \left(\{\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2} \rangle | x \in E\} \right)$$

$$= \left\{ \left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2} + \alpha \left(1 - \frac{\mu_A(x) + \mu_B(x)}{2} - \frac{\nu_A(x) + \nu_B(x)}{2}\right), \right.\right.$$
$$\left.\left. \frac{\nu_A(x) + \nu_B(x)}{2} + (1-\alpha)\left(1 - \frac{\mu_A(x) + \mu_B(x)}{2} - \frac{\nu_A(x) + \nu_B(x)}{2}\right) \right\rangle | x \in E \right\});$$

$$D_\alpha(O^*) = \{\langle x, 0 + \alpha(1 - 0 - 1), 1 + (1-\alpha)(1 - 0 - 1) | x \in E\} = O^*;$$

$$D_\alpha(E^*) = \{\langle x, 1 + \alpha(1 - 1 - 0), 0 + (1-\alpha)(1 - 1 - 0) | x \in E\} = E^*.$$

As we mentioned above, Condition D8 is valid because of the self-duality of the extended modal operator D_α.

This completes the proof. □.

6 Definition of a Second type of Intuitionistic fuzzy feeble topological structure with a self-dual operation

Below, we will call the object $\langle X, \mathcal{O}, \bullet \rangle$ a second type of a Feeble Topological Structure with a self-dual operation (FTSSDO2), when condition C1 is changed with the feeble condition:

C1' $\mathcal{O}(A \bullet B) \subseteq \mathcal{O}(A) \bullet \mathcal{O}(B)$.

keeping conditions C2, C3, C4, D2 and D3.

We must mention that now condition D1 here and thereafter is omitted because below we will use an operator that is not self-dual, while condition D2 must remain because the operation @ is a self-dual one. When this operator and the (self-dual) operation are intuitionistic fuzzy ones, then the structure is an Intuitionistic Fuzzy FTSSDO2 (IFFTSDO2).

Now, the used operator "$\mathcal{O}_1$" has a dual one – "$\mathcal{O}_2$" so that for each IFS A:

$$\neg\mathcal{O}_1(A) = \mathcal{O}_2(A).$$

The first operator will be an analogous of the standard topological operator "closure" (cl) and the second – of the standard topological operator "interior" (in). By this reason, we will denote the respective structures by cl- and in-ones.

Theorem 5. $\langle\mathcal{P}(E^*), \mathcal{C}, @\rangle$ is a cl-IFFTSDO2 with an (self-dual) operation "@".

Proof. Let the IFSs $A, B \in \mathcal{P}(E^*)$ be given. Then we check sequentially that conditions C1', C2, C3, C4 are valid:

$$\begin{aligned}
\mathcal{C}(A@B) &= \mathcal{C}(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}@\{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\
&= \mathcal{C}\left\{\left\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\mu_A(x)+\mu_B(x)}{2}\right\rangle | x \in E\right\} \\
&= \left\{\left\langle x, \sup_{y\in E}\frac{\mu_A(x)+\mu_B(x)}{2}, \inf_{y\in E} 1 - \frac{\mu_A(x)+\mu_B(x)}{2}\right\rangle | x \in E\right\} \\
&\subseteq \left\{\left\langle x, \frac{\sup\limits_{y\in E}\mu_A(x)+\sup\limits_{y\in E}\mu_B(x)}{2}, \frac{\inf\limits_{y\in E}\mu_A(x)+\inf\limits_{y\in E}\mu_B(x)}{2}\right\rangle | x \in E\right\} \\
&= \mathcal{C}(A)@\mathcal{C}(B),
\end{aligned}$$

i.e., condition C1' is valid, but obviously, condition C1 will not be always valid:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \subseteq \{\langle x, \sup_{y\in E}\mu_A(y), \inf_{y\in E}\nu_A(y)\rangle | x \in E\} = \mathcal{C}(A);$$

$$\mathcal{C}(O^*) = \mathcal{C}(\{\langle x, 0, 1\rangle | x \in E\}) = \{\langle x, \sup_{y\in E} 0, \inf_{y\in E} 1\rangle | x \in E\} = \{\langle x, 0, 1\rangle | x \in E\} = O^*;$$

$$\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(\{\langle x, K, L\rangle | x \in E\}) = \{\langle x, K, L\rangle | x \in E\} = \mathcal{C}(A);$$

Condition D2 is checked in Section 3 (Theorem 1), while the validity of condition D3 follows from:

$$\begin{aligned}
\mathcal{C}(E^*) &= \mathcal{C}(\{\langle x, 1, 0\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y\in E} 1, \inf_{y\in E} 0\rangle | x \in E\} = \{\langle x, 1, 0\rangle | x \in E\} = E^*.
\end{aligned}$$

This completes the proof. □

In the same manner, we can prove the next theorem.

Theorem 6. $\langle\mathcal{P}(E^*), \mathcal{I}, @\rangle$ is an in-IFFTSDO2 with an (self-dual) operation "@".

7 Definition of a Second type of Intuitionistic fuzzy modal feeble topological structure with a self-dual operation

We will call *Second type of Intuitionistic Fuzzy Modal Feeble Topological Structure with a Self-Dual Operation and operator (IFMFTSDO2)* the object

$$\langle \mathcal{P}(E^*), \mathcal{O}, \bullet, \circ \rangle,$$

where E is a fixed universe, $\mathcal{O} : IFS(E^*) \to IFS(E^*)$ is an operator, $\bullet : IFS(E^*) \times IFS(E^*) \to IFS(E^*)$ is a (self-dual) operation, $\circ : IFS(E^*) \to IFS(E^*)$ is a (standard) modal operator, and for every two IFSs $A, B \in \mathcal{P}(E^*)$: the conditions C1', C2, C3, C4, D2, …, D7 are satisfied.

Theorem 7. $\langle \mathcal{P}(E^*), \mathcal{C}, @, \Box \rangle$ is a cl-IFMFTSDO2 with the operator $\mathcal{W}$, the (self-dual) operation $@$, and with a (standard) modal operator $\Box$.
Proof. Conditions C1', C2, C3, C4, D2, D3, D5, D6 and D7 were checked above. So, here we check condition D4 as follows:

$$\Box \mathcal{C}(A) = \Box\{\langle x, K, L\rangle | x \in E\} = \{\langle x, K, 1 - K\rangle | x \in E\}) = \mathcal{C}(\Box A).$$

This completes the proof. □

By analogy, we can prove the validity of the following three assertions.

Theorem 8. $\langle \mathcal{P}(E^*), \mathcal{C}, @, \Diamond \rangle$ is a cl-IFMFTSDO2 with the operator $\mathcal{C}$, the (self-dual) operation $@$, and with the (standard) modal operator $\Diamond$.

Theorem 9. $\langle \mathcal{P}(E^*), \mathcal{I}, @, \Box \rangle$ is an in-IFMFTSDO2 with the operator $\mathcal{I}$, the (self-dual) operation $@$, and with the (standard) modal operator $\Box$.

Theorem 10. $\langle \mathcal{P}(E^*), \mathcal{I}, @, \Diamond \rangle$ is an in-IFMFTSDO2 with the operator $\mathcal{I}$, the (self-dual) operation $@$, and with the (standard) modal operator $\Diamond$.

8 Definition of a Second type of Intuitionistic fuzzy extended modal feeble topological structure with a self-dual operation and operator, and an extended self-dual modal operator

On the basis of the definitions in previouse sections, here, we define the second type of an Intuitionistic Fuzzy Extended Modal Feeble Topological Structure with a self-dual operation and with an extended self-dual modal operator (IFEMFTSDO2) as the object $\langle \mathcal{P}(E^*), \mathcal{O}, \bullet, \circ \rangle$ that satisfies the conditions C1', C2, C3, C4, D2, …, D8, where E is a fixed universe, $\mathcal{O} : IFS(E^*) \to IFS(E^*)$ is an operator, $\bullet : IFS(E^*) \times IFS(E^*) \to IFS(E^*)$ is an operation, $\circ : IFS(E^*) \to IFS(E^*)$ is an extended modal operator. In this case the following two theorems hold true.

Theorem 11. $\langle \mathcal{P}(E^*), \mathcal{C}, @, D_\alpha \rangle$ is a cl-IFEMFTSDO2 with a (self-dual) operator $\mathcal{W}$ and operation @, and with the extended modal operator D_α for each $\alpha \in [0, 1]$.

Theorem 12. $\langle \mathcal{P}(E^*), \mathcal{I}, @, D_\alpha \rangle$ is an in-IFEMFTSDO2 with a (self-dual) operator $\mathcal{W}$ and operation @, and with the extended modal operator D_α for each $\alpha \in [0, 1]$.

9 Conclusion and ideas for future research

The described above idea opens some directions for future research. Below, we discuss some of them.

First, we plan to introduce an intuitionistic fuzzy topological structure and an intuitionistic fuzzy modal topological structure, in which all standard conditions for topological structures are valid and operation is not self-dual (that is the reason for the above discussed change of the conditions for topological structures).

Second, we can extend the last type of an intuitionistic fuzzy topological structure with some of the extended modal operators that are not already self-dual ones.

References

[1] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[2] Atanassov, K. (1988). Two operators on intuitionistic fuzzy sets. *Comptes Rendus de l'Academie bulgare des Sciences*, 41(5), 35–38.

[3] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Springer, Heidelberg.

[4] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*. Springer, Berlin.

[5] Atanassov, K. (2017). *Intuitionistic Fuzzy Logics*. Springer, Cham.

[6] Atanassov, K., & Ban, A. (2000). On an operator over intuitionistic fuzzy sets. *Comptes Rendus de l'Academie bulgare des Sciences*, 53(5), 39–42.

[7] Blackburn, P., van Bentham, J., & Wolter, F. (2006). *Modal Logic*. North Holland, Amsterdam.

[8] Bourbaki, N. (1960). *Éléments De Mathématique, Livre III: Topologie Générale, Chapitre 1: Structures Topologiques, Chapitre 2: Structures Uniformes.* (3rd edition, French). Herman, Paris.

[9] Feys, R. (1965). *Modal Logics*. Gauthier, Paris.

[10] Fitting, M., & Mendelsohn, R. (1998). *First Order Modal Logic*. Kluwer, Dordrecht.

[11] Kuratowski, K. (1966). *Topology*, Vol. 1, New York, Academic Press.