Notes on Intuitionistic Fuzzy Sets
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R_1 and level R_1 separation axioms on intuitionistic fuzzy pairwise topological spaces using intuitionistic fuzzy open sets

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Abstract: This study investigates the notions of intuitionistic fuzzy (IF) R_1 pairwise topological space and α-R_1 type separation axioms on IF pairwise topological spaces in the sense of intuitionistic fuzzy open sets. We define four notions of IF R_1 pairwise topological space and

three notions of α-R_1 IF pairwise topological spaces and show the relationship among the notions separately with appropriate examples. In order to show the correctness of the proposed definitions, we show the good extension property for both types of the notions followed by examples as well. Finally, we show that our notions satisfy the hereditary and preserve the homeomorphism mapping properties.
Keywords: Intuitionistic fuzzy set (IFS), Intuitionistic fuzzy pairwise topological space, Separation axiom, R_1 Intuitionistic fuzzy pairwise topological space, Good extension.
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1 Introduction

Chang first investigated fuzzy topology [13], a prominent field of study in fuzzy mathematics, in 1968, expanding on the idea of fuzzy sets [30], which is the study of vagueness and uncertainty. Later, in 1983, Atanassov presented the idea of an intuitionistic fuzzy set [6]. Both membership and non-membership degrees are taken into account in this concept, provided that their aggregate does not exceed 1.

Coker and other colleagues created the fundamental ideas of intuitionistic fuzzy topological space (IFTS) [14, 16]. The analysis of these spaces using IFSs was then further advanced by a number of scholars, including Saadati and Park [28], Singh and Srivastava [29], Lee and Lee [18], Ahmed *et al.* [1], and Mahbub *et al.* [20]. In particular, the investigation of IFTS has received a lot of attention [17, 27]. Furthermore, the separation axioms [3, 5, 19, 22] in fuzzy, intuitionistic fuzzy [8, 23, 25], and fuzzy neutrosophic [24] topological spaces have been investigated by numerous fuzzy topologists [4, 7, 9, 26]. Notably, in the setting of IFTS, Al-Qubati [2] presented and investigated novel forms of b-separation axioms (b-$T_i, i = 0, 1, 2$).

In the literature, R_1-type separation axioms have fascinated researchers [1, 28] in different dimensions such as topology, fuzzy topology, fuzzy soft topology [21], and IF topology [10, 11]. In contrast to IFTSs, R_1 and α-R_1 intuitionistic fuzzy pairwise topological spaces (IFPTS) have not yet been examined in the literature, despite their possible importance in fuzzy mathematics.

In this study, we use IF open sets to define and investigate the notion of R_1 IFPTS in four distinct techniques, the notion of α-R_1 IFPTS in three distinct techniques, and we also look into their features. This is how the rest of the paper is organized: Basic concepts and preliminary findings of IF types, IFS, and their operations are presented in Section 2, along with pertinent concepts that are necessary to understand our work. The four new R_1 IFPTS and three α-R_1 IFPTS ideas are covered in Section 3, together with their numerous features and attributes as well as the implications between them. Hereditary and good extension properties have been discussed in Section 4, followed by some examples. Mappings between IFPTS have been discussed in the sense of bijective, intuitionistic fuzzy open, and intuitionistic fuzzy continuous functions in Section 5.

2 Preliminary results and basic notions

An intuitionistic fuzzy set $M \subseteq Y$ takes the form $M = \{\langle y, \mu_M(y), \nu_M(y)\rangle \mid y \in Y\}$, where μ_M and ν_M are functions in Y that represent the membership and non-membership degrees, respectively, with the condition $\mu_M(y) + \nu_M(y) \leq 1$. In this work, instead of the above notation for M, we use the simpler notation $M = \langle \mu_M, \nu_M\rangle$ for IFSs [12].

On a nonempty set X, the intuitionistic fuzzy topology t [15] is a family of IFSs in X, such that:

1. $\mathbf{0}_\sim, \mathbf{1}_\sim \in t$, where $\mathbf{0}_\sim = (\mathbf{0}^\sim, \mathbf{1}^\sim), \mathbf{1}_\sim = (\mathbf{1}^\sim, \mathbf{0}^\sim)$;
2. $M \cap N \in t$, for all $M, N \in t$; and
3. $\cup M_j \in t$ for any collection family $\{M_j \in t, j \in J\}$, where J is index set.

The pair (X, t) is called an IFTS, the members of the IFTS are called intuitionistic fuzzy open sets in X, and their complements are said to be intuitionistic fuzzy closed sets in X.

If $f : X \to Y$ is a function, with X and Y being IFTSs, where $M = \{\langle x, \mu_M(x), \nu_M(x)\rangle \mid x \in X\}$, $N = \{\langle y, \mu_N(y), \nu_N(y)\rangle \mid y \in Y\}$ are respectively IFSs in X and Y, the preimage [12] of N under f, denoted by $f^{-1}(N)$, is the IFS in X defined by:

$$\begin{aligned} f^{-1}(N) &= \{\langle x, (f^{-1}(\mu_N))(x), (f^{-1}(\nu_N))(x)\rangle \mid x \in X\} \\ &= \{\langle x, \mu_N(f(x)), \nu_N(f(x))\rangle \mid x \in X\}. \end{aligned}$$

The image of M, denoted as $f(M)$, is the IFS in X expressed as

$$f(M) = \{\langle y, (f(\mu_M))(y), (f(\nu_M))(y)\rangle \mid y \in Y\},$$

where for each $y \in Y$,

$$(f(\mu_M))(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_M(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

$$(f(\nu_M))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \nu_M(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

The function $f : (X, t) \to (Y, t')$ is said to be intuitionistic fuzzy continuous [15] if $f^{-1}(N) \in t$ for all $N \in t'$, and f is said to be fuzzy open if $f(M) \in t'$ for all $M \in t$.

3 R_1 and α-R_1 intuitionistic fuzzy pairwise topological spaces

In this section, we define R_1 separation axiom as intuitionistic fuzzy R_1 pairwise topological spaces (R_1-IFPTS) and α-R_1 separation axiom as intuitionistic fuzzy α-R_1 pairwise topological spaces (α-R_1-IFPTS) in the sense of intuitionistic fuzzy open sets and investigate the relationship among the results with appropriate examples.

Definition 3.1. An IFPTS (X, s, t) is called,

a. **IFPTS-R_1(i)** if for distinct $x, y \in X$, whenever there exists $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$, then there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that:

$$\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) = 1,\ \nu_P(y) = 0;\ \text{and } N \cap P = \mathbf{0}_{\sim}.$$

b. **IFPTS-R_1(ii)** if for distinct $x, y \in X$, whenever there exists $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$, then there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that:

$$\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) > 0,\ \nu_P(y) = 0;\ \text{and } N \cap P = (\mathbf{0}^{\sim}, \gamma^{\sim}),$$

where $\gamma \in (0, 1]$.

c. **IFPTS-R_1(iii)** if for distinct $x, y \in X$, whenever there exists $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$, then there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that:

$$\mu_N(x) > 0,\ \nu_N(x) = 0;\ \mu_P(y) = 1,\ \nu_P(y) = 0;\ \text{and } N \cap P = (\mathbf{0}^{\sim}, \gamma^{\sim}),$$

where $\gamma \in (0, 1]$.

d. **IFPTS-R_1(iv)** if for distinct $x, y \in X$, whenever there exists $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$, then there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that:

$$\mu_N(x) > 0,\ \nu_N(x) = 0;\ \mu_P(y) > 0,\ \nu_P(y) = 0;\ \text{and } N \cap P = (\mathbf{0}^{\sim}, \gamma^{\sim}),$$

where $\gamma \in (0, 1]$.

Definition 3.2. Let $\alpha \in (0, 1)$. An IFPTS (X, s, t) is called,

a. **IFPTS-α-R_1(i)** if for distinct $x, y \in X$, whenever there exists $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$, then there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that:

$$\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) \geq \alpha,\ \nu_P(y) = 0;\ \text{and } N \cap P = \mathbf{0}_{\sim}.$$

b. **IFPTS-α-R_1(ii)** if for distinct $x, y \in X$, whenever there exists $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$, then there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that:

$$\mu_N(x) \geq \alpha,\ \nu_N(x) = 0;\ \mu_P(y) \geq \alpha,\ \nu_P(y) = 0;\ \text{and } N \cap P = (\mathbf{0}^{\sim}, \gamma^{\sim}),$$

where $\gamma \in (0, 1]$.

c. **IFPTS-α-R_1(iii)** if for distinct $x, y \in X$, whenever there exists $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$, then there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that:

$$\mu_N(x) > 0,\ \nu_N(x) = 0;\ \mu_P(y) \geq \alpha,\ \nu_P(y) = 0;\ \text{and } N \cap P = (\mathbf{0}^{\sim}, \gamma^{\sim}),$$

where $\gamma \in (0, 1]$.

Theorem 3.1. *If (X, s, t) is an IFPTS, then the following implications hold (see Figure 1). The reverse implications are not generally true.*

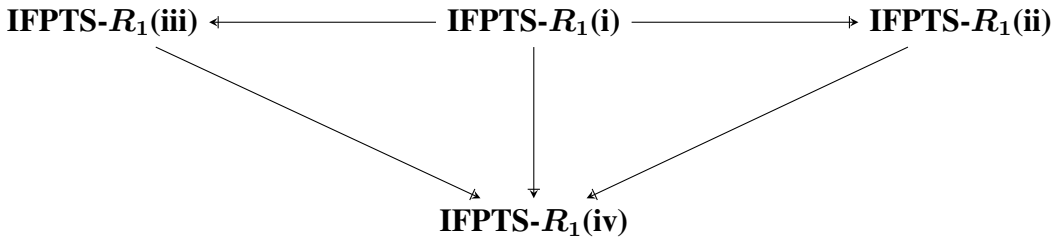

Figure 1. Implications among the notions of IFPTS.

Proof. Suppose (X, s, t) is an IFPT-R_1(i) space, and let $x, y \in X$ be distinct points, and $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ with $M(x) \neq M(y)$. Since (X, s, t) is IFPT-$R_1(i)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that, $\mu_N(x) = 1, \nu_N(x) = 0;\ \mu_P(y) = 1,$ $\nu_P(y) = 0;$ and $N \cap P = \mathbf{0}_\sim$. This implies $\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) > 0,\ \nu_P(y) = 0;$ and $N \cap P = (\mathbf{0}^\sim, \gamma^\sim)$, for some $\gamma \in (0, 1]$. Hence, (X, s, t) is IFPT-R_1(ii).

Again, assume (X, s, t) is IFPT-$R_1(i)$, and let $x, y \in X$ be distinct points, and $M = \langle \mu_M, \nu_M \rangle$ $\in (s \cup t)$ with $M(x) \neq M(y)$. Since (X, s, t) is IFPT-$R_1(i)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that $\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) = 1,\ \nu_P(y) = 0;$ and $N \cap P = \mathbf{0}_\sim$. This implies $\mu_N(x) > 0,\ \nu_N(x) = 0;\ \mu_P(y) = 1,\ \nu_P(y) = 0;$ and $N \cap P = (\mathbf{0}^\sim, \gamma^\sim)$, for some $\gamma \in (0, 1]$. Hence, (X, s, t) is IFPT-R_1(iii).

Furthermore, the proofs of IFPTS- R_1(i) $\Longrightarrow$ IFPTS-R_1(iv), IFPTS-R_1(ii) $\Longrightarrow$ IFPTS-R_1(iv), and IFPTS-R_1(iii) $\Longrightarrow$ IFPTS-R_1(iv) follow a similar approach. □

For the non-implications, we have the following examples:

Example 3.1. Let $X = \{x, y\}$ with two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.4, 0), (y, 0, 0.2)\}, N = \{(x, 1, 0), (y, 0, 0.5)\}$, $P = \{(x, 0, 0.6), (y, 0.7, 0)\}$. We conclude that (X, s, t) is IFPTS-R_1(ii) but not IFPTS-R_1(i).

Example 3.2. Let $X = \{x, y\}$ with two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.3, 0), (y, 0, 0.2)\}, N = \{(x, 0.5, 0), (y, 0, 0.7)\}$, $P = \{(x, 0, 0.6), (y, 1, 0)\}$. We conclude that (X, s, t) is IFPTS-R_1(iii) but not IFPTS-R_1(i).

Example 3.3. Let $X = \{x, y\}$ with two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.1, 0), (y, 0, 0.7)\}, N = \{(x, 1, 0), (y, 0, 0.6)\}$, $P = \{(x, 0, 0.3), (y, 0.9, 0)\}$. We conclude that (X, s, t) is IFPTS-R_1(ii) but not IFPTS-R_1(iii).

Example 3.4. Let $X = \{x, y\}$ with two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.2, 0), (y, 0, 0.4)\}, N = \{(x, 0.6, 0), (y, 0, 0.3)\}$, $P = \{(x, 0, 0.5), (y, 0.8, 0)\}$. We conclude that (X, s, t) is IFPTS-R_1(iii) but not IFPTS-R_1(ii).

Theorem 3.2. *If (X, s, t) IFPTS, the following implications hold, see Figure 2. The reverse is not generally true.*

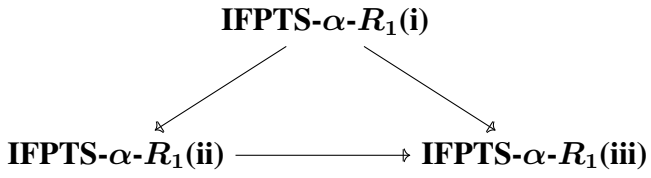

Figure 2. Implications among the notions of level *IFPTS*.

Proof. Suppose (X, s, t) is *IFPTS*-α-$R_1(i)$ space and $\alpha \in (0, 1)$. Again, for distinct $x, y \in X$, let $M = (\mu_M, \nu_M) \in (s \cup t)$ with $M(x) \neq M(y)$. Since (X, s, t) is *IFPTS*-α-$R_1(i)$, there exist $N = (\mu_N, \nu_N) \in (s \cup t)$ and $P = (\mu_P, \nu_P) \in (s \cup t)$ with $\mu_N(X) = 1,\ \nu_N(X) = 0;$ $\nu_P(Y) \geq \alpha,\ \nu_P(Y) = 0$, and $N \cap P = \mathbf{0}_\sim$. This implies $\mu_N(X) \geq \alpha,\ \nu_N(X) = 0;\ \mu_P(Y) \geq \alpha,$ $\mu_P(Y) = 0$ for any $\alpha \in (0, 1)$, and $N \cap P = (\{0\}, \gamma)$, where $\gamma \in (0, 1]$. Thus, (X, s, t) is *IFPTS*-R_1*(ii)*.

Again, suppose (X, s, t) is *IFPTS*-α-$R_1(ii)$ and $\alpha \in (0, 1)$. For distinct $x, y \in X$, let $M = (\mu_M, \nu_M) \in (s \cup t)$ with $M(x) \neq M(y)$. As (X, s, t) is *IFPTS*-α-$R_1(ii)$, there exist $N = (\mu_N, \nu_N) \in (s \cup t)$ and $P = (\mu_P, \nu_P) \in (s \cup t)$ with $\mu_N(X) \geq \alpha, \nu_N(X) = 0;\ \nu_P(Y) \geq \alpha,$ $\nu_P(Y) = 0$, and $N \cap P = \mathbf{0}_\sim$. This implies $\mu_N(X) \geq \alpha,\ \nu_N(X) = 0;\ \mu_P(Y) \geq \alpha, \mu_P(Y) = 0$ for any $\alpha \in (0, 1)$, and $N \cap P = (0^\sim, \gamma^\sim)$, where $\gamma \in (0, 1]$. Thus, (X, s, t) is *IFPTS*-α-$R_1(iii)$. Furthermore, we can show that *IFPTS*-α-$R_1(i) \implies$ *IFPTS*-α-$R_1(iii)$. □

For the non-implications, we have the examples below.

Example 3.5. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.2, 0), (y, 0, 0.3)\}, N = \{(x, 0.6, 0), (y, 0, 0.4)\},$ $P = \{(x, 0, 0.5), (y, 0.7, 0)\}$. For $\alpha = 0.4$, we conclude that (X, s, t) is *IFPTS*-α-$R_1(ii)$ but not *IFPTS*-α-$R_1(i)$.

Example 3.6. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.1, 0), (y, 0, 0.4)\}, N = \{(x, 0.3, 0), (y, 0, 0.7)\},$ $P = \{(x, 0, 0.6), (y, 0.5, 0)\}$. For $\alpha = 0.5$, we conclude that (X, s, t) is *IFPTS*-α-$R_1(iii)$ but not *IFPTS*-α-$R_1(i)$.

Theorem 3.3. *If (X, s, t) is an IFPTS and $0 < \alpha \leq \beta < 1$, the following implications hold. However, the reverse implications are not generally true.*

(a) IFPTS-β-$R_1(i) \implies$ IFPTS-α-$R_1(i)$,

(b) IFPTS-β-$R_1(ii) \implies$ IFPTS-α-$R_1(ii)$,

(c) IFPTS-β-$R_1(iii) \implies$ IFPTS-α-$R_1(iii)$.

Proof. Let $\beta \in (0, 1)$ and (X, s, t) be *IFPTS*-β-$R_1(i)$. For distinct $x, y \in X$, let $M = \langle \mu_M, \nu_M \rangle$ $\in (s \cup t)$ with $M(x) \neq M(y)$.

Since (X, s, t) is *IFPTS*-β-$R_1(i)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle$ $\in (s \cup t)$ such that

$$\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) \geq \beta,\ \nu_P(y) = 0,$$

and $N \cap P = \mathbf{0}_\sim$.

Since $0 < \alpha \leq \beta < 1$, it follows that

$$\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) \geq \alpha,\ \nu_P(y) = 0,$$

and $N \cap P = \mathbf{0}_\sim$. Thus, (X, s, t) is *IFPTS*-α-$R_1(i)$.

Similarly, we can prove that *IFPTS*-β-$R_1(ii) \implies$ *IFPTS*-α-$R_1(ii)$, and *IFPTS*-β-$R_1(iii) \implies$ *IFPTS*-α-$R_1(iii)$. □

The reverse implications can be seen from the examples below.

Example 3.7. Let $X = \{x, y\}$ with two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.3, 0), (y, 0, 0.7)\}, N = \{(x, 1, 0), (y, 0, 0.4)\}$, $P = \{(x, 0, 0.5), (y, 0.6, 0)\}$. For $\alpha = 0.5$ and $\beta = 0.7$, we conclude that (X, s, t) is *IFPTS*-α-$R_1(i)$ but not *IFPTS*-β-$R_1(i)$.

Example 3.8. Let $X = \{x, y\}$ with two topologies s and t in X generated by $\{A, B, C\}$ and a set of constants respectively, where $A = \{(x, 0.4, 0), (y, 0, 0.2)\}$,
$B = \{(x, 0.7, 0), (y, 0, 0.5)\},\ C = \{(x, 0, 0.3), (y, 0.6, 0)\}$. For $\alpha = 0.6$ and $\beta = 0.8$, we conclude that (X, s, t) is *IFPTS*-α-$R_1(ii)$ but not *IFPTS*-β-$R_1(ii)$.

Example 3.9. Let $X = \{x, y\}$ with two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.5, 0), (y, 0, 0.1)\}, N = \{(x, 0.4, 0), (y, 0, 0.6)\}$, $P = \{(x, 0, 0.3), (y, 0.5, 0)\}$. For $\alpha = 0.4$ and $\beta = 0.6$, we conclude that (X, s, t) is *IFPTS*-α-$R_1(iii)$ but not *IFPTS*-β-$R_1(iii)$.

4 Hereditary and good extension properties

In this section, hereditary and good extension properties are discussed for the notions of *IFPTS*-R_1 and *IFPTS*-α-R_1.

Theorem 4.1. *Let $\alpha \in (0, 1)$ and let (X, s, t) be an IFPTS. For $U \subseteq X$, define $s_U = \{M/U \mid M \in s\}$, $t_U = \{N/U \mid N \in t\}$. Then, the following implications hold*

a. *(X, s, t) is IFPTS-$R_1(j) \implies (U, s_U, t_U)$ is IFPTS-$R_1(j)$, where $j = i, ii, iii, iv$.*

b. *(X, s, t) is IFPTS-α-$R_1(k) \implies (U, s_U, t_U)$ is IFPTS-α-$R_1(k)$, where $k = i, ii, iii$.*

Proof. We will show the results for $j = i$ and for $k = i$. Suppose (X, s, t) is *IFPTS*-$R_1(i)$. Let $x, y \in U$ be distinct, with $M_U = (\mu_{M_U}, \nu_{M_U}) \in (s_U \cup t_U)$ such that $M_U(x) \neq M_U(y)$. Since $x, y \in U \subseteq X$, it follows that $x, y \in X$. Suppose $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ is the extension of M_U to X. Then $M(x) \neq M(y)$. As (X, s, t) is *IFPTS*-$R_1(i)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that $\mu_N(x) = 1,\ \nu_N(x) = 0;\ \mu_P(y) = 1,\ \nu_P(y) = 0$, and $N \cap P = \mathbf{0}_\sim$. This implies $(\mu_N|_U)(x) = 1,\ (\nu_N|_U)(x) = 0;\ (\mu_P|_U)(y) = 1,\ (\nu_P|_U)(y) = 0$, and $(\mu_N|_U, \nu_N|_U) \cap (\mu_P|_U, \nu_P|_U) = \mathbf{0}_\sim$. Hence, $\{(\mu_N|_U, \nu_N|_U) \cap (\mu_P|_U, \nu_P|_U)\} \in (s_U \cup t_U)$, which implies (U, s_U, t_U) is *IFPTS*-$R_1(i)$.

Now, assume (X, s, t) is *IFPTS*-α-$R_1(i)$. Let $x, y \in U$ be distinct, with $M_U = (\mu_{M_U}, \nu_{M_U}) \in (s_U \cup t_U)$ such that $M_U(x) \neq M_U(y)$.

Since $x, y \in U \subseteq X$, it follows that $x, y \in X$. Suppose $M = \langle \mu_M, \nu_M \rangle \in (s \cup t)$ is the extension of M_U to X. Then $M(x) \neq M(y)$. As (X, s, t) is *IFPTS*-α-$R_1(i)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (s \cup t)$ and $P = \langle \mu_P, \nu_P \rangle \in (s \cup t)$ such that $\mu_N(x) = 1,\ \nu_N(x) = 0$; $\mu_P(y) \geq \alpha,\ \nu_P(y) = 0$, and $N \cap P = (\mathbf{0}^\sim, \gamma^\sim)$, where $\gamma \in (0, 1]$. This implies $(\mu_N|_U)(x) = 1$, $(\nu_N|_U)(x) = 0$; $(\mu_P|_U)(y) \geq \alpha,\ (\nu_P|_U)(y) = 0$, and: $(\mu_N|_U, \nu_N|_U) \cap (\mu_P|_U, \nu_P|_U) = (\mathbf{0}^\sim, \gamma^\sim)$.

Hence, $\{(\mu_N|_U, \nu_N|_U) \cap (\mu_P|_U, \nu_P|_U)\} \in (s_U \cup t_U)$, which implies that (U, s_U, t_U) is *IFPTS*-α-$R_1(i)$. □

Definition 4.1. An intuitionistic pairwise topological space (X, τ_1, τ_2), where τ_1 and τ_2 are intuitionistic topologies in X, is called *IPTS*-R_1 if for all $x, y \in X,\ x \neq y$, whenever there exists $M = (M_1, M_2) \in \tau_1 \cup \tau_2$ with $M(x) \neq N(x)$, then there exist $L = (L_1, L_2) \in (\tau_1 \cup \tau_2)$ and $N = (N_1, N_2) \in (\tau_1 \cup \tau_2)$ such that $x \in L_1, x \notin L_2; y \in N_1, y \notin N_2$ and $L \cap N = \emptyset_\sim$.

Theorem 4.2. *Let (X, τ_1, τ_2) be an intuitionistic pairwise topological space IPTS, and let (X, t_1, t_2) be an IFPTS. Then (X, τ_1, τ_2) is IPTS-$R_1 \iff (X, t_1, t_2)$ is IFPTS-$R_1(j)$, where $j = i, ii, iii, iv$.*

Proof. We will prove this theorem for $j = ii$. Let (X, τ_1, τ_2) be IPTS-R_1 and $x, y \in X,\ x \neq y$ with $M = \langle \mu_M, \nu_M \rangle \in (t_1 \cup t_2)$ such that $M(x) \neq M(y)$, then let $(1_{P_1}(x) = 1,\ 1_{P_2}(y) = 1) \implies (x \in c_1,\ x \in c_2)$. Hence $(P_1, P_2) \in (\tau_1, \tau_2)$. Since (X, τ_1, τ_2) is IPTS-R_1, then there exist $F = (F_1, F_2),\ G = (G_1, G_2) \in (\tau_1, \tau_2)$ with $x \in F_1,\ x \notin F_2;\ y \in G_1,\ y \notin G_2$, and $F \cap G = \emptyset_\sim$. $\implies 1_{F_1}(x) = 1,\ 1_{F_2}(x) = 0;\ 1_{G_1}(y) > 0,\ 1_{G_2}(y) = 0$, and $F \cap G = \emptyset_\sim$. Let $\mu_N = 1_{F_1}$, $\nu_N = 1_{F_2}, \mu_P = 1_{G_1},\ \nu_P = 1_{G_2}$, where $N = \langle \mu_N, \nu_N \rangle \in (t_1 \cup t_2)$ and $P = \langle \mu_P, \nu_P \rangle \in (t_1 \cup t_2)$, which implies $\forall x, y \in X,\ x \neq y$ and $M = \langle \mu_M, \nu_M \rangle \in (t_1 \cup t_2)$ with $M(x) \neq M(y)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (t_1 \cup t_2)$ and $P = \langle \mu_P, \nu_P \rangle \in (t_1 \cup t_2)$ such that: $\mu_N(x) = 1,\ \nu_N(x) = 0$; $\mu_P(y) > 0,\ \nu_P(y) = 0,$ and $N \cap P = \mathbf{0}_\sim$, which implies that (X, t_1, t_2) is IFPTS-$R_1(ii)$.

Conversely, let (X, t_1, t_2) be IFPTS-$R_1(ii)$. We shall prove that (X, τ_1, τ_2) is IPTS-R_1. Let distinct $x, y \in X$, with $E = (E_1, E_2) \in (\tau_1 \cup \tau_2)$ such that $(x \in E_1, y \in E_2)$ or $(y \in E_1, x \in E_2)$. Since $(x \in E_1, y \in E_2)$ or $(y \in E_1, x \in E_2) \implies (1_{E_1}(x) = 1,\ 1_{E_2}(y) = 1)$ or $(1_{E_1}(y) = 1$, $1_{E_2}(x) = 1)$, hence $(1_{E_1}, 1_{E_2}) \in (t_1 \cup t_2)$ and $(1_{E_1}, 1_{E_2})(x) \neq (1_{E_1}, 1_{E_2})(y)$. Since (X, t_1, t_2) is IFPTS-$R_1(ii)$, there exist $(1_{C_1}, 1_{C_2}), (1_{D_1}, 1_{D_2}) \in (t_1 \cup t_2)$ such that $1_{C_1}(x) = 1, 1_{C_2}(y) = 0$; $1_{D_1}(x) > 0, 1_{D_2}(y) = 0$, and $\{(1_{C_1}, 1_{C_2}) \cap (1_{D_1}, 1_{D_2})\} = \emptyset_\sim$. This implies $x \in C_1,\ x \notin C_2$; $y \in D_1,\ y \notin D_2,$ and $(C_1, C_2) \cap (D_1, D_2) = \emptyset_\sim$. Hence, (X, τ_1, τ_2) is IPTS-R_1 and $\{(1_{C_1}, 1_{C_2}) \cap (1_{D_1}, 1_{D_2})\} = \emptyset_\sim \implies x \in C_1,\ x \notin C_2;\ y \in D_1,\ y \notin D_2$, and $(C_1, C_2) \cap (D_1, D_2) = \emptyset_\sim$. Hence, (X, τ_1, τ_2) is IPTS-R_1. Therefore IPTS-$R_1 \iff$ IFPTS-$R_1(ii)$. □

The reverse implications are not true in general which can be seen from the examples below.

Example 4.1. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.1, 0), (y, 0, 0.5)\}, N = \{(x, 1, 0), (y, 0, 0.7)\}$, $P = \{(x, 0, 0.4), (y, 0.8, 0)\}$. We conclude that (X, s, t) is *IFPTS*-$R_1(ii)$ but not the corresponding *IPTS*-R_1.

Example 4.2. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.2, 0), (y, 0, 0.3)\}, N = \{(x, 0.3, 0), (y, 0, 0.6)\}$, $P = \{(x, 0, 0.8), (y, 1, 0)\}$. We conclude that (X, s, t) is *IFPTS*-$R_1(iii)$ but not the corresponding *IPTS*-R_1.

Example 4.3. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where $M = \{(x, 0.4, 0), (y, 0, 0.3)\}, N = \{(x, 0.5, 0), (y, 0, 0.7)\}$, $P = \{(x, 0, 0.5), (y, 0.9, 0)\}$. We conclude that (X, s, t) is *IFPTS*-$R_1(iv)$ but not the corresponding *IPTS*-R_1.

Theorem 4.3. *If (X, τ_1, τ_2) be an intuitionistic topological space and let (X, t_1, t_2) be an IFPTS, the following implications hold, see Figure 3.*

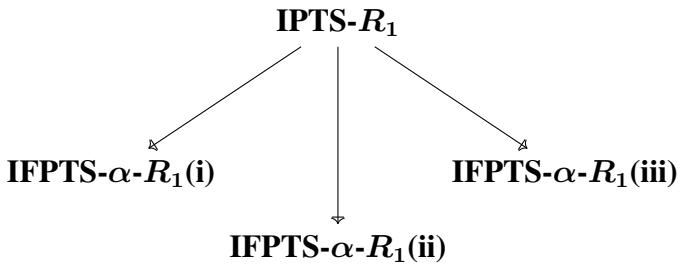

Figure 3. Implications among *IPTS* and the notions of level *IFPTS*.

Proof. We will prove this theorem for *IPTS*-$R_1 \rightarrow$ *IFPTS*-α-$R_1(ii)$. Let (X, τ_1, τ_2) be *IPTS*-R_1. Let $\alpha \in (0, 1)$ and distinct $x, y \in X$ with $M = \langle \mu_M, \nu_M \rangle \in (t_1 \cup t_2)$ such that $M(x) \neq M(y)$.

Since $M(x) \neq M(y)$, then let $(1_{P_1})(x) > 0,\ (1_{P_2})(y) = 0$ or $(1_{P_1})(y) = 0,\ (1_{P_2})(x) > 0$, which implies $(x \in P_1,\ y \in P_2)$ or $(y \in P_1,\ x \in P_2)$. Hence $(P_1, P_2) \in \tau$.
Since (X, τ_1, τ_2) is *IPTS*-R_1, there exist $L = (L_1, L_2),\ A = (A_1, A_2) \in (\tau_1, \tau_2)$ such that $x \in L_1,\ x \notin L_2;\ y \in A_1,\ y \notin A_2$ and $L \cap A = \emptyset_\sim$. This implies $1_{L_1}(x) = 1, 1_{L_2}(x) > 0$; $1_{A_1}(y) = 0,\ 1_{A_2}(y) = 0,$ and $L \cap A = \emptyset_\sim$. Let, $\mu_N = 1_{L_1},\ \nu_N = 1_{L_2},\ \mu_P = 1_{A_1},\ \nu_P = 1_{A_2}$, where $N = \langle \mu_N, \nu_N \rangle \in (t_1 \cup t_2)$ and $P = \langle \mu_P, \nu_P \rangle \in (t_1 \cup t_2)$. This implies that for all $x, y \in X,\ x \neq y$, and $M = \langle \mu_M, \nu_M \rangle \in (t_1 \cup t_2)$ with $M(x) \neq M(y)$: there exist $N = \langle \mu_N, \nu_N \rangle \in (t_1 \cup t_2), P = \langle \mu_P, \nu_P \rangle \in (t_1 \cup t_2)$such that: $\mu_N(x) = 1, \nu_N(x) = 0; \mu_P(y) \geq \alpha$, $\nu_P(y) = 0,$ and$N \cap P = \mathbf{0}_\sim$, for any $\alpha \in (0, 1)$. Thus, (X, τ_1, τ_2) is *IFPTS*-α-$R_1(ii)$. □

The reverse implications are not true in general, which can be seen from the examples below.

Example 4.4. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where: $M = \{(x, 0.3, 0), (y, 0, 0.4)\}, N = \{(x, 1, 0), (y, 0, 0.4)\}$, $P = \{(x, 0, 0.2), (y, 0.5, 0)\}$. For $\alpha = 0.5$, we conclude that (X, t_1, t_2) is *IFPTS*-α-$R_1(i)$ but not the corresponding *IPTS*-R_1.

Example 4.5. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where: $M = \{(x, 0.7, 0), (y, 0, 0.9)\}, N = \{(x, 0.6, 0), (y, 0, 0.4)\}$, $P = \{(x, 0, 0.5), (y, 0.4, 0)\}$. For $\alpha = 0.3$, we conclude that (X, t_1, t_2) is *IFPTS*-α-$R_1(ii)$ but not the corresponding *IPTS*-R_1.

Example 4.6. Let $X = \{x, y\}$ and the two topologies s and t in X generated by $\{M, N, P\}$ and a set of constants respectively, where: $M = \{(x, 0.2, 0), (y, 0, 0.6)\}, N = \{(x, 0.5, 0), (y, 0, 0.4)\}$, $P = \{(x, 0, 0.3), (y, 0.8, 0)\}$. For $\alpha = 0.7$, we conclude that (X, t_1, t_2) is *IFPTS*-α-$R_1(iii)$ but not the corresponding *IPTS*-R_1.

5 Mappings in R_1 intuitionistic fuzzy pairwise topological spaces

Mappings between IFPTS have been discussed in the sense of bijective, intuitionistc fuzzy open, and intuitionistic fuzzy continuous functions. In this circumstance, we study the image and pre-image of IF pairwise topological spaces.

Theorem 5.1. *Let (X, s_1, s_2) and (Y, t_1, t_2) be two IFPTS, and let $X \to Y$ be a one-to-one, onto, and IF continuous open mapping. Then*

- *(X, s_1, s_2) is IFPTS-$R_1(j)$ $\iff$ (Y, t_1, t_2) is IFPTS-$R_1(j)$, where $j = i, ii, iii, iv$; and*
- *(X, s_1, s_2) is IFPTS-α-$R_1(k)$ $\iff$ (Y, t_1, t_2) is IFPTS-α-$R_1(k)$, where $k = i, ii, iii$.*

Proof. We will prove this theorem for the cases $j = iii$ and $k = ii$, as the remaining cases canbe checked in the same manner.

- Proof for $j = iii$:

  ($\Rightarrow$) Suppose (X, s_1, s_2) is *IFPTS*-$R_1(iii)$ and $y_1, y_2 \in Y$, $y_1 \neq y_2$ with $M = \langle \mu_M, \nu_M \rangle \in (t_1 \cup t_2)$ such that $\mu_M(y_1) \neq \nu_M(y_2)$. As f is onto, there exist $x_1, x_2 \in X$ such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Since $y_1 \neq y_2$, then $f^{-1}(y_1) \neq f^{-1}(y_2)$ as f is one-to-one and onto. Hence $x_1 \neq x_2$. We have $N = \langle \mu_N, \nu_N \rangle \in (s_1 \cup s_2)$ such that $N = f^{-1}(M)$, that means $(\mu_N, \nu_N) = (f^{-1}(\mu_M), f^{-1}(\nu_M)) \in (s_1 \cup s_2)$, as f is intuitionistic fuzzy continuous.
  Now, if

  $$N(x_1) = \mu_N(x_1) = f^{-1}(\mu_M)(x_1) = \mu_M(f(x_1)) = \mu_M(y_1),$$
  $$\nu_N(x_1) = f^{-1}(\nu_M)(x_1) = \nu_M(f(x_1)) = \nu_M(y_1),$$

  and

  $$N(x_2) = \mu_N(x_2) = f^{-1}(\mu_M)(x_2) = \mu_M(f(x_2)) = \mu_M(y_2),$$
  $$\nu_N(x_2) = f^{-1}(\nu_M)(x_2) = \nu_M(f(x_2)) = \nu_M(y_2),$$

  hence $N(x_1) \neq N(x_2)$ as $M(y_1) \neq M(y_2)$. As (X, s_1, s_2) is *IFPTS*-$R_1(iii)$, there exist $P = \langle \mu_P, \nu_P \rangle \in (s_1 \cup s_2)$, $Q = (\mu_Q, \nu_Q) \in (s_1 \cup s_2)$ such that $\mu_P(x_1) > 0$, $\nu_P(x_1) = 0$; $\mu_Q(x_2) = 1$, $\nu_Q(x_2) = 0$, and $P \cap Q = \mathbf{0}_\sim$. Put $U = f(P)$ and $V = f(Q)$, where $U = (\mu_U, \nu_U)$, $V = (\mu_V, \nu_V) \in (t_1 \cup t_2)$ as f is intuitionistic fuzzy continuous.
  Now,

  $$\mu_U(y_1) = (f(\mu_P))(y_1) = \mu_P(f^{-1}(y_1)) = \mu_P(x_1) = 1,$$
  $$\nu_U(y_1) = (f(\nu_P))(y_1) = \nu_P(f^{-1}(y_1)) = \nu_P(x_1) = 0;$$
  $$\mu_V(y_2) = (f(\mu_Q))(y_2) = \mu_Q(f^{-1}(y_2)) = \mu_Q(x_2) = 1,$$
  $$\nu_V(y_2) = (f(\nu_Q))(y_2) = \nu_Q(f^{-1}(y_2)) = \nu_Q(x_2) = 0,$$

  and $U \cap V = \mathbf{0}_\sim$. Hence $(U, V) \in (t_1 \cup t_2)$. Thus, (Y, t_1, t_2) is *IFPTS*-$R_1(iii)$.

($\Leftarrow$) Conversely, suppose (Y, t_1, t_2) is *IFPTS*-$R_1(iii)$ and $x_1, x_2 \in X$, $x_1 \neq x_2$ with $M = (\mu_M, \nu_N) \in (s_1 \cup s_2)$ such that $y_i = f(x_i), i = 1, 2$. Hence $f(x_1) \neq f(x_2)$ implies $y_1 \neq y_2$ as f is one-to-one and: $(f(\mu_M), f(\nu_M)) \in (t_1 \cup t_2)$, as f is intuitionistic fuzzy continuous.
Now,

$$(f(\mu_M))(y_1) = (f(\mu_M))(f(x_1)) = \mu_M(f^{-1}(f(x_1))) = \mu_M(x_1) = 1,$$

and

$$(f(\nu_M))(y_2) = (f(\nu_M))(f(x_2)) = \nu_M(f^{-1}(f(x_2))) = \nu_M(x_2) = 1.$$

As (Y, t_1, t_2) is *IFPTS*-$R_1(iii)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (t_1 \cup t_2)$ such that $\mu_N(y_2) = 1$, $\nu_N(y_1) = 0$.
Now,

$$(f^{-1}(\mu_N))(x_2) = \mu_N(f(x_2)) = \mu_N(y_2) = 1,$$

and

$$(f^{-1}(\nu_N))(x_1) = \nu_N(f(x_1)) = \nu_N(y_1) = 1,$$

as f is one-to-one and onto. Hence $(f^{-1}(\mu_N), f^{-1}(\nu_N)) \in (s_1 \cup s_2)$. Thus, (X, s_1, s_2) is *IFPTS*-$R_1(iii)$.

- Proof for $k = ii$:

($\Rightarrow$) Now we will prove for $k = ii$,let (X, s_1, s_2) be *IFPTS*-α-$R_1(ii)$ and $y_1, y_2 \in Y$, $y_1 \neq y_2$ with $M = \langle \mu_M, \nu_M \rangle \in (t_1 \cup t_2)$ such that $\mu_M(y_1) \neq \nu_M(y_2)$. As f is onto, there exist $x_1, x_2 \in X$ such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.
Since $y_1 \neq y_2$, $f^{-1}(y_1) \neq f^{-1}(y_2)$ as f is one-to-one and onto, so $x_1 \neq x_2$. We have $N = \langle \mu_N, \nu_N \rangle \in (s_1 \cup s_2)$ such that $N = f^{-1}(M)$, that is: $(\mu_N, \nu_N) = (f^{-1}(\mu_M), f^{-1}(\nu_M)) \in (s_1 \cup s_2)$, as f is intuitionistic fuzzy continuous.
Now, if

$$N(x_1) = \mu_N(x_1) = f^{-1}(\mu_M)(x_1) = (\mu_M)(f(x_1)) = \mu_M(y_1),$$
$$\nu_N(x_1) = f^{-1}(\nu_M)(x_1) = (\nu_M)(f(x_1)) = \nu_M(y_1),$$

and

$$N(x_2) = \mu_N(x_2) = f^{-1}(\mu_M)(x_2) = (\mu_M)(f(x_2)) = \mu_M(y_2),$$
$$\nu_N(x_2) = f^{-1}(\nu_M)(x_2) = (\nu_M)(f(x_2)) = \nu_M(y_2),$$

hence, $N(x_1) \neq N(x_2)$ as $M(y_1) \neq M(y_2)$.
As (X, s_1, s_2) is *IFPTS*-α-$R_1(ii)$, there exist $P = \langle \mu_P, \nu_P \rangle \in (s_1 \cup s_2)$ and $Q = (\mu_Q, \nu_Q) \in (s_1 \cup s_2)$ such that: $\mu_P(x_1) \geq 0, \nu_P(x_1) = 0; \mu_Q(x_2) \geq 0, \nu_Q(x_2) = 0$, and $P \cap Q = \mathbf{0}_\sim$. Put $U = f(P)$ and $V = f(Q)$, where $U = (\mu_U, \nu_U)$ and $V = (\mu_V, \nu_V) \in (t_1 \cup t_2)$ as f is intuitionistic fuzzy continuous.

Now,

$$\mu_U(y_1) = (f(\mu_P))(y_1) = \mu_P(f^{-1}(y_1)) = \mu_P(x_1) = 1,$$
$$\nu_U(y_1) = (f(\nu_P))(y_1) = \nu_P(f^{-1}(y_1)) = \nu_P(x_1) = 0;$$
$$\mu_V(y_2) = (f(\mu_Q))(y_2) = \mu_Q(f^{-1}(y_2)) = \mu_Q(x_2) = 1,$$
$$\nu_V(y_2) = (f(\nu_Q))(y_2) = \nu_Q(f^{-1}(y_2)) = \nu_Q(x_2) = 0,$$

and $U \cap V = \mathbf{0}_\sim$.

Hence $(U, V) \in (t_1 \cup t_2)$. Therefore, (Y, t_1, t_2) is *IFPTS*-α-$R_1(ii)$.

($\Leftarrow$) Conversely, suppose (Y, t_1, t_2) is *IFPTS*-α-$R_1(ii)$ and $x_1, x_2 \in X$, $x_1 \neq x_2$ with $M = (\mu_M, \nu_N) \in (s_1 \cup s_2)$ such that $y_i = f(x_i), i = 1, 2$.

Hence $f(x_1) \neq f(x_2)$ implies $y_1 \neq y_2$ as f is one-to-one. We have $(f(\mu_M), f(\nu_M)) \in (t_1 \cup t_2)$ as f is intuitionistic fuzzy continuous.

Now,

$$(f(\mu_M))(y_1) = (f(\mu_M))(f(x_1)) = \mu_M(f^{-1}(f(x_1))) = \mu_M(x_1) = 1,$$

and

$$(f(\nu_M))(y_2) = (f(\nu_M))(f(x_2)) = \nu_M(f^{-1}(f(x_2))) = \nu_M(x_2) = 1.$$

As (Y, t_1, t_2) is *IFPTS*-α-$R_1(ii)$, there exist $N = \langle \mu_N, \nu_N \rangle \in (t_1 \cup t_2)$ such that $\mu_N(y_2) \geq \alpha$, $\nu_N(y_1) = 0$. Now

$$(f^{-1}(\mu_N))(x_2) = \mu_N(f(x_2)) = \mu_N(y_2) = 1,$$

and

$$(f^{-1}(\nu_N))(x_1) = \nu_N(f(x_1)) = \nu_N(y_1) = 1,$$

as f is one-to-one and onto. Hence $(f^{-1}(\mu_N), f^{-1}(\nu_N)) \in (s_1 \cup s_2)$. Therefore, (X, s_1, s_2) is *IFPTS*-α-$R_1(ii)$.

These prove that (X, s_1, s_2) is *IFPTS*-α-$R_1(ii) \iff (Y, t_1, t_2)$ is *IFPTS*-α-$R_1(ii)$.

This completes the proof. □

6 Conclusion

In conclusion, this study introduces and investigates new types of intuitionistic fuzzy R_1 pairwise topological spaces and α-R_1 separation axiom within intuitionistic fuzzy pairwise topological spaces, based on intuitionistic fuzzy open sets. We have defined four varieties of intuitionistic fuzzy R_1 pairwise topological spaces and three varieties of α-R_1 intuitionistic fuzzy pairwise topological spaces, analyzing their relationships with detailed examples. To verify these definitions, we have demonstrated their good extension properties with supporting examples. Additionally, we have established that these definitions are hereditary and retain homeomorphism properties, underscoring their coherence and practical relevance. Moreover, intuitionistic fuzzy R_1 pairwise

topological spaces and α-R_1 separation axioms within intuitionistic fuzzy pairwise topological spaces may inspire the researchers and academicians for further research in this trend. As future work, properties of productive and projective may be considered using the approach of quasi-coincidence following this work.

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