Notes on Intuitionistic Fuzzy Sets
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Some notes on intuitionistic fuzzy equivalence relations and their use on real data

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Abstract: The research presented in this paper is motivated by possibilities of using fuzzy equivalence relations to classify the data into the specific classes. We try to improve these results with the use of intuitionistic fuzzy equivalence relations. We define the basic structures and their properties, which are used in the paper. Then we present the data which we decided to classify and the methods of processing these data. At the end of the paper we discus obtained results and problems which occurred during the processing of the selected data.

Keywords: Intuitionistic fuzzy set, Intuitionistic fuzzy equivalence relation, Classification.
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1 Introduction

Classification of various data, structures or objects is one of the essential problems in number of different research fields. The research presented in this paper is motivated by possibilities of using fuzzy equivalence relations to classify the data into the specific classes. We use the properties of intuitionistic fuzzy relations and we apply them in order to construct the intuitionistic fuzzy equivalence relations on real data. During construction of such relations, we encountered some problems, which are mentioned in the following parts of this paper.

2 Preliminaries

Fuzzy sets were introduced by professor Lotfi A. Zadeh in 1965 [21]. Later as a natural extension of fuzzy sets, intuitionistic fuzzy sets (shortly IFSs) were introduced by Krassimir Atanassov [2]. Since then, many new properties and applications of this mathematical structures have been designed and implemented. In this section we define the mathematical functions and their properties which are used in this paper.

2.1 Fuzzy sets

Definition 2.1. *Let X be the universe. A set A is called a fuzzy set, if it holds*

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\},$$

where $\mu_A : X \to [0,1]$. Function μ_A is called the membership function of fuzzy set A.

Definition 2.2. *Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ be the finite sets. Then fuzzy relation R between the sets X and Y is defined by the fuzzy relation matrix*

$$R = \begin{pmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & \dots & \mu_R(x_1, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_R(x_n, y_1) & \mu_R(x_n, y_2) & \dots & \mu_R(x_n, y_m) \end{pmatrix}.$$

Then $\mu_R : X \times Y \to [0,1]$.

Definition 2.3. *Let X, Y, Z be finite sets. Suppose that R is a fuzzy relation on the Cartesian product $X \times Y$, S is a fuzzy relation on the Cartesian product $Y \times Z$ and T is a fuzzy relation on the Cartesian product $X \times Z$. Then the* $\max - \min$ *composition of fuzzy relations R and S is defined by the following way*

$$\mu_T(x, z) = \sup_{y \in Y} \min(\mu_R(x, y), \mu_S(y, z)).$$

We will shortly write

$$T = R \circ S.$$

Remark 1. There are special operations defined on fuzzy sets—t-norms and t-conorms, which are used to define various types of intersections and unions on fuzzy sets. The most used t-norm is minimum t-norm. By using this t-norm the intersection of two fuzzy sets is also the fuzzy set, which is equal to the minimum of the membership functions of input fuzzy sets at each point. In addition, t-norms and t-conorms are used to define various types of compositions of fuzzy relations (see for example [16]).

Remark 2. In this paper we will use one more type of composition of fuzzy relations, so called $\min - \max$ composition. Such composition P is defined for fuzzy relations R and S as follows

$$\mu_P(x, z) = \inf_{y \in Y} \max(\mu_R(x, y), \mu_S(y, z)).$$

We will shortly write

$$P = R \diamond S.$$

Definition 2.4. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let R be the fuzzy relation on Cartesian product $X \times X$. If fuzzy relation R satisfied the properties of*

- *reflexivity, i.e., $\mu_R(x_i, x_i) = 1$ holds for each $x_i \in X$,*
- *symmetry, i.e., $\mu_R(x_i, x_j) = \mu_R(x_j, x_i)$ holds for each $x_i, x_j \in X$,*
- *transitivity, i.e., $\min[\mu_R(x_i, x_j), \mu_R(x_j, x_k)] \leq \mu_R(x_i, x_k)$ holds for each $x_i, x_j, k_k \in X$,*

then R is called fuzzy equivalence relation. If fuzzy relation R satisfied the properties of reflexivity and symmetry, then R is called fuzzy tolerance relation.

Remark 3. In general, to determine transitivity, any t-norm can be used, and therefore the term t-transitivity is generally used.

While verification of the first two conditions is simple, further verification of the third condition is not. If for some fuzzy relation reflexivity holds, then all diagonal elements of its matrix are equal to 1. If some fuzzy relation is symmetric, then for its matrix $R = R^T$ holds, where R^T is a transposition of the matrix R. If we have fuzzy tolerance relation and we need to verify the transitivity of this relation, we should use properties resulting from the following Theorem (see [16]):

Theorem 1. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, let R be the fuzzy tolerance relation on Cartesian product $X \times X$ and let $\circ$ represent the $\max - \min$ composition of two fuzzy relations. Let us denote $R^2 = R \circ R$ and $R^k = R^{k-1} \circ R$. Then if

$$R^k = R^{k-1}$$

then the fuzzy relation R^{k-1} is the fuzzy equivalence relation.

Remark 4. Let R be the fuzzy tolerance relation on Cartesian product $X \times X$. From the previous Theorem it follows

- If $R^2 = R$, then relation R satisfied the property of transitivity and therefore R is the fuzzy equivalence relation.
- If $R^2 \neq R$, then relation R does not satisfy the property of transitivity. But if we need to generate the fuzzy equivalence relation from the relation R, we should use the $\max - \min$ composition of this relation with itself. After the finite number of steps we obtain the fuzzy equivalence relation. In addition, it was proved that the number of steps is always smaller or at most equal to the dimension n of the matrix R.

One of the ways to use the fuzzy equivalence relations is to use them for the classification of the data. From mathematical point of view, classification is equivalent to the decomposition of the set. For this purpose we define some more structures.

Definition 2.5. *Let X be a set and let $A_1, A_2, \ldots A_l$ be the system of crisp nonempty sets. Then the system of sets $A_1, A_2, \ldots A_l$ is called the decomposition of a set X, if the following properties are satisfied:*

- $A_1 \cup A_2 \cup \ldots \cup A_l = X$,
- $A_i \cap A_j = \emptyset, \;\; i, j \in \{1, 2, \ldots, l\}, i \neq j$.

There is the relationship between the relation of equivalence and the decomposition of a set, as mentioned in following Theorem (see [16]).

Theorem 2. Let X be a finite set and let R be the equivalence relation defined on X. Each decomposition of the set X defines the equivalence relation and on the other side, each equivalence relation defines the decomposition of the set X.

Definition 2.6. *Let X be a set and let $\alpha \in [0, 1]$. Then the α-cut of fuzzy set A is defined as the set with the following property:*

$$A^{(\alpha)} = \{x \in X, \; \mu_A(x) \geq \alpha\}.$$

In the next example we present decomposition of the set X with the use of fuzzy equivalence relation.

Example 1. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set and let R be the fuzzy equivalence relation defined on X by the following matrix

$$R = \begin{pmatrix} 1 & 0.3 & 0.9 & 0.9 \\ 0.3 & 1 & 0.3 & 0.3 \\ 0.9 & 0.3 & 1 & 1 \\ 0.9 & 0.3 & 1 & 1 \end{pmatrix}.$$

Let $\alpha = 0.9$. Then we could construct binary matrix $R^{(0.9)}$ from matrix R by using the value α in the following way

$$R^{(0.9)} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

From the binary matrix $R^{(0.9)}$ we get the following decomposition of the set X

$$\mathcal{A}^{(0.9)} = \{\{x_1, x_3, x_4\}, \{x_2\}\}.$$

By choosing different values of α we can get different decompositions of the set X. In this example we get three different decompositions of the set X

$$\mathcal{A}^{(1)} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}\}, \;\; \mathcal{A}^{(0.9)} = \{\{x_2\}, \{x_1, x_3, x_4\}\}, \;\; \mathcal{A}^{(0.3)} = \{\{x_1, x_2, x_3, x_4\}\}.$$

These decompositions of the set X could be shown by dendrogram (see Figure 1). It is well know, that by choosing different values of α, one could get different number of sets, which represent the clusters of classifications. In this paper we will use this idea.

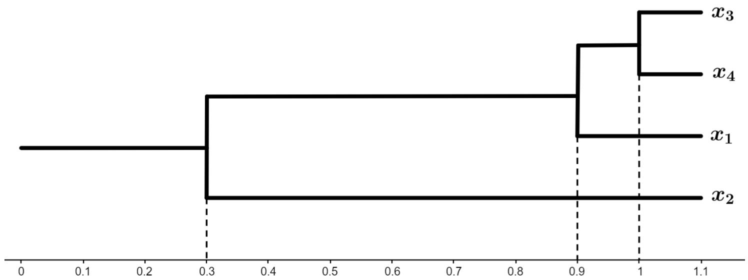


Figure 1. Dendrogram related to the Example 1

2.2 Intuitionistic fuzzy sets

Remark 5. In the rest of the paper the properties of intuitionistic fuzzy set will be used. Therefore in some of the following parts of the paper notations similar to the Section 2.1 will be used.

Definition 2.7. *Let X be the universe. An intuitionistic fuzzy set A is a set*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

of the functions $\mu_A : X \to [0,1], \nu_A : X \to [0,1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Function μ_A is called the membership function and function ν_A is called the non-membership function. $\mathcal{F}$ denotes the family of all intuitionistic fuzzy sets (shortly IFSs).

Definition 2.8. *Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets. Then it holds*

$$A = B \iff (\mu_A = \mu_B) \;\&\; (\nu_A = \nu_B),$$
$$A \leq B \iff (\mu_A \leq \mu_B) \;\&\; (\nu_A \geq \nu_B),$$
$$A \bigwedge B = ((\mu_A \wedge \mu_B),\ (\nu_A \vee \nu_B)),$$
$$A \bigvee B = ((\mu_A \vee \mu_B),\ (\nu_A \wedge \nu_B)).$$

In addition

$$(0,1) \leq A \leq (1,0),$$

which means, that element $(0,1)$ is the smallest element and element $(1,0)$ is the greatest element of the set $\mathcal{F}$.

Remark 6. In this text operations $\wedge$ and $\vee$ represent operations $\min$ and $\max$ respectively. In general any t-norm and t-conorm could be used instead of these operations.

Definition 2.9. *Let $R = (r_{i,j})_{n\times m}$ be a matrix. If all elements of matrix R belong to $\mathcal{F}$, then R is called an intuitionistic fuzzy matrix.*

It is obvious, that intuitionistic fuzzy matrix R represents the intuitionistic fuzzy relation between two sets, for example X and Y. In the following text we will work with the intuitionistic fuzzy relations defined on the Cartesian product $X \times X$ exclusively.

Definition 2.10. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. An intuitionistic fuzzy relation R on X is called*

- *reflexive, iff $R(x_i, x_i) = (1, 0)$ holds for each $x_i \in X$,*
- *symmetric, iff $R(x_i, x_j) = R(x_j, x_i)$ holds for each $x_i, x_j \in X$,*
- *transitive, iff $\sup_{x_j \in X} \min[R(x_i, x_j), R(x_j, x_k)] \leq R(x_i, x_k)$ holds for each $x_i, x_j, k_k \in X$.*

Definition 2.11. *Let X be a finite set and let R be the intuitionistic fuzzy relation on X. If intuitionistic fuzzy relation R is reflexive, symmetric and transitive, then R is called intuitionistic fuzzy equivalence relation. If intuitionistic fuzzy relation R satisfies the properties of reflexivity and symmetry, then R is called intuitionistic fuzzy tolerance relation.*

Definition 2.12. *The intuitionistic fuzzy relation matrix R could be written using two matrices $R = [R_\mu, R_\nu]$. Then the first matrix R_μ contains the membership degrees of elements of Cartesian product $X \times X$ and the second matrix R_ν contains the non-membership degrees of elements of Cartesian product $X \times X$.*

Definition 2.13. *Let X be a finite set and let $R = [R_\mu, R_\nu]$, $S = [S_\mu, S_\nu]$ and $T = [T_\mu, T_\nu]$ be the intuitionistic fuzzy relations on X. Let $\circ$ represent the* $\max - \min$ *composition and let $\diamond$ represent the* $\min - \max$ *composition of two fuzzy relations. Then the* $\max - \min$ *composition (denoted by $\star$) of intuitionistic fuzzy relations R and S is defined in the following way*

$$T = R \star S = [R_\mu \circ S_\mu, R_\nu \diamond S_\nu].$$

Theorem 3. Let X be a finite set, let R be the intuitionistic fuzzy tolerance relation on X and let $\star$ represent the $\max - \min$ composition of two intuitionistic fuzzy relations. Denote $R^2 = R \star R$ and $R^k = R^{k-1} \star R$. Then if

$$R^k = R^{k-1}$$

then the intuitionistic fuzzy relation R^{k-1} is the intuitionistic fuzzy equivalence relation.

Similar conclusion as in Remark 4 holds for this Theorem as well.

Definition 2.14. *(see [2]) Let X be a finite set and let $(\alpha, \beta) \in \mathcal{F}$. Then (α, β)-cut of the intuitionistic fuzzy set A is given by the following formula*

$$A^{(\alpha,\beta)} = \{x \in X, \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}.$$

Definition 2.15. *(see [6]) Let X be a finite set and let R be the intuitionistic fuzzy equivalence relation on X. Let a be any element of X. Then the IFS defined by*

$$aR = \{\langle x, (a\mu_R)(x), (a\nu_R)(x)\rangle | x \in X\}$$

where

$$(a\mu_R)(x) = \mu_R(a, x) \text{ and } (a\nu_R)(x) = \nu_R(a, x)$$

for each $x \in X$, is called an intuitionistic fuzzy equivalence class of a with respect to R.

Theorem 4. (see [6]) Let X be a finite set and let R be the intuitionistic fuzzy equivalence relation on X. Let a be any element of X. Then for any $(\alpha, \beta) \in \mathcal{F}$ the IFS defined by

$$R^{(\alpha,\beta)}(a) = [a]$$

is the equivalence class of a with respect to the intuitionistic fuzzy equivalence relation $R^{(\alpha,\beta)}$.

Theorem 5. (see [6]) Let X be a finite set and let R be the intuitionistic fuzzy equivalence relation on X. Let a, b be any element of X. $[a] = [b]$ denotes the equivalence classes of a and b with respect to the intuitionistic fuzzy equivalence relation $R^{(\alpha,\beta)}$. Then

$$[a] = [b] \text{ iff } (a, b) \in R^{(\alpha,\beta)}.$$

3 Processing of the data

Based on the properties mentioned in the previous section, it is clear that intuitionistic fuzzy sets could be used to classify real data. At the same time several questions arise. For example, do we get different results by using fuzzy relations and intuitionistic fuzzy relations? If we have an intuitionistic fuzzy tolerance relation $R = [R_\mu, R_\nu]$ based on known data and we want to create the intuitionistic fuzzy equivalence relation with the use of $\max - \min$ composition $\star$ of relation R with itself, is the number of required compositions of the matrix R_μ with itself and the number of required compositions of the matrix R_ν with itself, the same number? For answering these questions, author decided to use the data from their previous works. In the first part of this section, part of data from tire tread image database are processed. In the second part of the presented paper, the data from InterCriteria Analysis will be processed. We also offer discussion related to obtained results.

3.1 Processing of the data from tire tread image database

There are many real word problems, in which the classification of the objects into some specific classes is needed. This type of classification problem was used in the building of the database of tire tread prints which contains images of tire treads downloaded from various web pages ([18]). This set is comprised of images with different positions of tires. Therefore it is important to process the images in such way that the best possible position of tire tread sample will be obtained. On the behalf of this idea there was created number of methods (see [11–13, 19]) of classifying images into the chosen number of classes. In this part of the paper we use the results of classification of the images on the basis of intuitionistic fuzzy sets and we apply the conditions of intuitionistic fuzzy equivalence relations to obtain new results.

For pre-processing of the images we developed the specific algorithm (see [11]). After using this algorithm each image is represented by 16 coordinate vector. To calculate the value of membership and non-membership function of each coordinate of the image vector we used the procedure that was described in the paper [9]. Now each image is described by two 16 coordinate vectors (vector with its membership degrees and vector with its non-membership degrees).

We need to classify images into seven classes. For each of these seven classes we choose so called templates. Template images represent their classes. Three images from each class were selected as the templates. Therefore, we used 21 template images. To build the database of templates we used the same approach as mentioned in the paper [11]. The idea is to take any new image from the tire tread database and determine the intuitionistic fuzzy equivalence relation between this image and the template database. Then by using (α, β)-cuts of the created relation, we classify new image into the specific class.

During the processing of the data we realized that the greatest problem of this idea is method of calculation of the intuitionistic fuzzy relation elements. To describe the different types of intuitionistic fuzzy relation element construction and to discuss the specific problems related to these approaches, one of the images from the tire image database was randomly chosen. In this part of the paper we present some approaches that we used.

- In the paper [16] the way of calculation of the elements of fuzzy relation from the data represented by vectors was presented. Author used the cosine based similarity measure to calculate the similarity between each two vectors. The created matrix represents the fuzzy relation, which is reflexive and symmetric, but not transitive. We wanted to use the similar approach. We took 22 vectors which describe the membership part of all 22 images and we created the matrix of type 22×22 with the use of cosine based similarity measure. This way we obtained the fuzzy relation R_μ. In the next step we took 22 vectors which describe the non-membership part of all 22 images and we create the matrix of type 22×22 wit the use of cosine based similarity measure. Then we compute the elements of the fuzzy relation R_ν as $1-$ "$calculated\, values$". It is obvious, that the created relation $R = [R_\mu, R_\nu]$ is reflexive and symmetric. To verify the obtained results, we compute matrix $\bar{R} = R_\mu \oplus R_\nu$ where $\oplus$ represents the classical matrix summation and we found out, that some elements of the matrix $\bar{R}$ are greater than 1. Therefore created matrix $R = [R_\mu, R_\nu]$ does not represent the intuitionistic fuzzy relation.

- In the next step we try to use idea of cosine based similarity measure in another way. Since the formula $\sin^2(x) + \cos^2(x) = 1$ is well known, we modified the elements of relations R_μ and R_ν from previous point to satisfy the mentioned formula. We could conclude that the results similar to the previous point were obtained. The verification matrix $\bar{R} = R_\mu \oplus R_\nu$ contains some elements greater then 1.

To construct the intuitionistic fuzzy relations between the image vectors, we decided to use another type of (fuzzy) similarity and distance measures. Some results are listed below.

If we take 22 vectors (the membership part vectors and in the next step the non-membership part vectors) and as an example we compute the pairwise Euclidean distance between them, we get the matrix which also contains the values greater then 1. Therefore we need to normalize these elements into the interval $[0, 1]$. After the normalization we need to modify one of the matrices $\hat{R}_\mu, \hat{R}_\nu$ into the correct form. For example, if we use some of the similarity measures, then we get $\hat{R}_\mu(x_i, x_i) = 1$ and also $\hat{R}_\nu(x_i, x_i) = 1$. Therefore we need to modify the values of elements of non-membership matrix. In this situation we used the formulas $R_\mu = \hat{R}_\mu$ and

$R_\nu = 1 - \hat{R}_\nu$. By using this procedure we obtain the matrices for which holds that each element of the matrix $\bar{R} = R_\mu \oplus R_\nu$ is equal to 1.

Then we compose matrix R_μ with itself and also matrix R_ν with itself. When using various images from described dataset as inputs, the number of compositions of the relation R_μ with itself and the number of compositions of the relation R_ν with itself is equal. Furthermore, output from created equivalence relations $\bar{R}_\mu$ and $\bar{R}_\nu$ were equivalent dendrograms. Therefore it is not necessary to use intuitionistic fuzzy relations, since they are equal to fuzzy relations.

3.2 Processing of the data from InterCriteria Analysis

InterCriteria Analysis (shortly ICA) is a mathematical method that has been developed in Bulgaria in 2014 (see [4]) with the aim to support decision making in multiobject multicriteria problems, using the paradigms of intuitionistic fuzzy sets and index matrices. In the original formulation of the problem, part of the criteria in an industrial multicriteria decision making problem exhibit high complexity and cost of the measurement. In [4] authors designed and implemented a method which identifies existence of strong enough correlations between cost unfavorable criteria and the rest of the criteria with high enough precision. In ICA, the terminology "positive / negative consonance" or "dissonance" is used.

As input data of ICA a two-dimensional table with the measurements or evaluations of m objects against n criteria is required. This method returns an $n \times n$ matrix with intuitionistic fuzzy pairs, defining the degrees of consonance between each pair of criteria. The essence of the method is in the exhaustive pairwise comparison of the values of the measurements of all objects in the set against pairs of criteria, with all possible pairs being traversed, while counters being maintained for the percentage of the cases when the relations between the pairs of evaluations have been 'greater than', 'less than' or 'equal'. In these days ICA is used not just for comparison of criteria, but also for optimization of parameters (see [1, 8, 15, 17]) and it was applied in number of areas of life (see [7, 10, 14, 20, 22]).

In this part of the paper the data, which were presented in [5] are used. By using ICA, the relation $R = [R_\mu, R_\nu]$ was obtained. The computed matrices R_μ and R_ν are presented on Figure 2 and Figure 3. This example was chosen on the basis of small number of criteria – 12 criteria were used, while there were some elements of the relation R for which the sum of membership and non-membership degree is less then 1.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
1.	1.00	0.74	0.63	0.75	0.81	0.83	0.77	0.74	0.85	0.51	0.80	0.83
2.	0.74	1.00	0.50	0.70	0.74	0.70	0.65	0.61	0.77	0.66	0.81	0.79
3.	0.63	0.50	1.00	0.48	0.56	0.62	0.67	0.74	0.63	0.42	0.59	0.60
4.	0.75	0.70	0.48	1.00	0.79	0.70	0.63	0.59	0.70	0.53	0.74	0.77
5.	0.81	0.74	0.56	0.79	1.00	0.73	0.69	0.66	0.76	0.57	0.78	0.82
6.	0.83	0.70	0.62	0.70	0.73	1.00	0.81	0.72	0.80	0.50	0.75	0.75
7.	0.77	0.65	0.67	0.63	0.69	0.81	1.00	0.74	0.77	0.47	0.71	0.71
8.	0.74	0.61	0.74	0.59	0.66	0.72	0.74	1.00	0.72	0.53	0.70	0.72
9.	0.85	0.77	0.63	0.70	0.76	0.80	0.77	0.72	1.00	0.57	0.83	0.81
10.	0.51	0.66	0.42	0.53	0.57	0.50	0.47	0.53	0.57	1.00	0.63	0.60
11.	0.80	0.81	0.59	0.74	0.78	0.75	0.71	0.70	0.83	0.63	1.00	0.87
12.	0.83	0.79	0.60	0.77	0.82	0.75	0.71	0.72	0.81	0.60	0.87	1.00

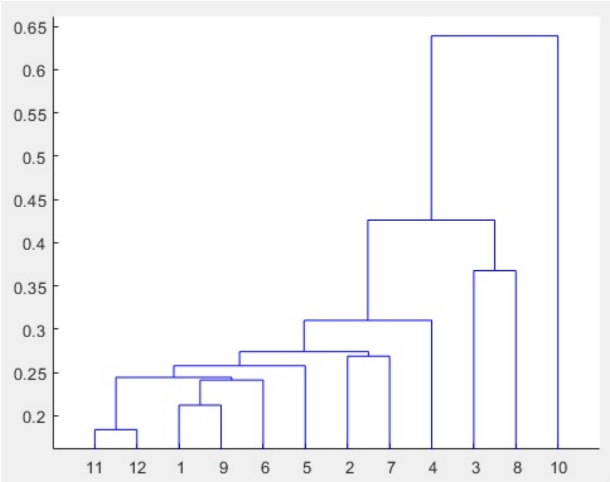
Figure 2. Membership part of intuitionistic fuzzy relation

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
1.	0.00	0.21	0.32	0.13	0.13	0.08	0.14	0.20	0.09	0.43	0.12	0.12
2.	0.21	0.00	0.46	0.19	0.20	0.22	0.28	0.34	0.17	0.29	0.12	0.15
3.	0.32	0.46	0.00	0.39	0.39	0.30	0.26	0.21	0.32	0.54	0.34	0.35
4.	0.13	0.19	0.39	0.00	0.08	0.14	0.21	0.28	0.17	0.35	0.12	0.10
5.	0.13	0.20	0.39	0.08	0.00	0.18	0.22	0.28	0.18	0.38	0.15	0.11
6.	0.08	0.22	0.30	0.14	0.18	0.00	0.09	0.19	0.11	0.42	0.15	0.15
7.	0.14	0.28	0.26	0.21	0.22	0.09	0.00	0.19	0.15	0.45	0.19	0.20
8.	0.20	0.34	0.21	0.28	0.28	0.19	0.19	0.00	0.21	0.42	0.23	0.22
9.	0.09	0.17	0.32	0.17	0.18	0.11	0.15	0.21	0.00	0.38	0.10	0.13
10.	0.43	0.29	0.54	0.35	0.38	0.42	0.45	0.42	0.38	0.00	0.30	0.34
11.	0.12	0.12	0.34	0.12	0.15	0.15	0.19	0.23	0.10	0.30	0.00	0.07
12.	0.12	0.15	0.35	0.10	0.11	0.15	0.20	0.22	0.13	0.34	0.07	0.00

Figure 3. Non-membership part of intuitionistic fuzzy relation

The matrices R_μ and R_ν represent the intuitionistic fuzzy relations, which are reflexive and symmetric. Therefore in the first step we compose these matrices with themselves to find out if they are transitive. The result was that these matrices are not transitive. To satisfy the equality $R_\mu^k = R_\mu^{k-1}$ six $\max - \min$ compositions of the matrix R_μ with itself were needed. Similarly, to satisfy the equality $R_\nu^k = R_\nu^{k-1}$ six $\min - \max$ compositions of the matrix R_ν with itself were needed.

In the next step we compare the dendrograms of the equivalence relations R_μ^{k-1} and R_ν^{k-1}. The results can be seen on the Figure 4. As it follows from this Figure, the dendrograms are not the same.



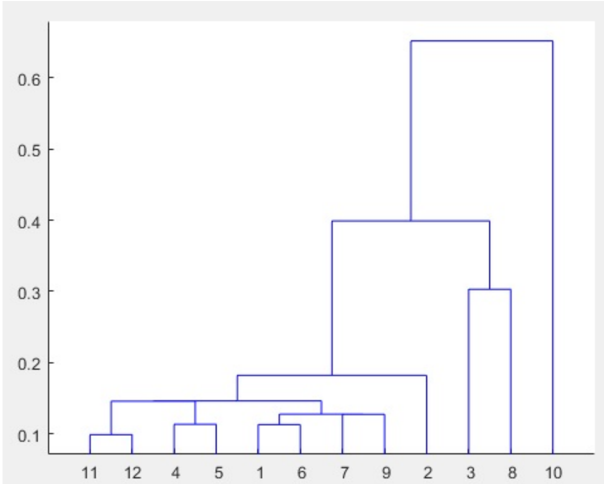


Figure 4. Dendrograms of membership and non-membership part of intuitionistic fuzzy relation

Depending on the problem which is being solved, one could deduce interesting conclusions. When classifying data with the use of fuzzy equivalence relation there is possibility of classifying an object to more then one class with the same degree. In such case, one could use intuitionistic fuzzy equivalence relation, since the non-membership part of this relation could help with the classification of the object to one specific class.

The paper [5] posed the question about choosing the criteria triples, which are in the strongest consonance. From the membership part of intuitionistic fuzzy equivalence relation (Figure 4, left) it is obvious that the criteria 11 and 12 are in the strongest consonance. The nearest criteria to the criteria 11 and 12 are criteria 1, 9 and 6. From the non-membership part of intuitionistic fuzzy equivalence relation (Figure 4, right) it is also obvious, that the criteria 11 and 12 are in the strongest consonance. The nearest criteria to the criteria 11 and 12 are criteria 4 and 5 and then criteria 1, 6, 7 and 9. From this observation we could conclude that the non-membership part of intuitionistic fuzzy equivalence relation could improve the evaluation of mutual relations of considered criteria.

4 Conclusions

In this paper, the properties of intuitionistic fuzzy equivalence relations were applied on real data. We pointed out the problems that may occur when processing such data with the use of intuitionistic fuzzy relations. We present the situation where there is not any difference between use of fuzzy relations and intuitionistic fuzzy relations. We also show an example, where the obtained results are different.

In future work, we would like to pre-process the data of tire tread images in different ways, so that we could use the advantages of intuitionistic fuzzy relations on these data. We would like to look deeper into the relationship between triples of criteria which follow from the obtained dendrograms (Figure 4).

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