New operations over intuitionistic fuzzy index matrices

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Abstract: In this paper, eight new operations over intuitionistic fuzzy index matrix are introduced and some of their basic properties are studied.

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1 Introduction

The concept of Index Matrix (IM) was discussed in a series of papers (see, e.g., [1, 2] and others) and it was extended to the concept of an Intuitionistic Fuzzy Index Matrix (IFIM) in [3].

Initially, in Section 2, we give some elements of Intuitionistic Fuzzy Logic (IFL, see, e.g., [4]). In Section 3, following [3], we give definition of IFIM and some operations over them.

Eight new operations over IFIMs are introduced in Section 4 and some of their properties are discussed.

2 Short remarks on intuitionistic fuzzy logic

In IFL, the truth-value function \( V \) assigns to each proposition \( p \) two real numbers \( \mu(p), \nu(p) \in [0, 1] \), called “degree of validity” and “degree of non-validity”, which satisfy condition (see [?]):

\[
\mu(p) + \nu(p) \leq 1.
\]
If for propositions \( x \) and \( y \), \( V(x) = \langle a, b \rangle \) and \( V(y) = \langle c, d \rangle \), where \( a, b, c, d, a + b, c + d \in [0, 1] \), then
\[
V(x) = V(y) \text{ if and only if } a = c \text{ and } b = d,
\]
\[
V(x) \leq V(y) \text{ if and only if } a \leq c \text{ and } b \geq d,
\]
\[
V(x) < V(y) \text{ if and only if } a \leq c \text{ and } b > d, \text{ or } a < c \text{ and } b \geq d,
\]
\[
V(x) \lor V(y) = V(x \lor y) = \langle \max(a, c), \min(b, d) \rangle,
\]
\[
V(x) \land V(y) = V(x \land y) = \langle \min(a, c), \max(b, d) \rangle.
\]

When we have \( n \) variables \( x_1, x_2, ..., x_n \) with truth-values \( \langle a_i, b_i \rangle \) for \( a_i, b_i, a_i + b_i \in [0, 1] \), where \( i = 1, 2, ..., n \), then
\[
V(\lor_i x_i) = \langle \max_i a_i, \min_i b_i \rangle,
\]
\[
V(\land_i x_i) = \langle \min_i a_i, \max_i b_i \rangle.
\]

For the needs for Section 4, we introduce also operation “average value” for the truth-values of \( n \geq 2 \) variables, that is analogous of operation \( @ \), defined over the intuitionistic fuzzy sets (see [4]):
\[
V(@_i x_i) = \langle \frac{1}{n} \sum_i a_i, \frac{1}{n} \sum_i b_i \rangle.
\]

### 3 Short remarks on index matrices and intuitionistic fuzzy index matrices

Let \( I \) be a fixed set of indices and \( \mathcal{R} \) be the set of the real numbers. In [2], by IM with index sets \( K \) and \( L \) \((K, L \subseteq I)\), we denoted the object:

\[
[K, L, \{a_{k,l_j}\}] \equiv \begin{array}{cccc}
  l_1 & l_2 & \ldots & l_n \\
  k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \ldots & a_{k_1,l_n} \\
  k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \ldots & a_{k_2,l_n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  k_m & a_{k_m,l_1} & a_{k_m,l_2} & \ldots & a_{k_m,l_n}
\end{array}
\]

where \( K = \{k_1, k_2, ..., k_m\} \), \( L = \{l_1, l_2, ..., l_n\} \), for \( 1 \leq i \leq m \), and \( 1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{R} \).

In [2], different operations, relations and operators are defined over IMs.

In [3], by IFIM with the above mentioned index sets, we denoted the object:

\[
[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}] \equiv \begin{array}{cccc}
  l_1 & l_2 & \ldots & l_n \\
  k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \langle \mu_{k_1,l_2}, \nu_{k_1,l_2}\rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\
  k_2 & \langle \mu_{k_2,l_1}, \nu_{k_2,l_1}\rangle & \langle \mu_{k_2,l_2}, \nu_{k_2,l_2}\rangle & \ldots & \langle \mu_{k_2,l_n}, \nu_{k_2,l_n}\rangle \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \langle \mu_{k_m,l_2}, \nu_{k_m,l_2}\rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle
\end{array}
\]

where for every \( 1 \leq i \leq m, 1 \leq j \leq n \): \( 0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1 \).
For the IMs \( A = [K, L, \{ \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle \}] \), \( B = [P, Q, \{ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle \}] \), operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other specific ones.

(a) **addition** \( A \oplus B = [K \cup P, L \cup Q, \{ \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle \}] \), where

\[
\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =
\begin{cases}
\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\
\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\
\langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \text{ and } v_w = l_j = q_s \in L \cap Q \\
\langle 0, 1 \rangle, & \text{otherwise}
\end{cases}
\]

(b) **termwise multiplication** \( A \odot B = [K \cap P, L \cap Q, \{ \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle \}] \), where

\[
\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =
\begin{cases}
\langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{if } t_u = k_i = p_r \in K \cap P \\
\langle 0, 1 \rangle, & \text{otherwise}
\end{cases}
\]

(c) **multiplication** \( A \odot B = [K \cup (P - L), Q \cup (L - P), \{ \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle \}] \), where

\[
\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =
\begin{cases}
\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\
\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\
\langle \max_{l_j=\rho_{p_r,q_s}}(\min(\mu_{k_i,l_j}, \rho_{p_r,q_s})), \min_{l_j=\rho_{p_r,q_s}}(\max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})) \rangle, & \text{if } t_u = k_i = p_r \in L \cap P \\
\langle 0, 1 \rangle, & \text{otherwise}
\end{cases}
\]

(d) **structural subtraction** \( A \ominus B = [K - P, L - Q, \{ \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle \}] \), where “−” is the set–theoretic difference operation and

\[
\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.
\]

(e) **negation of an IFIM** \( \neg A = [K, L, \{ \neg(\mu_{k_i,l_j}, \nu_{k_i,l_j}) \}] \), where \( \neg \) is one of the above (or another) negations.
(f) **termwise subtraction** \( A - B = A \oplus -B \).

Let the two IFIMs \( A = [K, L, \{a_{k,l}, b_{k,l}\}] \) and \( B = [P, Q, \{c_{p,q}, d_{p,q}\}] \) be given. We shall introduce the following (new) definitions where \( \subset \) and \( \subseteq \) denote the relations "strong inclusion" and "weak inclusion".

(a) **strict relation “inclusion about dimension”**

\[
A \subset_d B \iff ((K \subset P) \& (L \subset Q)) \lor (K \subseteq P) \& (L \subset Q) \lor (K \subset P) \& (L \subseteq Q) \\
\& (\forall k \in K) (\forall l \in L) (\langle a_{k,l}, b_{k,l} \rangle = \langle c_{k,l}, d_{k,l} \rangle).
\]

(b) **non-strict relation “inclusion about dimension”**

\[
A \subseteq_d B \iff (K \subseteq P) \& (L \subseteq Q) \& (\forall k \in K) (\forall l \in L) (\langle a_{k,l}, b_{k,l} \rangle = \langle c_{k,l}, d_{k,l} \rangle).
\]

(c) **strict relation “inclusion about value”**

\[
A \subset_v B \iff (K = P) \& (L = Q) \& (\forall k \in K) (\forall l \in L) (\langle a_{k,l}, b_{k,l} \rangle < \langle c_{k,l}, d_{k,l} \rangle).
\]

(d) **non-strict relation “inclusion about value”**

\[
A \subseteq_v B \iff (K = P) \& (L = Q) \& (\forall k \in K) (\forall l \in L) (\langle a_{k,l}, b_{k,l} \rangle \leq \langle c_{k,l}, d_{k,l} \rangle).
\]

(e) **strict relation “inclusion”**

\[
A \subset B \iff ((K \subset P) \& (L \subset Q)) \lor (K \subseteq P) \& (L \subset Q) \lor (K \subset P) \& (L \subseteq Q) \\
\& (\forall k \in K) (\forall l \in L) (\langle a_{k,l}, b_{k,l} \rangle < \langle c_{k,l}, d_{k,l} \rangle).
\]

(f) **non-strict relation “inclusion”**

\[
A \subseteq B \iff (K \subseteq P) \& (L \subseteq Q) \& (\forall k \in K) (\forall l \in L) (\langle a_{k,l}, b_{k,l} \rangle \leq \langle c_{k,l}, d_{k,l} \rangle).
\]

4 **Main results**

Let the IFIM \( A \) be given and let \( k_0 \not\in K \) and \( l_0 \not\in L \) are two indices. Now, we introduce the following eight operations over it:

(a) **max-row-aggregation**

\[
\rho_{\text{max}}(A, k_0) = \frac{1}{k_0} \left[ \begin{array}{cccc}
& l_1 & \ldots & l_n \\
\max_{1 \leq i \leq m} (a_{k_i, l_i}), \min_{1 \leq i \leq m} (b_{k_i, l_i}) & \max_{1 \leq i \leq m} (a_{k_i, l_n}), \min_{1 \leq i \leq m} (b_{k_i, l_n})
\end{array} \right],
\]

(b) **min-row-aggregation**

\[
\rho_{\text{min}}(A, k_0) = \frac{1}{k_0} \left[ \begin{array}{cccc}
& l_1 & \ldots & l_n \\
\min_{1 \leq i \leq m} (a_{k_i, l_i}), \max_{1 \leq i \leq m} (b_{k_i, l_i}) & \min_{1 \leq i \leq m} (a_{k_i, l_n}), \max_{1 \leq i \leq m} (b_{k_i, l_n})
\end{array} \right],
\]
(c) average-row-aggregation

$$\rho_{av}(A, k_0) = k_0 \begin{bmatrix} l_1 & \ldots & l_n \\ \frac{1}{m} \sum_{i=1}^{m} a_{k_i,l_1} \cdot \frac{1}{m} \sum_{i=1}^{m} b_{k_i,l_1} & \ldots & \frac{1}{m} \sum_{i=1}^{m} a_{k_i,l_n} \cdot \frac{1}{m} \sum_{i=1}^{m} b_{k_i,l_n} \end{bmatrix}$$

(d) max-column-aggregation

$$\sigma_{max}(A, l_0) = k_1 \begin{bmatrix} l_0 \\ \frac{\max (a_{k_1,l_j}, \min (b_{k_1,l_j}))}{1 \leq j \leq n} \\ \vdots \\ \frac{\max (a_{k_m,l_j}, \min (b_{k_m,l_j}))}{1 \leq j \leq n} \end{bmatrix}$$

(e) min-column-aggregation

$$\sigma_{min}(A, l_0) = k_1 \begin{bmatrix} l_0 \\ \frac{\min (a_{k_1,l_j}, \max (b_{k_1,l_j}))}{1 \leq j \leq n} \\ \vdots \\ \frac{\min (a_{k_m,l_j}, \max (b_{k_m,l_j}))}{1 \leq j \leq n} \end{bmatrix}$$

(f) average-column-aggregation

$$\sigma_{av}(A, l_0) = k_1 \begin{bmatrix} l_0 \\ \frac{\frac{1}{n} \sum_{j=1}^{n} a_{k_1,l_j} \cdot \frac{1}{n} \sum_{j=1}^{n} b_{k_1,l_j}}{1 \leq j \leq n} \\ \vdots \\ \frac{\frac{1}{n} \sum_{j=1}^{n} a_{k_m,l_j} \cdot \frac{1}{n} \sum_{j=1}^{n} b_{k_m,l_j}}{1 \leq j \leq n} \end{bmatrix}$$

We can see immediately, that for every IFIM $A$ and for every index $i$:

1. $\rho_{max}(\rho_{max}(A, i), i) = \rho_{max}(A, i)$,
2. $\rho_{min}(\rho_{min}(A, i), i) = \rho_{min}(A, i)$,
3. $\rho_{av}(\rho_{av}(A, i), i) = \rho_{av}(A, i)$,
4. $\sigma_{max}(\sigma_{max}(A, i), i) = \sigma_{max}(A, i)$,
5. $\sigma_{min}(\sigma_{min}(A, i), i) = \sigma_{min}(A, i)$,
6. $\sigma_{av}(\sigma_{av}(A, i), i) = \sigma_{av}(A, i)$.

and for every two indices $i$ and $j$:

1. $\rho_{max}(\sigma_{max}(A, j), i) = \sigma_{max}(\rho_{max}(A, i), j)$,
2. $\rho_{min}(\sigma_{min}(A, j), i) = \sigma_{min}(\rho_{min}(A, i), j)$,
(3) \( \rho_{av}(\sigma_{av}(A, j), i) = \sigma_{av}(\rho_{av}(A, i), j) \).

The following assertion is valid.

Theorem. For every two IFIMs \( A \) and \( B \) and for every index \( i \):

1. \( \rho_{max}(A \oplus B, i) \subseteq_v \rho_{max}(A, i) \oplus \rho_{max}(B, i) \),
2. \( \rho_{min}(A \oplus B, i) \supseteq_v \rho_{min}(A, i) \oplus \rho_{min}(B, i) \),
3. \( \rho_{av}(A \oplus B, i) = \rho_{av}(A, i) \oplus \rho_{av}(B, i) \),
4. \( \rho_{max}(A \otimes B, i) \supseteq_v \rho_{max}(A, i) \oplus \rho_{max}(B, i) \),
5. \( \rho_{min}(A \otimes B, i) \subseteq_v \rho_{min}(A, i) \oplus \rho_{min}(B, i) \),
6. \( \rho_{max}(A \ominus B, i) = \rho_{max}(A, i) \ominus \rho_{max}(B, i) \),
7. \( \rho_{min}(A \ominus B, i) = \rho_{min}(A, i) \ominus \rho_{min}(B, i) \),
8. \( \rho_{av}(A \ominus B, i) = \rho_{av}(A, i) \ominus \rho_{av}(B, i) \),
9. \( \sigma_{max}(A \oplus B, i) \subseteq_v \sigma_{max}(A, i) \oplus \sigma_{max}(B, i) \),
10. \( \sigma_{min}(A \oplus B, i) \supseteq_v \sigma_{min}(A, i) \oplus \sigma_{min}(B, i) \),
11. \( \sigma_{av}(A \oplus B, i) = \sigma_{av}(A, i) \oplus \sigma_{av}(B, i) \),
12. \( \sigma_{max}(A \otimes B, i) \supseteq_v \sigma_{max}(A, i) \oplus \sigma_{max}(B, i) \),
13. \( \sigma_{min}(A \otimes B, i) \subseteq_v \sigma_{min}(A, i) \oplus \sigma_{min}(B, i) \),
14. \( \sigma_{max}(A \ominus B, i) = \sigma_{max}(A, i) \ominus \sigma_{max}(B, i) \),
15. \( \sigma_{min}(A \ominus B, i) = \sigma_{min}(A, i) \ominus \sigma_{min}(B, i) \),
16. \( \sigma_{av}(A \ominus B, i) = \sigma_{av}(A, i) \ominus \sigma_{av}(B, i) \).

The operation \( \odot \) does not enter relation with the new eight operations.

5 Conclusion

In future, we will study the connectins between the new eight operations and the rest operations and relations defined over IFIMs.

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References


