Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy coloured multigraphs and their index matrix representation

Sevdalin Tsvetanov [1,2], **Peter Vassilev** [1] **and Krassimir Atanassov** [1]

[1] Department of Bioinformatics and Mathematical Modelling,
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria
e-mails: peter.vassilev@gmail.com, krat@bas.bg

[2] Department of Mathematics, Physics and Computers, Forest University
10 Kliment Ohridski Bld., 1756 Sofia, Bulgaria
e-mail: s.tsvetanov@ltu.bg

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Abstract: The concept of an Intuitionistic Fuzzy Coloured MultiGraph (IFCMG) is considered. An Index Matrix (IM) is constructed for each IFCMG. The operations "union" and "intersection" on IFCMGs via their IM interpretation are defined and some of their properties are studied.
Keywords: Coloured multigraph, Graph, Index matrix, Intersection, Intuitionistic fuzzy coloured multigraph, Union.
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1 Introduction

The concept of an Intuitionistic Fuzzy Graph (IFG) was introduced in September 1994 in [19]. Only a month later—in October 1994—the IFGs were given in [2] Index Matrix interpretation

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(IM, see [4, 6]). Ten years later, the concept of an intuitionistic fuzzy multigraph was introduced in [20] and its IM interpretation was discussed. These graphs are based on the concepts of Intuitionistic Fuzzy Sets (IFSs, see [3]) and Intuitionistic Fuzzy Pairs (IFPs, see [5, 7]).

In 1987, in [1], the concept of an Index Matrix (IM) was introduced as an extension of the well-known object "matrix" (see, e.g., [17, 18]). In [4, 6], the IMs (which were defined initially as two-dimensional objects) were extended to 3- and n-dimensional IMs, as well as to Intuitionistic Fuzzy Index Matrices (IFIMs) and to Extended IFIMs (EIFIMs).

In the present paper, we discuss the concept of an Intuitionistic Fuzzy Coloured MultiGraph (IFCMG), show its EIFIM interpretation and some of its properties.

2 Short remarks on EIFIMs

Following [5, 7], we will mention that the ordered pair $\langle a, b\rangle$ is called an IFP when $a, b, a + b \in [0, 1]$.

When there is a fixed universe E and its subset A, the corresponding IFS is an object with the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ to the set A, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

is called an IFS (see, e.g., [3]).

Let I be a fixed set. By EIFIM with index sets K and L $(K, L \subset I)$, we denote the object:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha^L_1, \beta^L_1\rangle & \ldots & l_j, \langle \alpha^L_j, \beta^L_j\rangle & \ldots & l_n, \langle \alpha^L_n, \beta^L_n\rangle \\ \hline k_1, \langle \alpha^K_1, \beta^K_1\rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j}\rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i, \langle \alpha^K_i, \beta^K_i\rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1}\rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n}\rangle \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m, \langle \alpha^K_m, \beta^K_m\rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j}\rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle \end{array},$$

where $K = \{k_1, k_2, ..., k_m\}$, $L = \{l_1, l_2, ..., l_n\}$ and

$$K^* = \{\langle k_i, \alpha^K_i, \beta^K_i\rangle \mid k_i \in K\} = \{\langle k_i, \alpha^K_i, \beta^K_i\rangle \mid 1 \leq i \leq m\},$$
$$L^* = \{\langle l_j, \alpha^L_j, \beta^L_j\rangle \mid l_j \in L\} = \{\langle l_j, \alpha^L_j, \beta^L_j\rangle \mid 1 \leq j \leq n\}$$

are IFSs and for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$\begin{aligned} \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} &\in [0, 1], \\ \alpha^K_i, \beta^K_i, \alpha^K_i + \beta^K_i &\in [0, 1], \\ \alpha^L_j, \beta^L_j, \alpha^L_j + \beta^L_j &\in [0, 1]. \end{aligned}$$

Following [4, 6], we will describe two operations over EIFIMs that are extensions of the standard operations over IMs, IFIMs and EIFIMs.

Let everywhere below $\circ, * \in \{\vee, \wedge\}$ and when $\circ$ or $*$ is $\vee$, then $\circ_1$ or $*_1$ is "$\max$", and $\circ_2$ or $*_2$ is "$\min$"; and when $\circ$ or $*$ is $\wedge$, then $\circ_1$ or $*_1$ is "$\min$", and $\circ_2$ or $*_2$ is "$\max$". Let $e_\vee = \langle 0, 1\rangle$ and $e_\wedge = \langle 1, 0\rangle$.

For the EIFIMs A and B operations that are analogous to the standard matrix operations of addition and multiplication are defined as follows:

Addition $\oplus^*_\circ$

$$A \oplus^*_\circ B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha^t_u, \beta^t_u\rangle \mid t_u \in K \cup P\},$$
$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha^v_w, \beta^v_w\rangle \mid v_w \in L \cup Q\},$$

$$\alpha^t_u = \begin{cases} \alpha^k_i, & \text{if } t_u \in K - P \\ \alpha^p_r, & \text{if } t_u \in P - K \\ *_1(\alpha^k_i, \alpha^p_r), & \text{if } t_u \in K \cap P \end{cases}, \quad \beta^v_w = \begin{cases} \beta^l_j, & \text{if } v_w \in L - Q \\ \beta^q_s, & \text{if } v_w \in Q - L \\ *_2(\beta^l_j, \beta^q_s), & \text{if } v_w \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \quad \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \sigma_{p_r,q_s}, \tau_{p_r,q_s}\rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \quad \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \circ_1(\mu_{k_i,l_j}, \sigma_{p_r,q_s}), \circ_2(\nu_{k_i,l_j}, \tau_{p_r,q_s})\rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \quad \text{and } v_w = l_j = q_s \in L \cap Q; \\ e_\circ, & \text{otherwise.} \end{cases}$$

Termwise multiplication $\otimes^*_\circ$

$$A \otimes^*_\circ B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha^t_u, \beta^t_u\rangle \mid t_u \in K \cap P\},$$
$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha^v_w, \beta^v_w\rangle \mid v_w \in L \cap Q\},$$
$$\alpha^t_u = *_1(\alpha^k_i, \alpha^p_r), \text{ for } t_u = k_i = p_r \in K \cap P,$$
$$\beta^v_w = *_2(\beta^l_j, \beta^q_s), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \langle \circ_1(\mu_{k_i,l_j}, \sigma_{p_r,q_s}), \circ_2(\nu_{k_i,l_j}, \tau_{p_r,q_s})\rangle.$$

3 Short remark on coloured multigraphs and intuitionistic fuzzy graphs

Let $\mathcal{V}$ be a finite set of vertex identifiers. Let $\mathcal{L}_{\mathcal{V}}$ and $\mathcal{L}_{\mathcal{H}}$ be a bounded lattices of, respectively, vertex and edge colours, with operations $\cap$ and $\cup$, least elements $\bot_{\mathcal{L}}$ and $\bot_{\mathcal{H}}$, and greatest elements $\top_{\mathcal{L}}$ and $\top_{\mathcal{H}}$, respectively. These operations describe the prescribed combination of vertex and edge labels and their interpretation can be algebraic. Let $\bot \notin \mathcal{L}_{\mathcal{V}} \cup \mathcal{L}_{\mathcal{H}}$ denote the absence of an edge in the corresponding colour or, as we used it above.

Following [8–16], the term Colour MultiGraph (CMG) will denotate the object $G = \langle V^*, H^* \rangle$ for which the following notations are introduced:

- $V^* \subseteq \mathcal{V} \times \mathcal{L}_{\mathcal{V}}$,
- $\alpha = |V|$ is the cardinality of the set of the G-vertices $V \subseteq \mathcal{V}$ and $\alpha \leq |\mathcal{V}|$,
- $V^* = \{\langle v_i, c_1 \rangle, \ldots, \langle v_\alpha, c_\alpha \rangle\}$,
- $\mathcal{H}^* \subseteq \mathcal{V} \times \mathcal{V} \times \mathcal{N} \times \{0,1\}^n$,
- $\mathcal{N}$ is the set of the natural numbers,
- $h_{i,j}$ is the number of the edges between vertices v_i and v_j of V,
- n is the number of all colours used for colouring of the edges of G and $n \leq |\mathcal{L}_{\mathcal{H}}|$,
- $\mathcal{H}^* = \{\langle v_i, v_j, h_{i,j}, \lambda(c^1_{i,j}), \ldots, \lambda(c^n_{i,j}) \rangle \mid v_i, v_j \in V \ \& \ (\forall k : 1 \leq k \leq n)(c^k_{i,j} \in \mathcal{L}_{\mathcal{H}})\}$.

Therefore, we can interpret the set $\mathcal{L}_{\mathcal{H}}$ as a fixed palette of edge colours for the graphs with vertices from $\mathcal{V}$ and for each k $(1 \leq k \leq n)$, for every two vertices v_i, v_j $(1 \leq i, j \leq \alpha)$:

$$\lambda(c) = \begin{cases} 1, & \text{if } c \in \mathcal{L}_{\mathcal{H}} \text{ is a colour of an edge between } v_i \text{ and } v_j \\ 0, & \text{if there is no edge} \end{cases}.$$

Then $h_{i,j} = \sum_{k=1}^{n} \lambda(c^k_{i,j})$.

The IFG is a graph which vertices and edges are indexed by IFPs.

Having in mind the relationship between the colours and IFSs discussed in [3], we can see immediately that each CMG (and, of course, each standard colour graph) can be interpreted as an IFG. Really, as it is shown in [3] all colours without black, white and their nuances can be represented by IFPs (see Figure 1).

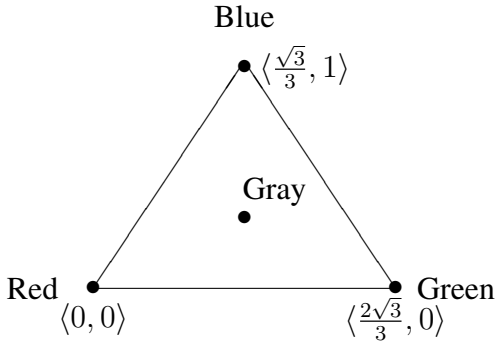


Figure 1. Intuitionistic fuzzy representation of the RGB colour triangle, with Red, Green, and Blue at its vertices and Gray at its centroid.

When we like to represent black and white colours, and all their nuances we can use a modification of Temporal IFS (see, [3]) where temporal scale corresponds to the dimension defined by the vertices "Black" and "White" on Figure 2. In the figure, depicting a triangular bipyramid, all colours without black, white and their nuances and with only gray colour lying on the base plane as the centroid determined by vertices "Red", "Green" and "Blue".

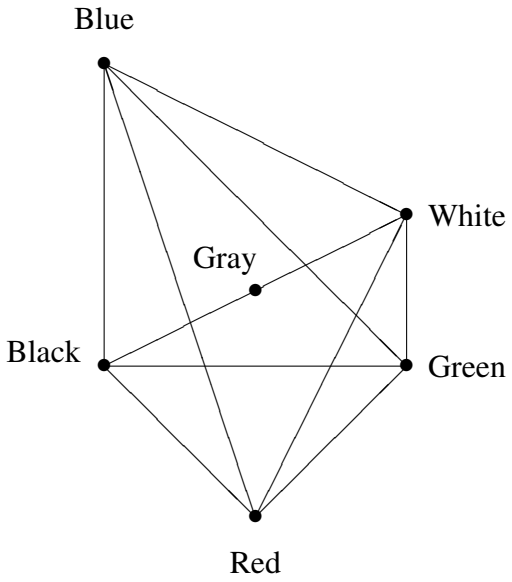


Figure 2. Schematic extension of the RGB colour triangle to a triangular bipyramid.
Red, Green, and Blue form the base plane; Black and White are the apices.
The Black–White axis meets the base at the RGB centroid Gray.

The triangles that lie in planes parallel to the base plane close to vertex "Black" will contain all colours from the base plane to which some quantity of black colour is added. Respectively, the triangles that lie in planes parallel to the base plane close to vertex "White" will contain all colours from the base plane to which some quantity of white colour is added.

We restrict our attention to colours in the RGB base triangle. In Figure 3, Red, Green, and Blue correspond to the vertices P, M, and N, respectively. If (w_R, w_G, w_B) are the barycentric coordinates of a point with respect to Red, Green, and Blue, respectively, we represent the point by the IFP $\langle w_G, w_B \rangle$, with degree of uncertainty (indeterminacy) $w_R = 1 - w_G - w_B$.

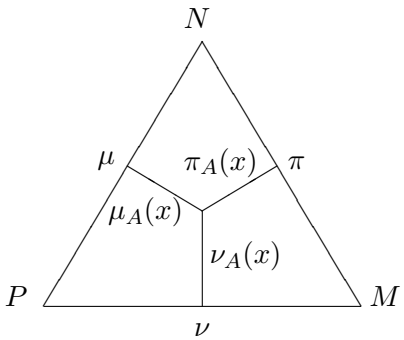


Figure 3. Membership, non-membership, and uncertainty degrees
in an equilateral version of the intuitionistic fuzzy interpretational triangle.

4 Intuitionistic fuzzy coloured multigraphs

Now, we can define: Intuitionstoc fuzzy coloured multigraph (IFCMG) is a CMG which vertices and edges are indexed by IFPs. It can be defined as $G = \langle V^*, H^* \rangle$, where set V has the above sense, but

$$V^* = \{\langle v_1, c_1, \mu_{v_1}, \nu_{v_1}\rangle, \ldots, \langle v_\alpha, c_\alpha, \mu_{v_\alpha}, \nu_{v_\alpha}\rangle\},$$

where the IFP $\langle \mu_{v_k}, \nu_{v_k} \rangle$ represents, e.g., the degrees of existence and non-existence of the k-th vertex v_k of the graph (for $1 \leq k \leq \alpha$).

For the set of edges H^* there are two cases that determine the type of the IFCMG.

For the Simple IFCMG (SIFCMG), H^* has the above form, but

$$\lambda(c_{i,j}^k) = \begin{cases} \langle \rho_{c_{i,j}^k}, \sigma_{c_{i,j}^k} \rangle, & \text{if } \langle \rho_{c_{i,j}^k}, \sigma_{c_{i,j}^k} \rangle \text{ is the IFP that corresponds to the colour } c_{i,j}^k \in \mathcal{L}_\mathcal{H} \\ & \quad \text{of an edge between } v_i \text{ and } v_j \\ 0, & \text{if there is no edge} \end{cases}.$$

For the Complex IFCMG (CIFCMG), H^* has the form

$$\mathcal{H}^\# = \{\langle v_i, v_j, h_{i,j}, \lambda(c_{i,j}^1), \langle \mu_{c_{i,j}^1}, \nu_{c_{i,j}^1}\rangle, \ldots, \lambda(c_{i,j}^n), \langle \mu_{c_{i,j}^n}, \nu_{c_{i,j}^n}\rangle\rangle \mid v_i, v_j \in V \\ \& \ (\forall k : 1 \leq k \leq n)(c_{i,j}^k \in \mathcal{L}_\mathcal{H})\}.$$

Therefore, in the complex case, the graph has as coloured edges, as well as degrees of existence and of non-existence of these colours. If the k-th edge between vertices v_i and v_j has a colour $c_{i,j}^k$ with a corresponding IFP $\langle \rho_{c_{i,j}^k}, \sigma_{c_{i,j}^k} \rangle$ and if this is valid for each edge of the graph, then we can transform the CIFCMG to a SIFCMG as follows: replace set $H^\#$ with H^*, but now function λ has the form

$$\lambda(c_{i,j}^k) = \begin{cases} O_{\rho_{c_{i,j}^k}, \sigma_{c_{i,j}^k}}(\langle \mu_{c_{i,j}^k}, \nu_{c_{i,j}^k} \rangle), & \text{if } \langle \rho_{c_{i,j}^k}, \sigma_{c_{i,j}^k} \rangle \text{ is the IFP that corresponds to the colour} \\ & \quad c_{i,j}^k \in \mathcal{L}_\mathcal{H} \text{ of an edge between } v_i \text{ and } v_j \\ 0, & \text{if there is no edge,} \end{cases}.$$

where the intuitionistic fuzzy operator $O_{\alpha,\beta}$ ($\alpha, \beta \in [0,1]$) is defined over the IFP $\langle a, b \rangle$ and it has one of the following forms (see [5]):

$$F_{\alpha,\beta}(\langle a, b\rangle) = \langle a + \alpha(1-a-b), b + \beta(1-a-b)\rangle \text{ and } \alpha + \beta \leq 1$$
$$G_{\alpha,\beta}(\langle a, b\rangle) = \langle \alpha a, \beta b\rangle$$
$$H_{\alpha,\beta}(\langle a, b\rangle) = \langle \alpha a, b + \beta(1-a-b)\rangle$$
$$H^*_{\alpha,\beta}(\langle a, b\rangle) = \langle \alpha a, b + \beta(1-\alpha a-b)\rangle$$
$$J_{\alpha,\beta}(\langle a, b\rangle) = \langle a + \alpha(1-a-b), \beta b\rangle$$
$$J^*_{\alpha,\beta}(\langle a, b\rangle) = \langle a + \alpha(1-a-\beta b), \beta b\rangle$$

for every $a, b, \alpha, \beta \in [0,1]$ such that $a + b \leq 1, \alpha, \beta \leq 1$.

Example 1. Consider a CIFCMG with two vertices v_1 and v_2 joined by one undirected edge of colour c, where c denotes a shade of green. The IFP associated with c in the palette is $\langle \rho_c, \sigma_c \rangle = \langle 0.6, 0.2 \rangle$. For this particular edge, let $\langle \mu_c, \nu_c \rangle = \langle 0.3, 0.4 \rangle$ be the additional IFP describing the degrees of existence and non-existence of the colour c on the edge.

The entry in the c-layer of the complex representation therefore stores the two pairs separately:

$$\big(\langle 0.6, 0.2 \rangle, \langle 0.3, 0.4 \rangle\big).$$

The first pair describes the colour in the palette; the second describes its occurrence on this particular edge.

To obtain a simple representation, choose the operator $O_{\rho_c,\sigma_c} = F_{\rho_c,\sigma_c}$. The uncertainty degree of the second pair is $1 - 0.3 - 0.4 = 0.3$, and hence

$$\begin{aligned} F_{0.6,0.2}(\langle 0.3, 0.4 \rangle) &= \langle 0.3 + 0.6 \cdot 0.3,\ 0.4 + 0.2 \cdot 0.3 \rangle \\ &= \langle 0.48, 0.46 \rangle. \end{aligned}$$

Thus, the operator assigns the value 0.18 of the original degree of uncertainty to membership and 0.06 to non-membership, hence leaving a degree of uncertainty of 0.06.

Since the edge is undirected, the entries at (v_1, v_2) and (v_2, v_1) coincide. In the complex representation, both entries store the two original IFPs; in the simple representation, both store the single IFP $\langle 0.48, 0.46 \rangle$. The edge still belongs to the c-layer, and its contribution to the total edge count remains one.

5 Representation of SIFCMGs and CIFCMGs by EIFIMs

When we have a SIFCMG, we can bijectively juxtapose to it the following EIFIM:

$$C = [V^*, V^*, \overline{N}, \langle h_{i,j}, \lambda(c^1_{i,j}), \ldots, \lambda(c^n_{i,j}) \} \rangle],$$

where

$$\overline{N} = \{c^1, \ldots, c^n, \bot\}$$

are the sets of all colours with which the edges of G_1 and G_2 are coloured, respectively, and the remaining components are described above. The vertex colours c_i are part of the indices in V^* and are omitted from the displayed headings for readability. Then C has the form

$$\left\{\begin{array}{c|cccccc} \bot & v_1, \langle \mu_{v_1}, \nu_{v_1} \rangle & \ldots & v_j, \langle \mu_{v_j}, \nu_{v_j} \rangle & \ldots & v_\alpha, \langle \mu_{v_\alpha}, \nu_{v_\alpha} \rangle \\ \hline v_1, \langle \mu_{v_1}, \nu_{v_1} \rangle & h_{1,1} & \ldots & h_{1,j} & \ldots & h_{1,\alpha} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_i, \langle \mu_{v_i}, \nu_{v_i} \rangle & h_{i,1} & \ldots & h_{i,j} & \ldots & h_{i,\alpha} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_\alpha, \langle \mu_{v_\alpha}, \nu_{v_\alpha} \rangle & h_{\alpha,1} & \ldots & h_{\alpha,j} & \ldots & h_{\alpha,\alpha} \end{array}\right\},$$

$$
\begin{array}{c|cccccc}
c^1 & v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & \dots & v_j,\langle\mu_{v_j},\nu_{v_j}\rangle & \dots & v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle \\
\hline
v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & \langle\rho_{c^1_{1,1}},\sigma_{c^1_{1,1}}\rangle & \dots & \langle\rho_{c^1_{1,j}},\sigma_{c^1_{1,j}}\rangle & \dots & \langle\rho_{c^1_{1,\alpha}},\sigma_{c^1_{1,\alpha}}\rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
v_i,\langle\mu_{v_i},\nu_{v_i}\rangle & \langle\rho_{c^1_{i,1}},\sigma_{c^1_{i,1}}\rangle & \dots & \langle\rho_{c^1_{i,j}},\sigma_{c^1_{i,j}}\rangle & \dots & \langle\rho_{c^1_{i,\alpha}},\sigma_{c^1_{i,\alpha}}\rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle & \langle\rho_{c^1_{\alpha,1}},\sigma_{c^1_{\alpha,1}}\rangle & \dots & \langle\rho_{c^1_{\alpha,j}},\sigma_{c^1_{\alpha,j}}\rangle & \dots & \langle\rho_{c^1_{\alpha,\alpha}},\sigma_{c^1_{\alpha,\alpha}}\rangle
\end{array},
$$

$$\dots,$$

$$
\left.\begin{array}{c|cccccc}
c^n & v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & \dots & v_j,\langle\mu_{v_j},\nu_{v_j}\rangle & \dots & v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle \\
\hline
v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & \langle\rho_{c^n_{1,1}},\sigma_{c^n_{1,1}}\rangle & \dots & \langle\rho_{c^n_{1,j}},\sigma_{c^n_{1,j}}\rangle & \dots & \langle\rho_{c^n_{1,\alpha}},\sigma_{c^n_{1,\alpha}}\rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
v_i,\langle\mu_{v_i},\nu_{v_i}\rangle & \langle\rho_{c^n_{i,1}},\sigma_{c^n_{i,1}}\rangle & \dots & \langle\rho_{c^n_{i,j}},\sigma_{c^n_{i,j}}\rangle & \dots & \langle\rho_{c^n_{i,\alpha}},\sigma_{c^n_{i,\alpha}}\rangle \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle & \langle\rho_{c^n_{\alpha,1}},\sigma_{c^n_{\alpha,1}}\rangle & \dots & \langle\rho_{c^n_{\alpha,j}},\sigma_{c^n_{\alpha,j}}\rangle & \dots & \langle\rho_{c^n_{\alpha,\alpha}},\sigma_{c^n_{\alpha,\alpha}}\rangle
\end{array}\right\}.
$$

When we have a CIFCMG, we can bijective juxtapose to it the following EIFIM:

$$C = [V^*, V^*, \overline{N}, \langle h_{i,j}, (\lambda(c^1_{i,j}), \langle\mu_{c^1_{i,j}}, \nu_{c^1_{i,j}}\rangle), \dots, (\lambda(c^n_{i,j}), \langle\mu_{c^n_{i,j}}, \nu_{c^n_{i,j}}\rangle\}\rangle],$$

then C has the form

$$
\left\{\begin{array}{c|ccccc}
\perp & v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & \dots & v_j,\langle\mu_{v_j},\nu_{v_j}\rangle & \dots & v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle \\
\hline
v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & h_{1,1} & \dots & h_{1,j} & \dots & h_{1,\alpha} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
v_i,\langle\mu_{v_i},\nu_{v_i}\rangle & h_{i,1} & \dots & h_{i,j} & \dots & h_{i,\alpha} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle & h_{\alpha,1} & \dots & h_{\alpha,j} & \dots & h_{\alpha,\alpha}
\end{array}\right.,
$$

$$
\begin{array}{c|ccc}
c^1 & v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & \dots & v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle \\
\hline
v_1,\langle\mu_{v_1},\nu_{v_1}\rangle & (\langle\rho_{c^1_{1,1}},\sigma_{c^1_{1,1}}\rangle,\langle\mu_{c^1_{1,1}},\nu_{c^1_{1,1}}\rangle) & \dots & (\langle\rho_{c^1_{1,\alpha}},\sigma_{c^1_{1,\alpha}}\rangle,\langle\mu_{c^1_{1,\alpha}},\nu_{c^1_{1,\alpha}}\rangle) \\
\vdots & \vdots & \ddots & \vdots \\
v_i,\langle\mu_{v_i},\nu_{v_i}\rangle & (\langle\rho_{c^1_{i,1}},\sigma_{c^1_{i,1}}\rangle,\langle\mu_{c^1_{i,1}},\nu_{c^1_{i,1}}\rangle) & \dots & (\langle\rho_{c^1_{i,\alpha}},\sigma_{c^1_{i,\alpha}}\rangle,\langle\mu_{c^1_{i,\alpha}},\nu_{c^1_{i,\alpha}}\rangle) \\
\vdots & \vdots & \ddots & \vdots \\
v_\alpha,\langle\mu_{v_\alpha},\nu_{v_\alpha}\rangle & (\langle\rho_{c^1_{\alpha,1}},\sigma_{c^1_{\alpha,1}}\rangle,\langle\mu_{c^1_{\alpha,1}},\nu_{c^1_{\alpha,1}}\rangle) & \dots & (\langle\rho_{c^1_{\alpha,\alpha}},\sigma_{c^1_{\alpha,\alpha}}\rangle,\langle\mu_{c^1_{\alpha,\alpha}},\nu_{c^1_{\alpha,\alpha}}\rangle)
\end{array},
$$

$$\dots,$$

$$
\left.\begin{array}{c|ccc}
c^n & v_1, \langle \mu_{v_1}, \nu_{v_1} \rangle & \cdots & v_\alpha, \langle \mu_{v_\alpha}, \nu_{v_\alpha} \rangle \\
\hline
v_1, \langle \mu_{v_1}, \nu_{v_1} \rangle & (\langle \rho_{c^n_{1,1}}, \sigma_{c^n_{1,1}} \rangle, \langle \mu_{c^n_{1,1}}, \nu_{c^n_{1,1}} \rangle) & \cdots & (\langle \rho_{c^n_{1,\alpha}}, \sigma_{c^n_{1,\alpha}} \rangle, \langle \mu_{c^n_{1,\alpha}}, \nu_{c^n_{1,\alpha}} \rangle) \\
\vdots & \vdots & \ddots & \vdots \\
v_i, \langle \mu_{v_i}, \nu_{v_i} \rangle & (\langle \rho_{c^n_{i,1}}, \sigma_{c^n_{i,1}} \rangle, \langle \mu_{c^n_{i,1}}, \nu_{c^n_{i,1}} \rangle) & \cdots & (\langle \rho_{c^n_{i,\alpha}}, \sigma_{c^n_{i,\alpha}} \rangle, \langle \mu_{c^n_{i,\alpha}}, \nu_{c^n_{i,\alpha}} \rangle) \\
\vdots & \vdots & \ddots & \vdots \\
v_\alpha, \langle \mu_{v_\alpha}, \nu_{v_\alpha} \rangle & (\langle \rho_{c^n_{\alpha,1}}, \sigma_{c^n_{\alpha,1}} \rangle, \langle \mu_{c^n_{\alpha,1}}, \nu_{c^n_{\alpha,1}} \rangle) & \cdots & (\langle \rho_{c^n_{\alpha,\alpha}}, \sigma_{c^n_{\alpha,\alpha}} \rangle, \langle \mu_{c^n_{\alpha,\alpha}}, \nu_{c^n_{\alpha,\alpha}} \rangle)
\end{array}\right\}.
$$

In the example in Section 4, the green-layer entries in row v_1, column v_2, and in row v_2, column v_1, both store $(\langle 0.6, 0.2\rangle, \langle 0.3, 0.4\rangle)$ in the complex representation and $\langle 0.48, 0.46\rangle$ after reduction to the simple representation. Both corresponding count entries are 1.

For the CMGs considered here, we use a common edge-colour palette C_H and allow no loops and at most one edge of each colour between any pair of vertices. We match vertices by identifier and identify each coloured edge by its endpoints and colour. The operations $\oplus$ and $\otimes$ take the union and intersection of the vertex and coloured edge sets, respectively. At common vertices they combine the colours by join and meet, respectively. Under union, a vertex occurring in only one graph retains its colour. The total-count slice is recomputed from the resulting binary colour layers.

Following [8], we can define object $\mathcal{G}(\mathcal{V}, \mathcal{L}_\mathcal{V}, \mathcal{L}_\mathcal{H})$ as the class of CMGs defined in Section 3, with vertex sets contained in $\mathcal{V}$. Let $G_\varnothing$ be the graph with no vertices and no edges. Let G_{full} have vertex set $\mathcal{V}$, vertex colour $\mathcal{L}_\mathcal{L}$ at every vertex, and one edge of each colour in $\mathcal{L}_\mathcal{H}$ between every pair of distinct vertices.

Therefore, we can prove the following theorem.

Theorem 2. *The structures*

$$\langle \mathcal{G}(\mathcal{V}, \mathcal{L}_\mathcal{V}, \mathcal{L}_\mathcal{H}), \oplus^*_\circ, G_\varnothing \rangle \;\; \textit{and} \;\; \langle \mathcal{G}(\mathcal{V}, \mathcal{L}_\mathcal{V}, \mathcal{L}_\mathcal{H}), \otimes^*_\circ, G_{\text{full}} \rangle$$

are commutative monoids.

Consequently, the corresponding EIFIM representations of these CMGs form commutative monoids under operations $\oplus$ and $\otimes$, respectively.

Proof. For a graph G, let V_G be its set of vertex identifiers and let N_G be its set of coloured edges. The operations satisfy:

$$V_{G\oplus H} = V_G \cup V_H, \quad N_{G\oplus H} = N_G \cup N_H,$$
$$V_{G\otimes H} = V_G \cap V_H, \quad N_{G\otimes H} = N_G \cap N_H.$$

At a common vertex, the resulting colour is the join of the two colours for $\oplus^*_\circ$ and their meet for $\otimes^*_\circ$. Under $\oplus^*_\circ$, a vertex occurring in only one operand retains its colour.

Both operations are closed: the endpoints of each retained edge belong to the resulting vertex set, the vertex colours remain in $\mathcal{L}_\mathcal{V}$, and there is at most one edge of each colour for any admissible pair of endpoints. Commutativity follows from the commutativity of set union and intersection and of lattice join and meet.

For the union of three graphs, either bracketing gives the same vertex and edge sets. At each resulting vertex, the colour is the join of its colours in the operands containing that vertex.

This join is independent of the bracketing. For the intersection of three graphs, either bracketing retains the same common vertices and edges, and each surviving vertex receives the meet of its three colours. Thus, both operations are associative.

The empty graph $G_{\varnothing}$ is an identity for $\oplus^*_\circ$. The graph G_{full} contains every admissible vertex and coloured edge. Its vertex colours are all $\mathcal{L}_\mathcal{V}$, so taking the meet with these colours leaves the colours of the other operand unchanged. Hence G_{full} is an identity for $\otimes^*_\circ$.

Finally, the index-matrix encoding is injective: the indices record the vertex identifiers and colours, and the binary colour layers record the coloured edges. On the image of this encoding, the matrix operations act on the same identifiers and colour layers by the rules above. The two monoid structures therefore carry over to the corresponding index-matrix representations. □

6 Conclusion

The present paper is the first step to realize the idea from [8] to introduce IFCMG.

In the next study, we will discuss other operations, as well as relations and modal and level type operators over CMGs and their IM interpretations.

The basic concepts related to graphs (e.g. path, diameter, etc.) and properties (e.g. planarity, (dis)connectedness, symmetry, etc.) of IFCMGs in their IM-forms will be discussed.

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References

[1] Atanassov, K. (1987). Generalized index matrices. *Comptes rendus de l'Academie Bulgare des Sciences*, 40(11), 15–18.

[2] Atanassov, K. (1994). Index matrix representation of the intuitionistic fuzzy graphs. In: *Fifth Scientific Session of the Math. Foundations of Artificial Intelligence Seminar*, 5 October 1994, Sofia. Preprint MRL-MFAIS-10-94, pp. 36–41.

[3] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets*. Springer, Berlin.

[4] Atanassov, K. (2014). *Index Matrices: Towards an Augmented Matrix Calculus*. Springer, Cham.

[5] Atanassov, K. (2017). *Intuitionistic Fuzzy Logics*. Springer, Cham.

[6] Atanassov, K. (2025). *On Index Matrices*. Academic Publishing House "Acad. Marin Drinov", Sofia.

[7] Atanassov, K., Szmidt, E., & Kacprzyk, J. (2013). On intuitionistic fuzzy pairs. *Notes on Intuitionistic Fuzzy Sets*, 19(3), 1–13.

[8] Atanassov, K., & Tsvetanov, S. (2026) Coloured multigraphs and their index matrix representation. *Notes on Number Theory and Discrete Mathematics*, 32(4) (in press).

[9] Bretto, A. (2013). *Hypergraph Theory: An Introduction*. Springer, Cham.

[10] De Domenico, M., Solé-Ribalta, A., Cozzo, E., Kivelä, M., Moreno, Y., Porter, M. A., Gómez, S., & Arenas, A. (2013). Mathematical formulation of multilayer networks. *Physical Review X*, 3, Article ID 041022.

[11] Díaz-Boils, J., & Galdames-Bravo, O. (2023). *A lattice-ordered monoid on multilayer networks*. Preprint. Available online at: `https://arxiv.org/abs/2308.16630`.

[12] Diestel, R. (2025). *Graph Theory*. Springer, Heidelberg.

[13] Harary, F. (1969). *Graph Theory*. Addison-Wesley Publishing Company, Massachusetts.

[14] Harary, F., & Palmer, E. (1973). *Graphical Enumeration*. Academic Press, New York.

[15] Jensen, T., & Toft, B. (1995). *Graph Coloring Problems*. John Wiley & Sons, New York.

[16] Kivelä, M., Arenas, A., Barthelemy, M., Gleeson, J. P., Moreno, Y., & Porter, M. A. (2014). Multilayer networks. *Journal of Complex Networks*, 2(3), 203–271.

[17] Lancaster, P. (1969). *Theory of Matrices*. Academic Press, New York.

[18] MacLane, S., & Birkhoff, G. (1999). *Algebra* (Third Edition). AMS Chelsea Publishing, Providence.

[19] Shannon, A. G., & Atanassov, K. (1994). A first step to a theory of the intuitionistic fuzzy graphs. In: *Lakov, D. (Ed.). Proceedings of the First Workshop on Fuzzy Based Expert Systems*, 28–30 September 1994, Sofia, Bulgaria, pp. 59–61.

[20] Shannon, A. G., & Atanassov, K. (2004). On intuitionistic fuzzy multigraphs and their index matrix interpretations. In: *Proceedings of Second International IEEE Conference Intelligent Systems*, 22–24 June 2004, Varna, Bulgaria, Vol. 2, pp. 440–443.