Notes on Intuitionistic Fuzzy Sets
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Solvability of an intuitionistic fuzzy fractional differential equation of second type

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Abstract: In this research, we provide certain novel requirements for the existence and uniqueness of fuzzy solutions for a type of non-linear intuitionistic fuzzy fractional equations with intuitionistic fuzzy initial conditions under the intuitionistic fuzzy fractional derivative of order $n \in (0, 3)$

in Caputo sense. The required findings are demonstrated by employing the Banach fixed point theorem, the intuitionistic fuzzy Laplace transform and the Mittag-Leffler function. An example is provided to demonstrate the reliability of our findings.
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1 Introduction

In 1965, Lotfi Zadeh [28] introduced the notion of fuzzy sets as an extension of traditional set theory using membership. Zadeh defines a fuzzy subset (FS) A of X as the following:

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\},$$

with $\mu_A : X \longrightarrow [0, 1]$.

This concept demonstrates various qualities, and writers are writing papers to adapt classical notions into fuzzy theories. In addition, fractional calculus and fractional differential equations are becoming increasingly significant in mathematical computation and applications. They are more beneficial in a variety of domains including engineering, physics, and chemistry. Hukuhara's generalized difference was employed to develop the generalized derivative, and scientists constructed fractional operators (integral and derivatives) in this theory to be used later in many articles to study fuzzy fractional differential equations [1, 4–7, 11–19, 24].

In 1986, Atanassov [9] introduced the intuitionistic fuzzy set as an extension of Zadeh's work. An intuitionistic fuzzy subset (IFS) A of X is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

with $\mu_A, \nu_A : X \longrightarrow [0, 1]$, $\mu_A(x) + \nu_A(x) \leq 1$.

An ensemble of investigators have concentrated on constructing the features and collaborating on the fuzzy set in the intuitionistic fuzzy case. In a comparable manner in fractional calculus, authors have succeeded in improve the operators of integrals and fractional derivatives in the intuitionistic fuzzy case, and have labored in multiple studies on the existence of a solution to intuitionistic fuzzy fractional differential equations [20, 21].

The ideas behind the fuzzy set's generalization do not end here. In 2013, Yager [27], possibly unknowingly, rediscovered the ”intuitionistic fuzzy set of second type” which was introduced back in 1989 by Atanassov in [8]. He coined the name ”Pythagorean fuzzy set” for that object, which unfortunately, seems to have gained widespread popularity, despite the fact that from historical point of view Atanassov's definition predates this of Yager by 24 years. In this sense, the original author should be credited in full, despite the fact that Yager helped in making the concept a popular research topic.

Atanassov's definition of intuitionistic fuzzy set of second type, rediscovered by Yager (under the new name "Fuzzy Pythagorean Subset (PFS)") A of X is as follows:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\},$$

with $\mu_A, \nu_A : X \longrightarrow [0,1], \mu_A^2(x) + \nu_A^2(x) \leq 1.$

Now we are drawn to the theory of intuitionistic fuzzy set of second type fractional calculus equations. Before attempting to solve fractional differential equations, it is required to first teach basic fractional computation ideas.

In 2022, [2] proposes the definitions of the Riemann–Liouville fractional integral, Riemann–Liouville fractional derivative, and Caputo fractional derivative of a "Pythagorean" fuzzy-valued function. The solution of fractional differential equations is not achievable. Other additional concepts are used. In [2], we find the definition of the "Pythagorean" fuzzy Laplace transform and some properties. In [3], we find the definition of the "Pythagorean" fuzzy Fourier transform and some properties. In the same article, we find the resolution of the "Pythagorean" fuzzy partial fractional differential equation based on the previous notions. In [2], several "Pythagorean" fuzzy fractional differential equations have been solved using Caputo's "Pythagorean" fuzzy fractional derivative.

In this paper, we will adhere to the historically first introduced notion of intuitionistic fuzzy sets of second type (instead of the predatory "Pythagorean fuzzy sets"), and will discuss the solution of the intuitionistic fuzzy fractional differential equation of second type:

$$\begin{cases} ^{C}D_{0+}^{n} h(\xi) = \mathcal{F}(\xi, h(\xi), ^{C}D_{0+}^{m} h(\xi)) \quad , \quad \xi \in [0,T] \\ h(0) = h_0 \in \mathbb{F}, \end{cases} \tag{1}$$

where $^{C}D_{0+}^{n}$ is the intuitionistic fuzzy Caputo fractional derivative of second type of order $0 < n < 3$, $^{C}D_{0+}^{m}$ is the intuitionistic fuzzy Caputo fractional derivative of second type of h of order $m \in (0,1)$, the function $\mathcal{F} : [0,T] \times \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ is continuous, $h_{0(\alpha,\beta)}$ is an intuitionistic fuzzy number of second type, and the function $h : [0,T] \longrightarrow \mathbb{F}$ an intuitionistic fuzzy-valued function of second type. To solve problem (1), we first try to locate the solution using the intuitionistic fuzzy Laplace transform of second type, then prove the existence and uniqueness of this solution based on the concept of contraction.

Our paper is structured as follows: In Section 2, we start with the basic notions related to the intuitionistic fuzzy sets and numbers of second type. In Section 3, we discuss the concepts of the Caputo fractional derivative and Laplace transform for the case of intuitionistic fuzzy sets of second type, and their characteristics. In Section 4, we offer an approach to solve the issue (1) and show the existence and uniqueness of its solution. Finally, in Section 5, we provide an example.

2 Preliminaries

In this section, we will present the basic notions that will be used hereafter. We will note that essentially all definitions and preliminaries in this paper follow [2], while using here the historically correct term "intuitionistic fuzzy set of second type" instead of "Pythagorean fuzzy set".

Definition 1 (cf. [26]). *The intuitionistic fuzzy set of second type $\mathcal{P}$ in X is defined in the following form:*

$$\mathcal{P} = \{\langle \xi, u_{b_t}(\xi), u_{b_f}(\xi)\rangle \mid \xi \in X\}$$

with, $u_{b_t} : X \longrightarrow [0,1]$ *and* $u_{b_f} : X \longrightarrow [0,1]$ *represent the degrees of membership and non-membership of* ξ *in* X.
And with the following condition: $u_{b_t}^2(\xi) + u_{b_f}^2(\xi) \leq 1$.
Therefore, $\varrho = \sqrt{1 - (u_{b_t}^2(\xi) + u_{b_f}^2(\xi))}$ *is expressed as the hesitancy degree.*

Definition 2 (cf. [22,23]). *The intuitionistic fuzzy number of second type $b = (b_t, b_f)$ is a non-empty subset of X.*

1- For all $\langle \xi, u_{b_t}(\xi), u_{b_f}(\xi)\rangle \in \mathcal{P}$ is convex intuitionistic fuzzy set of second type, i.e.,

i) $u_{b_t} : X \longrightarrow [0,1]$ *fuzzy convex:*
For all $\lambda \in [0,1]$ *and for all* $\xi_1, \xi_2 \in X$ *we have:*
$u_{b_t}(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \min\{u_{b_t}(\xi_1), u_{b_t}(\xi_2)\}$.

ii) $u_{b_f} : X \longrightarrow [0,1]$ *fuzzy concave:*
For all $\lambda \in [0,1]$ *and for all* $\xi_1, \xi_2 \in X$ *we have:*
$u_{b_f}(\lambda\xi_1 + (1-\lambda)\xi_2) \geq \max\{u_{b_f}(\xi_1), u_{b_f}(\xi_2)\}$.

2- For all $\langle \xi, u_{b_t}(\xi), u_{b_f}(\xi)\rangle \in \mathcal{P}$ is normal, i.e., there exists ξ such that:

$$u_{b_t}(\xi) = 1 \ \ \textit{and} \ \ u_{b_f}(\xi) = 0.$$

Definition 3 (cf. [10]). *Let $b = (b_t, b_f)$. We define (α, β)-cut representation of the intuitionistic fuzzy-valued function of second type b as follows:*

$$[b]^{(\alpha,\beta;t,f)} = \left\langle \left[b_{1(\alpha,t)}, b_{2(\alpha,t)}\right] ; \left[b_{1(\beta,f)}, b_{2(\beta,f)}\right] \right\rangle$$

with,

$$\begin{cases} b_{1(\alpha,t)} = \min\{s \ , \ s \in [b]^{(\alpha,\beta;t,f)}\}; \\ b_{2(\alpha,t)} = \max\{s \ , \ s \in [b]^{(\alpha,\beta;t,f)}\}; \\ b_{1(\beta,f)} = \min\{s \ , \ s \in [b]^{(\alpha,\beta;t,f)}\}; \\ b_{2(\beta,f)} = \max\{s \ , \ s \in [b]^{(\alpha,\beta;t,f)}\}. \end{cases}$$

Definition 4 (cf. [10]). *Let $b^*, b^{**} \in \mathbb{F}$, there exists $b^{***} \in \mathbb{F}$ such that: $b^* = b^{**} + b^{***}$. Then, we say that b^{***} is $gH-$difference of b^* and b^{**}, and we denote it by: $b^{***} = b^* \ominus_{gH} b^{**}$.*
If $b^* \ominus_{gH} b^{**}$ exists, then:

$$[b^* \ominus_{gH} b^{**}]^{(\alpha,\beta;t,f)} = \left\langle \left[b^*_{1(\alpha,t)} - b^{**}_{1(\alpha,t)}, b^*_{2(\alpha,t)} - b^{**}_{2(\alpha,t)}\right] ; \left[b^*_{1(\beta,f)} - b^{**}_{1(\beta,f)}, b^*_{2(\beta,f)} - b^{**}_{2(\beta,f)}\right] \right\rangle .$$

Remark 1. *We have: $b^* \ominus_{gH} b^{**} \neq b^* + (-1)b^{**}$.*

Now, we will define a distance on $\mathbb{F}$:

$$d_{\mathcal{H}} : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{R}^+ \cup \{0\},$$

such that, for all $b^*, b^{**} \in \mathcal{F}$ we have:

$$d_{\mathcal{H}}(b^*, b^{**}) = \sup_{\alpha,\beta\in[0,1]} \max\{\mid b^*_{1(\alpha,t)} - b^{**}_{1(\alpha,t)} \mid, \mid b^*_{2(\alpha,t)} - b^{**}_{2(\alpha,t)} \mid; \mid b^*_{1(\beta,f)} - b^{**}_{1(\beta,f)} \mid, \mid b^*_{2(\beta,f)} - b^{**}_{2(\beta,f)} \mid\}.$$

Therefore, b is continuous.

Subsequently, we give the notions related to the differentiation of intuitionistic fuzzy-valued functions of second type.

Definition 5 (cf. [10]). *Let $b : X \longrightarrow \mathbb{F}$ be an intuitionistic fuzzy-valued function of second type, b is said to be strongly generalized differentiable at $\xi \in X$, if there exists $\mathcal{D}b(\xi) \in \mathbb{F}$ such that one of the following conditions is verified:*

1) $\forall \varepsilon > 0$, expressions $b(\xi + \varepsilon) \ominus_{gH} b(\xi)$ and $b(\xi) \ominus_{gH} b(\xi - \varepsilon)$ exist, such that:

$$\begin{aligned} \mathcal{D}b(\xi) &= \lim_{\varepsilon \to 0^+} \frac{b(\xi + \varepsilon) \ominus_{gH} b(\xi)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{b(\xi) \ominus_{gH} b(\xi - \varepsilon)}{\varepsilon}, \end{aligned}$$

2) $\forall \varepsilon > 0$, expressions $b(\xi) \ominus_{gH} b(\xi + \varepsilon)$ and $b(\xi - \varepsilon) \ominus_{gH} b(\xi)$ exist, such that:

$$\begin{aligned} \mathcal{D}b(\xi) &= \lim_{\varepsilon \to 0^+} \frac{b(\xi) \ominus_{gH} b(\xi + \varepsilon)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{b(\xi - \varepsilon) \ominus_{gH} b(\xi)}{\varepsilon}, \end{aligned}$$

Therefore, b is differentiable on X if b is differentiable for all $\xi \in X$.

Definition 6 (cf. [2, Def. 8]). *Let $b : X \longrightarrow \mathbb{F}$ be an intuitionistic fuzzy-valued function of second type.*

1- We say that b is $l-$differentiable on X if b is differentiable according to type 1 of Definition 5, and we note: $\mathcal{D}^{(l)}b(\xi)$.

2- We say that b is $u-$differentiable on X if b is differentiable according to type 2 of Definition 5, and we note: $\mathcal{D}^{(u)}b(\xi)$.

Definition 7 (cf. [2, Def. 9]). *Let $b : X \longrightarrow \mathbb{F}$ be an intuitionistic fuzzy-valued function of second type.*

1- If b is $l-$differentiable, then $b_{1(\alpha,t)}, b_{2(\alpha,t)}, b_{1(\beta,t)}$ and $b_{2(\beta,t)}$ are differentiable and:

$$\left[\mathcal{D}^{(l)}b(\xi)\right]^{(\alpha,\beta;t,f)} = \left\langle \left[\mathcal{D}b_{1(\alpha,t)}(\xi), \mathcal{D}b_{2(\alpha,t)}(\xi)\right]; \left[\mathcal{D}b_{1(\beta,t)}(\xi), \mathcal{D}b_{2(\beta,t)}(\xi)\right] \right\rangle.$$

2- If b is $u-$differentiable, then $b_{1(\alpha,t)}, b_{2(\alpha,t)}, b_{1(\beta,t)}$ and $b_{2(\beta,t)}$ are differentiable and:

$$\left[\mathcal{D}^{(u)}b(\xi)\right]^{(\alpha,\beta;t,f)} = \left\langle \left[\mathcal{D}b_{2(\alpha,t)}(\xi), \mathcal{D}b_{1(\alpha,t)}(\xi)\right]; \left[\mathcal{D}b_{2(\beta,t)}(\xi), \mathcal{D}b_{1(\beta,t)}(\xi)\right] \right\rangle.$$

Definition 8 ([25]). *The Mittag-Leffler function of two variables is defined by the following formula:*

$$E_{\alpha,\beta}(s) = \sum_{i=0}^{\infty} \frac{s^i}{\Gamma(\alpha i + \beta)}.$$

3 Intuitionistic fuzzy Caputo fractional derivative and Laplace transform of second type

In this part, we give some important concepts concerning the Caputo fractional derivative and the Laplace transform in the fuzzy set of second type case.

Before starting, we have noted:

- $C^{\mathbb{F}}[a,b]$: the space of continuous functions with intuitionistic fuzzy values on $[a,b]$.
- $L^{\mathbb{F}}[a,b]$: the space of Lebesgue integrable functions with intuitionistic fuzzy values on $[a,b]$.

Definition 9 (cf. [2, Def. 10]). *Let $h \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ be an intuitionistic fuzzy-valued function of second type. Then the intuitionistic fuzzy Riemann–Liouville fractional integral of second type of h of order $\gamma \in \mathbb{C}$, $Re(\gamma) > 0$ is defined by:*

$$\left(I_{a^+}^{\gamma} h\right)(\xi) = \frac{1}{\Gamma(\gamma)} \int_a^{\xi} \frac{h(s)ds}{(\xi - s)^{1-\gamma}} \ , \ \forall \xi > a.$$

Using the (α,β)-cut representation of h, we give the following theorem.

Theorem 1 (cf. [2, Th. 1]). *Let $h \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ be an intuitionistic fuzzy-valued function of second type. Then the intuitionistic fuzzy Riemann–Liouville fractional integral of second type of h of order γ is defined as follows:*

$$\left(I_{a^+}^{\gamma} h\right)_{(\alpha,\beta)}(\xi) = \left[\left(I_{a^+}^{\gamma} h_1\right)_{(\alpha)}(\xi), \left(I_{a^+}^{\gamma} h_2\right)_{(\alpha)}(\xi); \left(I_{a^+}^{\gamma} h_1\right)_{(\beta)}(\xi), \left(I_{a^+}^{\gamma} h_2\right)_{(\beta)}(\xi)\right] \ , \forall \alpha, \beta \in [0,1].$$

With,

$$\begin{cases} \left(I_{a^+}^{\gamma} h_1\right)_{(\alpha)}(\xi) = \dfrac{1}{\Gamma(\gamma)} \int_a^{\xi} \dfrac{h_{1(\alpha)}(s)ds}{(\xi - s)^{1-\gamma}}; \\ \left(I_{a^+}^{\gamma} h_2\right)_{(\alpha)}(\xi) = \dfrac{1}{\Gamma(\gamma)} \int_a^{\xi} \dfrac{h_{2(\alpha)}(s)ds}{(\xi - s)^{1-\gamma}}; \\ \left(I_{a^+}^{\gamma} h_1\right)_{(\beta)}(\xi) = \dfrac{1}{\Gamma(\gamma)} \int_a^{\xi} \dfrac{h_{1(\beta)}(s)ds}{(\xi - s)^{1-\gamma}}; \\ \left(I_{a^+}^{\gamma} h_2\right)_{(\beta)}(\xi) = \dfrac{1}{\Gamma(\gamma)} \int_a^{\xi} \dfrac{h_{2(\beta)}(s)ds}{(\xi - s)^{1-\gamma}}. \end{cases}$$

Example 1. *Let us have the following intuitionistic fuzzy-valued function of second type h:*

$$[h(\xi)]_{(\alpha,\beta)} = \left[h_{1(\alpha)}(\xi), h_{2(\alpha)}(\xi); h_{1(\beta)}(\xi), h_{2(\beta)}(\xi)\right] = \left[1 \ , \ \xi \ ; \ 0 \ , \ \xi^2\right],$$

then, the intuitionistic fuzzy Riemann–Liouville fractional integral of second type of h of order $\gamma = \dfrac{1}{2}$ is given by:

$$\left(I_{0^+}^{\frac{1}{2}}h_1\right)_{(\alpha)}(\xi) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\xi}(\xi-s)^{\frac{1}{2}-1}1\ ds$$
$$= \frac{2}{\Gamma(\frac{1}{2})}\sqrt{\xi} = \frac{2}{\sqrt{\pi}}\sqrt{\xi},$$

$$\left(I_{0^+}^{\frac{1}{2}}h_2\right)_{(\alpha)}(\xi) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\xi}(\xi-s)^{\frac{1}{2}-1}s\ ds$$
$$= \frac{4}{3\Gamma(\frac{1}{2})}\sqrt{\xi^3} = \frac{4}{3\sqrt{\pi}}\sqrt{\xi^3},$$

$$\left(I_{0^+}^{\frac{1}{2}}h_1\right)_{(\beta)}(\xi) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\xi}(\xi-s)^{\frac{1}{2}-1}0\ ds = 0,$$

$$\left(I_{0^+}^{\frac{1}{2}}h_2\right)_{(\beta)}(\xi) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\xi}(\xi-s)^{\frac{1}{2}-1}s^2\ ds$$
$$= \frac{16}{15\Gamma(\frac{1}{2})}\sqrt{\xi^5} = \frac{16}{15\sqrt{\pi}}\sqrt{\xi^5}.$$

Hence,
$$\left(I_{0^+}^{\frac{1}{2}}h\right)_{(\alpha,\beta)}(\xi) = \left[\frac{2}{\sqrt{\pi}}\sqrt{\xi}\ ,\ \frac{4}{3\sqrt{\pi}}\sqrt{\xi^3}\ ;\ 0\ ,\ \frac{16}{15\sqrt{\pi}}\sqrt{\xi^5}\right].$$

Definition 10 (cf. [2, Def. 17])**.** *Let $h \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ be an intuitionistic fuzzy-valued function of second type. Then, the intuitionistic fuzzy Caputo fractional derivative of h of order $\gamma \in \mathbb{C}$, $Re(\gamma) \geq 0$ of second type is defined by:*
$$\left({}^c\mathcal{D}_{a^+}^{\gamma}h\right)(\xi) = \left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h\right)\right)(\xi)\ ,\ \forall\xi > 0,$$
with n being the integer for which: $n-1 \leq Re(\gamma) < n$.
Especially, if $\gamma \in (0,1)$ and $a = 0$, then the intuitionistic fuzzy Caputo fractional derivative of h of second type becomes:
$$\left({}^c\mathcal{D}_{0^+}^{\gamma}h\right)(\xi) = \frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'(s)ds}{(\xi-s)^{\gamma}}\ ,\ \forall\xi > 0.$$

Theorem 2 (cf. [2, Th. 3])**.** *Let $h \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ be an intuitionistic fuzzy-valued function of second type.*

1- If h is $(l)-$differentiable, then the intuitionistic fuzzy Caputo fractional derivative of order $\gamma \in \mathbb{C}, Re(\gamma) \geq 0$ of second type is defined as follows:
$$\left({}^c\mathcal{D}_{a^+}^{\gamma}h\right)_{(\alpha,\beta)}(\xi) = [\left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{1(\alpha)}\right)\right)(\xi), \left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{2(\alpha)}\right)\right)(\xi);$$
$$\left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{1(\beta)}\right)\right)(\xi), \left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{2(\beta)}\right)\right)(\xi)].$$

2- *If h is $(u)-$differentiable, then the intuitionistic fuzzy Caputo fractional derivative of order $\gamma \in \mathbb{C}, Re(\gamma) \geq 0$ of second type is defined as follows:*

$$\left({}^{c}\mathcal{D}_{a^+}^{\gamma} h\right)_{(\alpha,\beta)}(\xi) = [\left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{2(\alpha)}\right)\right)(\xi), \left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{1(\alpha)}\right)\right)(\xi);$$

$$\left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{2(\beta)}\right)\right)(\xi), \left(I_{a^+}^{n-\gamma}\left(\frac{d^n}{d\xi^n}h_{1(\beta)}\right)\right)(\xi)].$$

where the integral is defined in Definition 9 and n is a natural number such that $n-1 \leq Re(\gamma) < n$.

Remark 2. *If $\gamma \in (0,1)$ and $a = 0$, then the intuitionistic fuzzy Caputo fractional derivative of h of second type becomes :*

1- *If h is $(l)-$differentiable, then*

$$\left({}^{c}\mathcal{D}_{0^+}^{\gamma} h\right)_{(\alpha,\beta)}(\xi) = \left[\frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{1(\alpha)}(s)ds}{(\xi-s)^{\gamma}}, \frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{2(\alpha)}(s)ds}{(\xi-s)^{\gamma}}; \frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{1(\beta)}(s)ds}{(\xi-s)^{\gamma}}, \frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{2(\beta)}(s)ds}{(\xi-s)^{\gamma}}\right].$$

2- *If h is $(u)-$differentiable, then*

$$\left({}^{c}\mathcal{D}_{0^+}^{\gamma} h\right)_{(\alpha,\beta)}(\xi) = \left[\frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{2(\alpha)}(s)ds}{(\xi-s)^{\gamma}}, \frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{1(\alpha)}(s)ds}{(\xi-s)^{\gamma}}; \frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{2(\beta)}(s)ds}{(\xi-s)^{\gamma}}, \frac{1}{\Gamma(1-\gamma)}\int_0^{\xi}\frac{h'_{1(\beta)}(s)ds}{(\xi-s)^{\gamma}}\right].$$

Example 2. *Let the following intuitionistic fuzzy-valued function of second type h:*

$$[h(\xi)]_{(\alpha,\beta)} = \left[h_{1(\alpha)}(\xi), h_{2(\alpha)}(\xi); h_{1(\beta)}(\xi), h_{2(\beta)}(\xi)\right] = \left[\xi\ ,\ 3\ ;\ \xi^2\ ,\ -2\xi\right],$$

then, the intuitionistic fuzzy Caputo fractional derivative of second type of h of order $\gamma = \dfrac{2}{3}$ is given by:

$$\left({}^{c}\mathcal{D}_{0^+}^{\frac{2}{3}} h_1\right)_{(\alpha)}(\xi) = \frac{1}{\Gamma(1-\frac{2}{3})}\int_0^{\xi}(\xi-s)^{-\frac{2}{3}}(s)'\ ds$$
$$= \frac{3}{\Gamma(1-\frac{2}{3})}\sqrt[3]{\xi} = \frac{3}{\sqrt{\pi}}\sqrt[3]{\xi},$$

$$\left({}^{c}\mathcal{D}_{0^+}^{\frac{2}{3}} h_2\right)_{(\alpha)}(\xi) = \frac{1}{\Gamma(1-\frac{2}{3})}\int_0^{\xi}(\xi-s)^{-\frac{2}{3}}(3)' = 0,$$

$$\left({}^{c}\mathcal{D}_{0^+}^{\frac{2}{3}} h_1\right)_{(\beta)}(\xi) = \frac{1}{\Gamma(1-\frac{2}{3})}\int_0^{\xi}(\xi-s)^{-\frac{2}{3}}(s^2)'\ ds$$
$$= \frac{6}{\Gamma(1-\frac{2}{3})}\sqrt[3]{\xi^4} = \frac{6}{\sqrt{\pi}}\sqrt[3]{\xi^4},$$

$$\left({}^{c}\mathcal{D}_{0^+}^{\frac{2}{3}}h_2\right)_{(\beta)}(\xi) = \frac{1}{\Gamma(1-\dfrac{2}{3})}\int_0^{\xi}(\xi-s)^{-\frac{2}{3}}(-2s)'\,ds$$
$$= \frac{-6}{\Gamma(1-\dfrac{2}{3})}\sqrt[3]{\xi} = \frac{-6}{\sqrt{\pi}}\sqrt[3]{\xi}.$$

Hence,

$$\left({}^{c}\mathcal{D}_{0^+}^{\frac{2}{3}}h\right)_{(\alpha,\beta)}(\xi) = \left[\frac{3}{\sqrt{\pi}}\sqrt[3]{\xi}\ ,\ 0\ ;\ \frac{6}{\sqrt{\pi}}\sqrt[3]{\xi^4}\ ,\ \frac{-6}{\sqrt{\pi}}\sqrt[3]{\xi}\right].$$

Now we move on to define the Laplace transform in the case of intuitionistic fuzzy sets of second type, and investigate its properties.

Definition 11 (cf. [2, Def. 18])**.** *Let $h \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ be an intuitionistic fuzzy-valued function of second type.*

Suppose that $e^{-q\xi}h(\xi)$ is improper intuitionistic fuzzy Riemann integrable of second type on $[0,\infty)$, then the integral $\int_0^\infty e^{-q\xi}h(\xi)d\xi$ is the intuitionistic fuzzy Laplace transform of second type of h, and we write:

$$\mathcal{L}(q) = \mathfrak{L}[h(\xi)] = \int_0^\infty e^{-q\xi}h(\xi)d\xi\ ,\ q>0.$$

Then,

$$\mathfrak{L}[h(\xi)] = \left\langle \left[l[h_{1(\alpha)}(\xi)], l[h_{2(\alpha)}(\xi)]\right]\ ;\ \left[l[h_{1(\beta)}(\xi)], l[h_{2(\beta)}(\xi)]\right]\right\rangle$$
$$= \left\langle \left[\int_0^\infty e^{-q\xi}h_{1(\alpha)}(\xi)d\xi, \int_0^\infty e^{-q\xi}h_{2(\alpha)}(\xi)d\xi\right]; \left[\int_0^\infty e^{-q\xi}h_{1(\beta)}(\xi)d\xi, \int_0^\infty e^{-q\xi}h_{2(\beta)}(\xi)d\xi\right]\right\rangle.$$

Example 3. *Let the following intuitionistic fuzzy-valued function of second type h:*

$$[h(\xi)]_{(\alpha,\beta)} = \left[h_{1(\alpha)}(\xi), h_{2(\alpha)}(\xi); h_{1(\beta)}(\xi), h_{2(\beta)}(\xi)\right] = \left[2\ ,\ \xi\ ;\ -3\xi\ ,\ \xi^2\right],$$

then, the intuitionistic fuzzy Laplace transform of second type of h is defined by:

Let $\xi \in [0,1]$ and for all $q>0$ we have,

$$l[h_{1(\alpha)}(\xi)] = \int_0^\infty e^{-q\xi}h_{1(\alpha)}(\xi)d\xi$$
$$= \int_0^\infty e^{-q\xi}\ 2\ d\xi = 2\xi^{-1},$$

$$l[h_{2(\alpha)}(\xi)] = \int_0^\infty e^{-q\xi}h_{2(\alpha)}(\xi)d\xi$$
$$= \int_0^\infty e^{-q\xi}\ \xi\ d\xi = 1,$$

$$l[h_{1(\beta)}(\xi)] = \int_0^\infty e^{-q\xi}h_{1(\beta)}(\xi)d\xi$$
$$= \int_0^\infty e^{-q\xi}\ (-3\xi)\ d\xi = -3,$$

and

$$l[h_{2(\beta)}(\xi)] = \int_0^{\infty} e^{-q\xi} h_{2(\beta)}(\xi) d\xi$$
$$= \int_0^{\infty} e^{-q\xi}\, \xi^2 \, d\xi = \xi.$$

Hence,

$$\mathfrak{L}[h(\xi)] = \left[2\xi^{-1} \, , \, 1 \, ; \, -3 \, , \, \xi\right].$$

Theorem 3 (cf. [2, Th. 4]). *(Linearity property) Let h and g be continuous intuitionistic fuzzy-valued function of second type and let a and b be two constants then:*

$$\mathfrak{L}[a \odot h(\xi) \oplus b \odot g(\xi)] = a \odot \mathfrak{L}[h(\xi)] \oplus b \odot \mathfrak{L}[g(\xi)],$$

where $\oplus$ and $\odot$ are the addition and multiplication extensions.

Theorem 4 (cf. [2, Th. 5]). *Let $h(\xi)$ be continuous intuitionistic fuzzy-valued function of second type and $g(\xi) \geq 0$ a real-valued function. Suppose that $(h(\xi) \odot g(\xi))e^{-q\xi}$ is improper intuitionistic fuzzy Riemann integrable of second type integrable, then:*

$$\int_0^{\infty} (h(\xi) \odot g(\xi))e^{-q\xi} d\xi =$$
$$\left[\int_0^{\infty} g(\xi)e^{-q\xi} h_{1(\alpha)}(\xi)d\xi, \int_0^{\infty} g(\xi)e^{-q\xi} h_{2(\alpha)}(\xi)d\xi; \int_0^{\infty} g(\xi)e^{-q\xi} h_{1(\beta)}(\xi)d\xi, \int_0^{\infty} g(\xi)e^{-q\xi} h_{2(\beta)}(\xi)d\xi\right].$$

Theorem 5 (cf. [2, Th. 7]). *Let $h \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ be an intuitionistic fuzzy-valued function of second type.*

Then, the Intuitionistic fuzzy Laplace transform of second type of the intuitionistic fuzzy Caputo fractional derivative is given by:

1- If h is $(l)-$differentiable, then:

$$\mathfrak{L}[{}^c\mathcal{D}^{\gamma}_{a^+} h(\xi)] = q^{\gamma}\mathcal{L}(q) \ominus_{gH} q^{\gamma-1}h(0).$$

2- If h is $(u)-$differentiable, then:

$$\mathfrak{L}[{}^c\mathcal{D}^{\gamma}_{a^+} h(\xi)] = (-1)q^{\gamma-1}h(0) \ominus_{gH} (-1)q^{\gamma}\mathcal{L}(q).$$

4 Intuitionistic fuzzy integral solution of second type

For fractional problem (1), we pose for the two functions h and $\mathcal{F}$:

$$[h(\xi)]_{(\alpha,\beta)} = \left[h_{1(\alpha)}(\xi), h_{2(\alpha)}(\xi); h_{1(\beta)}(\xi), h_{2(\beta)}(\xi)\right],$$
$$[\mathcal{F}]_{(\alpha,\beta)} = \left[\mathcal{F}_{1(\alpha)}, \mathcal{F}_{2(\alpha)}; \mathcal{F}_{1(\beta)}, \mathcal{F}_{2(\beta)}\right],$$

We now propose the following two hypotheses:

(H_1) The function $\mathcal{F} : [0,T] \times \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$ is continuous and differentiable.

(H_2) There is a constant $M > 0$ such that, $\forall \xi \in [0,1]$ and $x, y, u, v \in \mathbb{F}$ we have:

$$d_{\mathcal{H}}\left(\mathcal{F}(\xi, x, u), \mathcal{F}(\xi, y, v)\right) \leq M\left(d_{\mathcal{H}}(x,y) + d_{\mathcal{H}}(u,v)\right).$$

Theorem 6. *Let $h \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionistic fuzzy-valued function of second type.*

1- When h is $(l)-$differentiable, the fractional problem (1) *allows for the following solution:*

$$h(\xi) = h_0 E_{n,1}(\xi^n) \oplus \int_0^\xi (\xi - s)^{n-1} E_{n,n}((\xi - s)^n)\mathcal{F}(s, h(s), {}^C D_{0+}^m h(s))ds,$$

2- When h is $(u)-$differentiable, the fractional problem (1) *allows for the following solution:*

$$h(\xi) = h_0 E_{n,1}(\xi^n) \ominus_{gH} \int_0^\xi (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \left(-\mathcal{F}(s, h(s), {}^C D_{0+}^m h(s))\right) ds.$$

where $\ominus_{gH}$ is the Hukuhara generalised difference.

Proof. We have in problem (1): ${}^C D_{0+}^n h(\xi) = \mathcal{F}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))$

We make use of the intuitionistic fuzzy Laplace transform of second type for both sides,

$$\mathfrak{L}[{}^C D_{0+}^n h(\xi)] = \mathfrak{L}[\mathcal{F}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))],$$

Now, if h is $(l)-$differentiable and according to Theorem 5, we have:

$$q^n \mathfrak{L}[h(\xi)] \ominus_{gH} q^{n-1} h_{(\alpha,\beta)}(0) = \mathfrak{L}[\mathcal{F}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))],$$

then,

$$\begin{aligned}
q^n l[h_{1(\alpha)}(\xi)] - q^{n-1} h_{1(\alpha)}(0) &= l[\mathcal{F}_{1(\alpha)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))], \\
q^n l[h_{2(\alpha)}(\xi)] - q^{n-1} h_{2(\alpha)}(0) &= l[\mathcal{F}_{2(\alpha)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))], \\
q^n l[h_{1(\beta)}(\xi)] - q^{n-1} h_{1(\beta)}(0) &= l[\mathcal{F}_{1(\beta)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))], \\
q^n l[h_{2(\beta)}(\xi)] - q^{n-1} h_{2(\beta)}(0) &= l[\mathcal{F}_{2(\beta)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))],
\end{aligned}$$

then, we obtain:

$$\begin{aligned}
l[h_{1(\alpha)}(\xi)] &= \frac{q^{n-1}}{q^n} h_{1(\alpha)}(0) + \frac{1}{q^n} l[\mathcal{F}_{1(\alpha)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))], \\
l[h_{2(\alpha)}(\xi)] &= \frac{q^{n-1}}{q^n} h_{2(\alpha)}(0) + \frac{1}{q^n} l[\mathcal{F}_{2(\alpha)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))], \\
l[h_{1(\beta)}(\xi)] &= \frac{q^{n-1}}{q^n} h_{1(\beta)}(0) + \frac{1}{q^n} l[\mathcal{F}_{1(\beta)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))], \\
l[h_{2(\beta)}(\xi)] &= \frac{q^{n-1}}{q^n} h_{2(\beta)}(0) + \frac{1}{q^n} l[\mathcal{F}_{2(\beta)}(\xi, h(\xi), {}^C D_{0+}^m h(\xi))].
\end{aligned}$$

We are currently using the intuitionistic fuzzy inverse Laplace transform of second type on both sides:

$$\begin{aligned}
h_{1(\alpha)}(\xi) &= h_{0,1(\alpha)} E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1} E_{n,n}((\xi - s)^n)\mathcal{F}_{1(\alpha)}(s, h(s), {}^C D_{0+}^m h(s))ds, \\
h_{2(\alpha)}(\xi) &= h_{0,2(\alpha)} E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1} E_{n,n}((\xi - s)^n)\mathcal{F}_{2(\alpha)}(s, h(s), {}^C D_{0+}^m h(s))ds, \\
h_{1(\beta)}(\xi) &= h_{0,1(\beta)} E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1} E_{n,n}((\xi - s)^n)\mathcal{F}_{1(\beta)}(s, h(s), {}^C D_{0+}^m h(s))ds,
\end{aligned}$$

$$h_{2(\beta)}(\xi) = h_{0,2(\beta)}E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1}E_{n,n}((\xi - s)^n)\mathcal{F}_{2(\beta)}(s, h(s), ^C D^m_{0^+}h(s))ds,$$

This can be summarized as follows:

$$[h_{1(\alpha)}(\xi), h_{2(\alpha)}(\xi); h_{1(\beta)}(\xi), h_{2(\beta)}(\xi)] = [h_{0,1(\alpha)}, h_{0,2(\alpha)}; h_{0,1(\beta)}, h_{0,2(\beta)})] E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1}E_{n,n}((\xi - s)^n)$$
$$\left[\mathcal{F}_{1(\alpha)}(s, h(s), ^C D^m_{0^+}h(s)), \mathcal{F}_{2(\alpha)}(s, h(s), ^C D^m_{0^+}h(s)); \mathcal{F}_{1(\beta)}(s, h(s), ^C D^m_{0^+}h(s)), \mathcal{F}_{2(\beta)}(s, h(s), ^C D^m_{0^+}h(s))\right] ds.$$

Since,

$$[h(\xi)]_{(\alpha,\beta)} = [h_0]_{(\alpha,\beta)} E_{n,1}(\xi^n) + \left[\int_0^\xi \mathcal{F}(s, h(s), ^C D^m_{0^+}h(s))(\xi - s)^{n-1}E_{n,n}((\xi - s)^n)ds\right]_{(\alpha,\beta)},$$

hence, the final expression of the solution is as follows:

$$h(\xi) = h_0E_{n,1}(\xi^n) \oplus \int_0^\xi (\xi - s)^{n-1}E_{n,n}((\xi - s)^n)\mathcal{F}(s, h(s), ^C D^m_{0^+}h(s))ds.$$

On the flip side, h is $(u)-$differentiable and similarly, as per Theorem 5, we have:

$$-q^{n-1}h_{(\alpha,\beta)}(0) \ominus_{gH} (-q^n\mathfrak{L}[h(\xi)]) = \mathfrak{L}[\mathcal{F}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))].$$

Then,

$$q^n l[h_{1(\alpha)}(\xi)] - q^{n-1}h_{1(\alpha)}(0) = l[\mathcal{F}_{2(\alpha)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))],$$
$$q^n l[h_{2(\alpha)}(\xi)] - q^{n-1}h_{2(\alpha)}(0) = l[\mathcal{F}_{1(\alpha)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))],$$
$$q^n l[h_{1(\beta)}(\xi)] - q^{n-1}h_{1(\beta)}(0) = l[\mathcal{F}_{2(\beta)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))],$$
$$q^n l[h_{2(\beta)}(\xi)] - q^{n-1}h_{2(\beta)}(0) = l[\mathcal{F}_{1(\beta)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))],$$

then, we have:

$$l[h_{1(\alpha)}(\xi)] = \frac{q^{n-1}}{q^n}h_{1(\alpha)}(0) + \frac{1}{q^n}l[\mathcal{F}_{2(\alpha)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))],$$
$$l[h_{2(\alpha)}(\xi)] = \frac{q^{n-1}}{q^n}h_{2(\alpha)}(0) + \frac{1}{q^n}l[\mathcal{F}_{1(\alpha)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))],$$
$$l[h_{1(\beta)}(\xi)] = \frac{q^{n-1}}{q^n}h_{1(\beta)}(0) + \frac{1}{q^n}l[\mathcal{F}_{2(\beta)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))],$$
$$l[h_{2(\beta)}(\xi)] = \frac{q^{n-1}}{q^n}h_{2(\beta)}(0) + \frac{1}{q^n}l[\mathcal{F}_{1(\beta)}(\xi, h(\xi), ^C D^m_{0^+}h(\xi))].$$

Now, we apply the intuitionistic fuzzy inverse Laplace transform of second type on both sides, and we obtain:

$$h_{1(\alpha)}(\xi) = h_{0,1(\alpha)}E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1}E_{n,n}((\xi - s)^n)\mathcal{F}_{2(\alpha)}(s, h(s), ^C D^m_{0^+}h(s))ds,$$
$$h_{2(\alpha)}(\xi) = h_{0,2(\alpha)}E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1}E_{n,n}((\xi - s)^n)\mathcal{F}_{1(\alpha)}(s, h(s), ^C D^m_{0^+}h(s))ds,$$
$$h_{1(\beta)}(\xi) = h_{0,1(\beta)}E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1}E_{n,n}((\xi - s)^n)\mathcal{F}_{2(\beta)}(s, h(s), ^C D^m_{0^+}h(s))ds,$$
$$h_{2(\beta)}(\xi) = h_{0,2(\beta)}E_{n,1}(\xi^n) + \int_0^\xi (\xi - s)^{n-1}E_{n,n}((\xi - s)^n)\mathcal{F}_{1(\beta)}(s, h(s), ^C D^m_{0^+}h(s))ds.$$

Therefore, in the same way we obtain the following form:

$$h(\xi) = h_0 E_{n,1}(\xi^n) \ominus_{gH} \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \left(-\mathcal{F}(s, h(s), {}^C D_{0+}^m h(s))\right) ds. \qquad \square$$

Theorem 7. *Suppose that the two hypotheses* (H_1) *and* (H_2) *are verified, then the fractional problem* (1) *admits a unique solution on* $[0, T]$ *for the two directions of the differentiability, if the following condition is verified:*

$$\frac{MT^n}{\Gamma(n+1)} \left(1 + \frac{T^{1-m}}{\Gamma(2-m)}\right) < 1.$$

Proof. Before initiating the demonstration, we establish the following distance:

$$D_{\mathcal{H}}(h, g) = \sup_{\xi \in [0,T]} d_{\mathcal{H}}(h(\xi), g(\xi)) \, , \, \forall h, g \in C^{\mathbb{F}} \cap L^{\mathbb{F}}.$$

Let us have an operator ψ such that, if h is $(l)-$differentiable, then:

$$\psi(h)(\xi) = h_0 E_{n,1}(\xi^n) \oplus \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}(s, h(s), {}^C D_{0+}^m h(s)) ds.$$

Let $h, g \in C^{\mathbb{F}} \cap L^{\mathbb{F}}$ be an intuitionistic fuzzy-valued function of second type and for all $\xi \in [0, T]$ we have:

$$
\begin{aligned}
& d_{\mathcal{H}}(\psi(h)(\xi), \psi(g)(\xi)) \\
&= \sup_{\alpha, \beta \in (0,1)} \max \Bigg(\Big| \left(h_{0,1(\alpha)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{1(\alpha)}(s, h(s), {}^C D_{0+}^m h(s)) ds \right) \\
&\qquad - \left(g_{0,1(\alpha)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{1(\alpha)}(s, g(s), {}^C D_{0+}^m g(s)) ds \right) \Big|, \\
&\qquad \Big| \left(h_{0,2(\alpha)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{2(\alpha)}(s, h(s), {}^C D_{0+}^m h(s)) ds \right) \\
&\qquad - \left(g_{0,2(\alpha)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{2(\alpha)}(s, g(s), {}^C D_{0+}^m g(s)) ds \right) \Big|; \\
&\qquad \Big| \left(h_{0,1(\beta)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{1(\beta)}(s, h(s), {}^C D_{0+}^m h(s)) ds \right) \\
&\qquad - \left(g_{0,1(\beta)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{1(\beta)}(s, g(s), {}^C D_{0+}^m g(s)) ds \right) \Big|, \\
&\qquad \Big| \left(h_{0,2(\beta)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{2(\beta)}(s, h(s), {}^C D_{0+}^m h(s)) ds \right) \\
&\qquad - \left(g_{0,2(\beta)} E_{n,1}(\xi^n) + \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \mathcal{F}_{2(\beta)}(s, g(s), {}^C D_{0+}^m g(s)) ds \right) \Big| \Bigg) \\
&\leq \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) d_{\mathcal{H}} \left(\mathcal{F}(s, h(s), {}^C D_{0+}^m h(s)), \mathcal{F}(s, g(s), {}^C D_{0+}^m g(s)) \right) ds \\
&\leq M \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \left(d_{\mathcal{H}} \left(h(s), g(s) \right) + d_{\mathcal{H}} \left({}^C D_{0+}^m h(s), {}^C D_{0+}^m g(s) \right) \right) ds \\
&\leq M \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) ds D_{\mathcal{H}}(h, g) \\
&\quad + \frac{M}{\Gamma(1-m)} \int_0^{\xi} (\xi - s)^{n-1} E_{n,n}((\xi - s)^n) \int_0^s (s - t)^{-m} dt ds D_{\mathcal{H}}(h, g),
\end{aligned}
$$

After reducing the complexity of the calculations, we observe the subsequent increment,

$$\begin{aligned} d_{\mathcal{H}}(\psi(h)(\xi),\psi(g)(\xi)) &\leq \frac{MT^n}{\Gamma(n+1)}D_{\mathcal{H}}(h,g)+\frac{MT^nT^{1-m}}{\Gamma(n+1)\Gamma(2-m)}D_{\mathcal{H}}(h,g)\\ &\leq \frac{MT^n}{\Gamma(n+1)}\left(1+\frac{T^{1-m}}{\Gamma(2-m)}\right)D_{\mathcal{H}}(h,g), \end{aligned}$$

Now, if h is $(u)-$differentiable, then:

$$\psi(h)(\xi)=h_0E_{n,1}(\xi^n)\ominus_{gH}\int_0^\xi(\xi-s)^{n-1}E_{n,n}((\xi-s)^n)\left(-\mathcal{F}(s,h(s),^C D^m_{0+}h(s))\right)ds.$$

Let $h,g\in C^{\mathbb{F}}\cap L^{\mathbb{F}}$ be an intuitionistic fuzzy-valued function of second type and for all $\xi\in[0,T]$, using the same previous approach, we have:

$$\begin{aligned} d_{\mathcal{H}}(\psi(h)(\xi),\psi(g)(\xi)) &\leq M\int_0^\xi(\xi-s)^{n-1}E_{n,n}((\xi-s)^n)\left(d_{\mathcal{H}}\left(h(s),g(s)\right)\right.\\ &\quad \left.+d_{\mathcal{H}}\left(^C D^m_{0+}h(s),^C D^m_{0+}g(s)\right)\right)ds\\ &\leq M\int_0^\xi(\xi-s)^{n-1}E_{n,n}((\xi-s)^n)dsD_{\mathcal{H}}(h,g)\\ &\quad +\frac{M}{\Gamma(1-m)}\int_0^\xi(\xi-s)^{n-1}E_{n,n}((\xi-s)^n)\int_0^s(s-t)^{-m}dtdsD_{\mathcal{H}}(h,g)\\ &\leq \frac{MT^n}{\Gamma(n+1)}D_{\mathcal{H}}(h,g)+\frac{MT^nT^{1-m}}{\Gamma(n+1)\Gamma(2-m)}D_{\mathcal{H}}(h,g)\\ &\leq \frac{MT^n}{\Gamma(n+1)}\left(1+\frac{T^{1-m}}{\Gamma(2-m)}\right)D_{\mathcal{H}}(h,g), \end{aligned}$$

In both cases, if $\frac{MT^n}{\Gamma(n+1)}\left(1+\frac{T^{1-m}}{\Gamma(2-m)}\right)<1$, then operator ψ is a contraction, therefore, it admits a unique fixed point.

Hence, the fractional problem (1) admits a unique solution. □

5 Example

As an example, we propose the following fractional problem:

$$\begin{cases} ^C D^{\frac{5}{2}}_{0+}h(\xi)=\mathcal{F}(\xi,h(\xi),^C D^{\frac{1}{2}}_{0+}h(\xi)), & \xi\in[0,1]\\ h(0)=h_0\in\mathbb{F}, \end{cases} \tag{2}$$

with $n=\frac{5}{2}$, $m=\frac{1}{2}$ and $T=1$.

The function $\mathcal{F}$ is given by the following formula:

$$\mathcal{F}(\xi,h(\xi),^C D^{\frac{1}{2}}_{0+}h(\xi))=h(\xi)\oplus^C D^{\frac{1}{2}}_{0+}h(\xi),$$

for all $\alpha,\beta\in(0,1)$, we have the initial condition of the problem is given by:

$$\begin{aligned} [h_0] &= \left[h_{0,1(\alpha)},h_{0,2(\alpha)};h_{0,1(\beta)},h_{0,2(\beta)}\right]\\ &= [\alpha+3,1-\alpha;5-2\beta,5+2\beta]. \end{aligned}$$

To find the solution to this problem, we apply the method described in the previous part.

We apply the intuitionistic fuzzy Laplace transform of second type to both sides of (2).

Case 1: If h is $(l)-$differentiable, then:

$$\begin{aligned} q^{\frac{5}{2}}\mathfrak{L}(h(\xi)) \ominus_{gH} q^{\frac{5}{2}-1}h_{(\alpha,\beta)}(0) &= \mathfrak{L}\left(h(\xi) \oplus^{C} D^{\frac{1}{2}}_{0^+} h(\xi)\right) \\ &= \mathfrak{L}(h(\xi)) \oplus \mathfrak{L}\left({}^{C}D^{\frac{1}{2}}_{0^+} h(\xi)\right) \\ &= \mathfrak{L}(h(\xi)) \oplus q^{\frac{1}{2}}\mathfrak{L}(h(\xi)) \ominus_{gH} q^{\frac{1}{2}-1}h_{(\alpha,\beta)}(0) \end{aligned}$$

then,

$$\begin{aligned} q^{\frac{5}{2}}l[h_{1(\alpha)}(\xi)] - q^{\frac{5}{2}-1}h_{1(\alpha)}(0) &= l[h_{1(\alpha)}(\xi)] + q^{\frac{1}{2}}l[h_{1(\alpha)}(\xi)] - q^{\frac{1}{2}-1}h_{1(\alpha)}(0), \\ q^{\frac{5}{2}}l[h_{2(\alpha)}(\xi)] - q^{\frac{5}{2}-1}h_{2(\alpha)}(0) &= l[h_{2(\alpha)}(\xi)] + q^{\frac{1}{2}}l[h_{2(\alpha)}(\xi)] - q^{\frac{1}{2}-1}h_{2(\alpha)}(0), \\ q^{\frac{5}{2}}l[h_{1(\beta)}(\xi)] - q^{\frac{5}{2}-1}h_{1(\beta)}(0) &= l[h_{1(\beta)}(\xi)] + q^{\frac{1}{2}}l[h_{1(\beta)}(\xi)] - q^{\frac{1}{2}-1}h_{1(\beta)}(0), \\ q^{\frac{5}{2}}l[h_{2(\beta)}(\xi)] - q^{\frac{5}{2}-1}h_{2(\beta)}(0) &= l[h_{2(\beta)}(\xi)] + q^{\frac{1}{2}}l[h_{2(\beta)}(\xi)] - q^{\frac{1}{2}-1}h_{2(\beta)}(0), \end{aligned}$$

after simplifying the calculations, we obtain:

$$\begin{aligned} l[h_{1(\alpha)}(\xi)] &= \frac{q^{\frac{5}{2}} - q^{\frac{1}{2}}}{q\left(q^{\frac{3}{2}} - q^{\frac{1}{2}} - 1\right)}h_{0,1(\alpha)} = \frac{q^2-1}{q^{\frac{1}{2}}\left(q^{\frac{1}{2}}\left(q^2-1\right)-1\right)}h_{0,1(\alpha)}, \\ l[h_{2(\alpha)}(\xi)] &= \frac{q^{\frac{5}{2}} - q^{\frac{1}{2}}}{q\left(q^{\frac{3}{2}} - q^{\frac{1}{2}} - 1\right)}h_{0,2(\alpha)} = \frac{q^2-1}{q^{\frac{1}{2}}\left(q^{\frac{1}{2}}\left(q^2-1\right)-1\right)}h_{0,2(\alpha)}, \\ l[h_{1(\beta)}(\xi)] &= \frac{q^{\frac{5}{2}} - q^{\frac{1}{2}}}{q\left(q^{\frac{3}{2}} - q^{\frac{1}{2}} - 1\right)}h_{0,1(\beta)} = \frac{q^2-1}{q^{\frac{1}{2}}\left(q^{\frac{1}{2}}\left(q^2-1\right)-1\right)}h_{0,1(\beta)}, \\ l[h_{2(\beta)}(\xi)] &= \frac{q^{\frac{5}{2}} - q^{\frac{1}{2}}}{q\left(q^{\frac{3}{2}} - q^{\frac{1}{2}} - 1\right)}h_{0,2(\beta)} = \frac{q^2-1}{q^{\frac{1}{2}}\left(q^{\frac{1}{2}}\left(q^2-1\right)-1\right)}h_{0,2(\beta)}. \end{aligned}$$

Then, we apply the intuitionistic fuzzy inverse Laplace transform of second type, we find the following solution:

$$\begin{aligned} h_{1(\alpha)}(\xi) &= E_{(2,1)}(\xi^2)(\alpha+3), \\ h_{2(\alpha)}(\xi) &= E_{(2,1)}(\xi^2)(1-\alpha), \\ h_{1(\beta)}(\xi) &= E_{(2,1)}(\xi^2)(5-2\beta), \\ h_{2(\beta)}(\xi) &= E_{(2,1)}(\xi^2)(5+2\beta). \end{aligned}$$

Therefore, the solution is given by:

$$[h(\xi)]_{(\alpha,\beta)} = E_{(2,1)}(\xi^2)\left[\alpha+3, 1-\alpha; 5-2\beta, 5+2\beta\right].$$

Case 2: If h is $(u)-$differentiable, then:

$$-q^{\frac{5}{2}-1}h_{(\alpha,\beta)}(0)\ominus_{gH}\left(-q^{\frac{5}{2}}\mathfrak{L}(h(\xi))\right)=\mathfrak{L}\left(h(\xi)\oplus^{C}D_{0^+}^{\frac{1}{2}}h(\xi)\right)$$
$$=\mathfrak{L}(h(\xi))\oplus\mathfrak{L}\left({}^{C}D_{0^+}^{\frac{1}{2}}h(\xi)\right)$$
$$=\mathfrak{L}(h(\xi))\oplus\left(-q^{\frac{1}{2}-1}h_{(\alpha,\beta)}(0)\right)\ominus_{gH}\left(-q^{\frac{1}{2}}\mathfrak{L}(h(\xi))\right).$$

Then,

$$q^{\frac{5}{2}}l[h_{1(\alpha)}(\xi)]-q^{\frac{5}{2}-1}h_{1(\alpha)}(0)=l[h_{2(\alpha)}(\xi)]+q^{\frac{1}{2}}l[h_{1(\alpha)}(\xi)]-q^{\frac{1}{2}-1}h_{1(\alpha)}(0),$$
$$q^{\frac{5}{2}}l[h_{2(\alpha)}(\xi)]-q^{\frac{5}{2}-1}h_{2(\alpha)}(0)=l[h_{1(\alpha)}(\xi)]+q^{\frac{1}{2}}l[h_{2(\alpha)}(\xi)]-q^{\frac{1}{2}-1}h_{2(\alpha)}(0),$$
$$q^{\frac{5}{2}}l[h_{1(\beta)}(\xi)]-q^{\frac{5}{2}-1}h_{1(\beta)}(0)=l[h_{2(\beta)}(\xi)]+q^{\frac{1}{2}}l[h_{1(\beta)}(\xi)]-q^{\frac{1}{2}-1}h_{1(\beta)}(0),$$
$$q^{\frac{5}{2}}l[h_{2(\beta)}(\xi)]-q^{\frac{5}{2}-1}h_{2(\beta)}(0)=l[h_{1(\beta)}(\xi)]+q^{\frac{1}{2}}l[h_{2(\beta)}(\xi)]-q^{\frac{1}{2}-1}h_{2(\beta)}(0),$$

after the calculations we find the following system:

$$\begin{cases} l[h_{1(\alpha)}(\xi)]=\left(q^{\frac{5}{2}}-q^{\frac{1}{2}}\right)l[h_{2(\alpha)}(\xi)]-\left(q^{\frac{5}{2}-1}-q^{\frac{1}{2}-1}\right)h_{0,2(\alpha)},\\ l[h_{2(\alpha)}(\xi)]=\left(q^{\frac{5}{2}}-q^{\frac{1}{2}}\right)l[h_{1(\alpha)}(\xi)]-\left(q^{\frac{5}{2}-1}-q^{\frac{1}{2}-1}\right)h_{0,1(\alpha)},\\ l[h_{1(\beta)}(\xi)]=\left(q^{\frac{5}{2}}-q^{\frac{1}{2}}\right)l[h_{2(\beta)}(\xi)]-\left(q^{\frac{5}{2}-1}-q^{\frac{1}{2}-1}\right)h_{0,2(\beta)},\\ l[h_{2(\beta)}(\xi)]=\left(q^{\frac{5}{2}}-q^{\frac{1}{2}}\right)l[h_{1(\beta)}(\xi)]-\left(q^{\frac{5}{2}-1}-q^{\frac{1}{2}-1}\right)h_{0,1(\beta)}. \end{cases}$$

We solve the system and we obtain:

$$l[h_{1(\alpha)}(\xi)]=\frac{q^4-1}{q^5-q-1}h_{0,1(\alpha)}+\frac{q^2-1}{q^{\frac{1}{2}}(q^5-q-1)}h_{0,2(\alpha)}=\frac{q^4-1}{q(q^4-1)-1}h_{0,1(\alpha)}+\frac{q^2-1}{q^{\frac{1}{2}}(q(q^4-1)-1)}h_{0,2(\alpha)},$$
$$l[h_{2(\alpha)}(\xi)]=\frac{q^4-1}{q^5-q-1}h_{0,2(\alpha)}+\frac{q^2-1}{q^{\frac{1}{2}}(q^5-q-1)}h_{0,1(\alpha)}=\frac{q^4-1}{q(q^4-1)-1}h_{0,2(\alpha)}+\frac{q^2-1}{q^{\frac{1}{2}}(q(q^4-1)-1)}h_{0,1(\alpha)},$$
$$l[h_{1(\beta)}(\xi)]=\frac{q^4-1}{q^5-q-1}h_{0,1(\beta)}+\frac{q^2-1}{q^{\frac{1}{2}}(q^5-q-1)}h_{0,2(\beta)}=\frac{q^4-1}{q(q^4-1)-1}h_{0,1(\beta)}+\frac{q^2-1}{q^{\frac{1}{2}}(q(q^4-1)-1)}h_{0,2(\beta)},$$
$$l[h_{2(\beta)}(\xi)]=\frac{q^4-1}{q^5-q-1}h_{0,2(\beta)}+\frac{q^2-1}{q^{\frac{1}{2}}(q^5-q-1)}h_{0,1(\beta)}=\frac{q^4-1}{q(q^4-1)-1}h_{0,2(\beta)}+\frac{q^2-1}{q^{\frac{1}{2}}(q(q^4-1)-1)}h_{0,1(\beta)}.$$

Afterwards, we apply the intuitionistic fuzzy inverse Laplace transform of second type, and we obtain the following solution:

$$h_{1(\alpha)}(\xi)=E_{(4,1)}(\xi^4)(\alpha+3)+E_{(4,\frac{7}{2})}(\xi^4)(1-\alpha),$$
$$h_{2(\alpha)}(\xi)=E_{(4,1)}(\xi^4)(1-\alpha)+E_{(4,\frac{7}{2})}(\xi^4)(\alpha+3),$$
$$h_{1(\beta)}(\xi)=E_{(4,1)}(\xi^4)(5-2\beta)+E_{(4,\frac{7}{2})}(\xi^4)(5+2\beta),$$
$$h_{2(\beta)}(\xi)=E_{(4,1)}(\xi^4)(5+2\beta)+E_{(4,\frac{7}{2})}(\xi^4)(5-2\beta).$$

Therefore, the solution is given by:

$$[h(\xi)]_{(\alpha,\beta)}=E_{(4,1)}(\xi^4)\left[\alpha+3,1-\alpha;5-2\beta,5+2\beta\right]+E_{(4,\frac{7}{2})}(\xi^4)\left[1-\alpha,\alpha+3;5+2\beta,5-2\beta\right].$$

For the existence and uniqueness of this solution, we have the following condition:

$$\begin{aligned}\frac{MT^n}{\Gamma(n+1)}\left(1+\frac{T^{1-m}}{\Gamma(2-m)}\right) &= \frac{1}{\Gamma\left(\frac{5}{2}+1\right)}\left(1+\frac{1}{\Gamma\left(2-\frac{1}{2}\right)}\right)\\ &= \frac{1}{\frac{15}{8}\sqrt{\pi}}\left(1+\frac{1}{\frac{1}{2}\sqrt{\pi}}\right)\\ &= 0.64 < 1,\end{aligned}$$

Therefore, according to Theorem 7, problem (2) admits a unique solution.

6 Conclusion and future work

In this article, we studied the solution of a fractional equation with the intuitionistic fuzzy fractional Caputo derivative of second type, we found the solution using the intuitionistic fuzzy Laplace transform of second type, then we showed the existence and uniqueness of this solution. We illustrated the proposed approach with a particular example.

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