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Universal algebra in intuitionistic fuzzy set theory

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Abstract: Several algebraic structures on intuitionistic fuzzy sets as group, ideal, ring, etc., have been studied by researchers. In this study, the concept of universal algebra on intuitionistic fuzzy sets were introduced and some basic theorems were proved.
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1 Introduction

The fuzzy sets was introduced as an extension of crisp sets by Zadeh [5]. In fuzzy set theory, if the membership degree of an element x is $\mu(x)$ then the nonmembership degree is $1 - \mu(x)$ and thus it is fixed.

Atanassov introduced the intuitionistic fuzzy set concept in 1983 [1] and form an extension of fuzzy sets by enlarging the truth value set to the lattice $[0, 1] \times [0, 1]$ is defined as follows.

Definition 1. *Let* $L = [0, 1]$ *then*

$$L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\}$$

is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".

For $(x_1, y_1), (x_2, y_2) \in L^*$, the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as follows:

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$$

$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$$

For each $J \subseteq L^*$:

$$\sup J = (sup\{x : (x, y \in [0,1]), ((x, y) \in J)\}, inf\{y : (x, y \in [0,1])((x, y) \in J)\})$$

and

$$\inf J = (inf\{x : (x, y \in [0,1])((x, y) \in J)\}, sup\{y : (x, y \in [0,1])((x, y) \in J)\}).$$

Definition 2. *[1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A", $\nu_A(x), (\nu_A : X \to [0,1])$ is called the "degree of non- membership of x in A ", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 3. *[1] An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4. *[1] Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the above set is callede the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

2 Intuitionistic fuzzy universal algebra

Intuitionistic fuzzy algebraic structures like intuitionistic fuzzy subgroup, intuitionistic fuzzy ideal, intuitionistic fuzzy ring, intuitionistic fuzzy BG-algebras have been studied by several authors. In this paper, we studied universal algebras on intuitionistic fuzzy sets. The concept of universal algebra in crisp set theory defined as follows:

Definition 5. *[2] A universal algebra (or an algebra) A is a pair $[S, F]$ where S is a non-empty set and F is a specified set of operatians f_α, each mapping a power $S^{n(\alpha)}$ of S into S, for some appropriate nonnegative finite integer $n(\alpha)$.*

Otherwise stated, each operation f_α assigns to every $n(\alpha)$-ple $\left(x_1, ..., x_{n(\alpha)}\right)$ of elements of S, a value $f_\alpha\left(x_1, ..., x_{n(\alpha)}\right)$ in S, the result of performing the operation f_α on the sequence $x_1, ..., x_{n(\alpha)}$. If $n(\alpha) = 1$, the operation f_α is called unary; if $n(\alpha) = 2$, it is called binary; if $n(\alpha) = 3$, it is called ternary, etc. When $n(\alpha) = 0$, the operation f_α is called nullary; it selects a fixed element of S.

Fuzzy universal algebra introduced by Murali [3] using Zadeh's extension principle [6]. Fuzzy subalgebras and homomorphism between fuzzy algebras defined by same author. Here, we extend the concept of fuzzy universal algebra on intuitionistic fuzzy sets.

Definition 6. *Let $S = [X, F]$ be a universal algebra where X is a non-empty set and F is a specified set of finite operations f_α, each mapping a power $X^{n(\alpha)}$ of X into X, for some appropriate nonnegative finite integer $n(\alpha)$. For each f_α, a corresponding operation ω_α on $IFS(X)$ as follows;*

$$\omega_\alpha : IFS(X) \times IFS(X) \times ... \times IFS(X) \to IFS(X), \omega_\alpha \left(A_1, A_2, ..., A_{n(\alpha)}\right) = A,$$

such that

$$A(x) = \sup \left\{A_1(x_1) \wedge A_2(x_2) \wedge ... \wedge A_{n(\alpha)}(x_{n(\alpha)}) : f_\alpha \left(x_1, x_2, ..., x_{n(\alpha)}\right) = x\right\},$$

otherwise, $A(x) = \Theta$. It will be shown that $A = \omega_\alpha \left(A_1, A_2, ..., A_{n(\alpha)}\right)$. Let $\Omega = \{\omega_\alpha$: corresponding operation for each $f_\alpha \in F\}$ then $S^* = \left[(I \times I)^X, \Omega\right]$ is called intuitionistic fuzzy universal algebra (or algebra).

If $n(\alpha) = 0$ then $f_\alpha : X \to X$, $f_\alpha(x) = e$ that e is a fixed element of X. So, ω_α is defined as following:

$$\omega_\alpha : IFS(X) \to IFS(X), \omega_\alpha(A) = A_e$$

$$A_e(x) = \begin{cases} \sup\limits_{x \in X} A(x), & x = e \\ (0, 1), & x \neq e \end{cases}$$

Example 1. *A group $S = [G, F]$ is a universal algebra where $F = \{., e\}$ include one binary operation and one nullary operation, respectively. Let $S^* = [IFS(G), \Omega]$ and $A_1, A_2 \in IFS(G), x, x_1, x_2 \in G$ then with corresponding operations defined as follows:*

$$A_1A_2\left(x\right) = \left(\mu_{A_1}\mu_{A_2}\left(x\right), \nu_{A_1}\nu_{A_2}\left(x\right)\right),$$

so that

$$\begin{aligned} \mu_{A_1}\mu_{A_2}\left(x\right) &= \sup_{x = x_1x_2} \left(\mu_{A_1}\left(x_1\right) \wedge \mu_{A_2}\left(x_2\right)\right), \\ \nu_{A_1}\nu_{A_2}\left(x\right) &= \inf_{x = x_1x_2} \left(\nu_{A_1}\left(x_1\right) \vee \nu_{A_2}\left(x_2\right)\right). \end{aligned}$$

S^* is an intuitionistic fuzzy universal algebra.

Definition 7. *Let X be a non-empty set and $A \in IFS(X)$. A is called an intuitionistic fuzzy subalgebra (IF-subalgebra) of $S^* = [IFS(X), \Omega]$ intuitionistic fuzzy universal algebra if and only if for nonnegative finite integer $n(\alpha), \omega_\alpha\left(A, A, ..., A\right) \leq A$, for every ω_α.*

Theorem 1. *Let $S = [X, F]$ be a universal algebra, $f_\alpha \in F$ and $A, A_1, A_2, ..., A_{n(\alpha)}$ be $n(\alpha) + 1$ IF-subalgebras. $\omega_\alpha \left(A_1, A_2, ..., A_{n(\alpha)}\right) \leq A$ if and only if*

$$A\left(f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right)\right) \geq \min_{1 \leq i \leq n(\alpha)} A_i\left(x_i\right)$$

is true for every $\left(x_1, x_2, ..., x_{n(\alpha)}\right) \in S^{n(\alpha)}$.

Proof. (1) Let $n(\alpha) \neq 0$ and $\omega_\alpha\left(A_1, A_2, ..., A_{n(\alpha)}\right) \leq A$.

$$\omega_\alpha\left(A_1, A_2, ..., A_{n(\alpha)}\right)(x) \leq A(x), \text{ for all } x \in X.$$

So,

$$\sup_{f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right)=x} \left(A_1(x_1) \wedge A_2(x_2) \wedge ... \wedge A_{n(\alpha)}(x_{n(\alpha)})\right) \leq A(x),$$

for all $\left(x_1, ..., x_{n(\alpha)}\right) \in X^{n(\alpha)}$.

$$\begin{aligned} A\left(f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right)\right) &\geq \omega_\alpha\left(A_1, A_2, ..., A_{n(\alpha)}\right)\left(f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right)\right) \\ &\geq A_1(x_1) \wedge A_2(x_2) \wedge ... \wedge A_{n(\alpha)}(x_{n(\alpha)}) \\ &= \min_{1 \leq i \leq n(\alpha)} A_i(x_i) \end{aligned}$$

Conversely, let $f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right) = x$. It is clear that, since $\min_{1 \leq i \leq n(\alpha)} A_i(x_i) \leq A(x)$ for all $f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right) = x$, then $\sup_x(\min_{1 \leq i \leq n(\alpha)} A_i(x_i)) \leq A(x)$.

That is, $\omega_\alpha\left(A_1, A_2, ..., A_{n(\alpha)}\right)(x) \leq A(x)$, for all $f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right) = x$.

If for some x there exist no such $n(a)$-tuples, then $\omega_\alpha\left(A_1, A_2, ..., A_{n(\alpha)}\right)(x) = (0, 1) \leq A(x)$.

(2) If $n(\alpha) = 0$, then $f_\alpha(x) = e$, e is a fixed element of X.

$$\begin{aligned} \omega_\alpha(A_1)(x) &\leq A(x) \Leftrightarrow A(e) \geq \omega_\alpha(A_1) = \sup_x A_1(x) \\ &\Leftrightarrow A(\omega_\alpha(x)) \geq A_1(x) \text{ for all } x \in X. \end{aligned}$$

This completes the proof. □

Example 2. *Let G be a group. $A \in IFS(G)$ intuitionistic fuzzy subgroup defined as follow: for all $x, y \in G$,*

$$\begin{aligned} A(xy) &\geq A(x) \wedge A(y) \\ A\left(x^{-1}\right) &\geq A(x) \end{aligned}$$

that is,

$$\begin{aligned} \mu_A(xy) &\geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y) \\ \mu_A\left(x^{-1}\right) &\geq \mu_A(x) \text{ and } \nu_A\left(x^{-1}\right) \leq \nu_A(x) \end{aligned}$$

This definition coincides with [4].

Theorem 2. *Let $S^* = [IFS(X), \Omega]$ be an IF-algebra. If $\{A_i\}_{i \in \Lambda}$ is a family of IF-subalgebras of S^* then*

$$A = \bigcap_{i \in \Lambda} A_i$$

is an IF-subalgebra of S^*.

Proof. Let $f_\alpha \in F$ and $\left(x_1, x_2, ..., x_{n(\alpha)}\right) \in S^{n(\alpha)}$ for the corresponding $n(\alpha)$.

$$\begin{aligned} A\left(f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right)\right) &= \bigcap_{i\in\Lambda} A_i\left(f_\alpha\left(x_1, x_2, ..., x_{n(\alpha)}\right)\right) \\ &\geq \bigcap_{i\in\Lambda}\left(\min_{1\leq j\leq n(\alpha)} A_i\left(x_j\right)\right) \\ &= \min_{1\leq j\leq n(\alpha)}\left(\inf_{i\in\Lambda} A_i\left(x_j\right)\right) \\ &= \min_{1\leq j\leq n(\alpha)} A\left(x_j\right) \end{aligned}$$

So, A is an IF-subalgebra of S^*. □

3 Conclusion

In this paper, we discussed the concept of universal algebra on intuitionistic fuzzy sets. The subalgebra was defined and some fundamental properties of substructures were given. In continuation, homomorphism between intuitionistic fuzzy universal algebras will be defined. The effect of this homomorphism on IF-algebras will be studied.

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