Notes on Intuitionistic Fuzzy Sets
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On the $(\min, \max)$-transitivity of some intuitionistic fuzzy binary relations associated with probability distributions on pre-orders

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Abstract: In this paper, we introduce an intuitionistic fuzzy binary relation on a finite universe generated by a probability distribution on a set of complete pre-orders of the universe. We establish necessary and sufficient conditions on the probability distribution under which the obtained relation is $(\min, \max)$-transitive. We study two specific cases where the relation is generated by the generalized parametric Mallows and Plackett–Luce probability distributions.

Keywords: Probability distribution on pre-orders, Intuitionistic fuzzy binary relation, $(\min, \max)$-transitivity.
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1 Introduction

In the theory of intuitionistic fuzzy set, an intuitionistic fuzzy binary relation (IFR for short) is an interesting tool that expresses ambiguous preferences. Thus many authors ([6], [3]) have developed theoretical results and applications on IFR. For example, Fono *et al.* [3] proposed a factorization of an intuitionistic fuzzy relation in an indifference component and a strict component, they studied the transitivity of the obtained strict components and they applied these results to establish intuitionistic fuzzy versions of some well-known arrowian results. Up to now, authors considered a given intuitionistic fuzzy relation. As the best of our knowledge, there is not a method to derive such relation from a primary underlying preference. A new trend is to introduce and to study intuitionistic fuzzy relations (IFRs for short) on a finite universe X from a probability distribution on the set of complete pre-orders (reflexive, transitive and complete crisp binary relations) of X. An underlying idea of this new method is to link pairwise comparisons of elements of X and the occurrences of orderings with possible ties of these elements. Note that we are inspired from works of some scholars (Kamdem *et al.* [4]) who introduce and study choice functions generated by fuzzy binary relations of X generated by probability distributions on the set of linear orders. Let us illustrate the usefulness of this method of the determination of an IFR in the voting theory. Let $X = \{a, b, c\}$ be the set of candidates and N the finite set of voters who express their preferences on X by ordering with possible ties, that is, each voter will choose one complete pre-order of X (choose one on the set of complete pre-orders $P_3 = \{cab, (ac)b, acb, a(cb), abc, c(ab), (cab), (ab)c, cba, (cb)a, bca, b(ac), bac\}$). A probability distribution $\mathcal{P}$ on P_3 is interpreted as the proportions of voters who choose each complete pre-order. One interesting question is to model and analyze pairwise comparisons of candidates of X based on their rankings (with possible ties) by voters. One answer is to introduce an IFR associated with $\mathcal{P}$ to model such comparisons and the analysis will focus on the study of the transitivity property of the obtained IFR based on the probability distribution.

The modest contribution of the paper is to introduce and study the transitivity of IFRs derived from probability distributions on the set of complete pre-orders. More precisely, we introduce, by means of the sum operator, an intuitionistic fuzzy relation of a universe generated by a probability distribution on the set of complete pre-orders of the universe. We establish necessary and/or sufficient conditions on the probability distributions under which the relation is $(\min, \max)$-transitive. We study the specific cases where the probability distribution is uniform and the probability distribution is restricted to the set of linear orders of the universe X. We consider the generalized parametric Mallows probability distribution and the generalized parametric Plackett–Luce probability distribution on the set of complete pre-orders as suggested Ngibasona *et al.* [5]. We rewrite the relation associated with these two probability distributions. We deduce, from the first results, conditions on parameters under which the generated relation satisfies the transitivity.

The paper is organized as follows. Section 2 concerns probability distributions on the set of complete pre-orders on a finite universe. As an example, we recall the generalized parametric Mallows and Plackett–Luce probability distributions of that set. In Section 3, we introduce, by means of the sum operator, an IFR generated by a probability distribution on the set of complete pre-orders and we deduce its two expressions for the parametric probability distributions. We study necessary and/or sufficient conditions on probability distribution under which these three relations are $(\min, \max)$-transitive. Section 4 gives some concluding remarks.

2 Preliminaries

2.1 Set of crisp complete pre-orders: Definitions, notations, examples and its probability distributions

P_n is the set of complete pre-orders on X with n elements $x_1, \ldots, x_n$. Note that the set S_n of linear orders (reflexive, antisymmetric, transitive and complete crisp binary relations) on X is a subset of P_n. For a complete pre-order $\pi \in P_n$ and $x, y \in X$, the notation $x \succ_\pi y$ means x is strictly preferred to y, and the notation $x \sim_\pi y$ means that x and y are indifferent. For $x, y, z \in X$, the complete pre-order $\pi = xzy$ is an order with x the first element, z the second and y the third. The complete pre-order $\pi = z(yx)$ has two blocks (positions) with z at the first position and the block of indifferent elements x and y at the last and second position. For $a, b \in X$, $P_n^{(a,b)} \subset P_n$ denotes the set of complete pre-orders π such that $a \succ_\pi b$ or $a \sim_\pi b$ and, $P_n^{(a>b)} \subset P_n^{(a,b)}$ denotes the set of complete pre-orders π such that $a \succ_\pi b$.

In this paper, we use probability distributions on P_n as illustrated on P_3 in the following example.

Example 1. *Let $X = \{a, b, c\}$. A probability distribution $\mathcal{P}$ on P_3 is specified by the 13 probabilities as shown in the following Table 1:*

Table 1. Probability distribution $\mathcal{P}$ on P_3

$\boldsymbol{\pi_i \in P_3}$	cab	$(ac)b$	acb	$a(cb)$	abc	$c(ab)$	(abc)
$\boldsymbol{\mathcal{P}(\pi_i)}$	p_1	p_2	p_3	p_4	p_5	p_6	p_7

$\boldsymbol{\pi_i \in P_3}$	$(ab)c$	cba	$(cb)a$	bca	$b(ac)$	bac
$\boldsymbol{\mathcal{P}(\pi_i)}$	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}

Note that those probabilities satisfy $p_1, \ldots, p_{13} \geq 0$ and $p_1 + \cdots + p_{13} = 1$. Moreover, $p_i = \mathcal{P}(\pi_i)$ is the probability to observe the complete pre-order π_i.

Let us complement the vote-theoretical significance of the previous probability distribution on P_3 given in the Introduction. If we assume that n decision makers (voters) express their preferences in term of complete pre-orders on the set X of outcomes (candidates), then P_3 is the set of complete pre-orders where the decision maker can choose and $p_i = \mathcal{P}(\pi_i)$ is the proportion of decision makers who choose the complete pre-order π_i.

We will use the following two notations: $\mathcal{P}(x \succ y \succeq z) = \mathcal{P}(xyz) + \mathcal{P}(x(yz))$ the probability that x is the unique first and $\mathcal{P}(x \succeq y \succ z) = \mathcal{P}((xy)z) + \mathcal{P}(xyz)$ the probability that z is the unique third (last).

In the following, we recall two probability distributions introduced by Ngibasona *et al.* [5].

2.2 Parametric probability distributions on crisp complete pre-orders

Let us start by the generalized Mallows probability distribution on complete pre-orders.

Firstly inspired from framework of Diffo *et al.* [2], Ngibasona *et al.* [5] have recently proposed the metric d_α on P_n as follows: The metric $d_\alpha : P_n^2 \to \mathbb{R}^+$ with $\alpha \in [\frac{1}{2}, 1]$ on P_n is defined by: $\forall(\pi_1, \pi_2) \in P_n^2$,

$$d_\alpha(\pi_1, \pi_2) = \sum_{\{a,b\} \in P_2(X)} d_\alpha^{\{a;b\}}(\pi_1, \pi_2) \tag{1}$$

where $P_2(X)$ is the set of all the subsets of X having two elements and $d_\alpha^{\{a,b\}}$ is a mapping on P_n^2 measuring the distance between π_1 and π_2 on $\{a, b\} \in P_2(X)$ as in the following Table 2.

Table 2. Distance d_α between π_1 and π_2

$d_\alpha^{\{a,b\}}(\pi_1, \pi_2)$	$a \succ_{\pi_1} b$	$a \sim_{\pi_1} b$	$b \succ_{\pi_1} a$
$a \succ_{\pi_2} b$	0	α	1
$a \sim_{\pi_2} b$	α	0	α
$b \succ_{\pi_2} a$	1	α	0

Note that the restriction of d_α on S_n becomes the well-known kendall metric on S_n and d_α generalizes the Kemeny distance on complete pre-orders. Secondly, Ngibasona *et al.* [5] used the metric d_α to generalize the Mallows' parametric distribution on P_n (simply called P_n-Mallows distribution) as follows: given $\alpha \in [\frac{1}{2}, 1], \pi_0 \in P_n$ and the spread parameter $m \geq 0$, the probability $\mathcal{P}_{\pi_0,m,d_\alpha}$ is defined by: $\forall\pi \in P_n, \mathcal{P}_{\pi_0,m,d_\alpha}(\pi) = \frac{\exp(-md_\alpha(\pi_0,\pi))}{\Phi(m)}$ where $\Phi(m) = \sum_{\pi \in P_n} e^{-md_\alpha(\pi_0,\pi)}$ is the normalization factor.

Let us illustrate the P_3-Mallows distribution in the following Example.

Example 2. (see Ngibasona *et al.* [5]) *Let $X = \{a, b, c\}$ and $\pi_0 = abc$. By using the notations of Table 1 of Example 1, we obtain:*

$$\begin{cases} p_1 = \mathcal{P}_{\pi_0,m,d_\alpha}(cab) = p_{11} = \mathcal{P}_{\pi_0,m,d_\alpha}(bca) = \frac{e^{-2m}}{\Phi(m)}, \\ p_{12} = \mathcal{P}_{\pi_0,m,d_\alpha}(b(ca)) = p_2 = \mathcal{P}_{\pi_0,m,d_\alpha}((ca)b) = \frac{e^{-m(1+\alpha)}}{\Phi(m)}, \\ p_{13} = \mathcal{P}_{\pi_0,m,d_\alpha}(acb) = p_3 = \mathcal{P}_{\pi_0,m,d_\alpha}(bac) = \frac{e^{-m}}{\Phi(m)}, \\ p_4 = \mathcal{P}_{\pi_0,m,d_\alpha}(a(cb)) = p_8 = \mathcal{P}_{\pi_0,m,d_\alpha}((ab)c) = \frac{e^{-m\alpha}}{\Phi(m)}, \\ p_5 = \mathcal{P}_{\pi_0,m,d_\alpha}(abc) = \frac{1}{\Phi(m)}, \\ p_6 = \mathcal{P}_{\pi_0,m,d_\alpha}(c(ba)) = p_{10} = \mathcal{P}_{\pi_0,m,d_\alpha}((bc)a) = \frac{e^{-m(2+\alpha)}}{\Phi(m)}, \\ p_7 = \mathcal{P}_{\pi_0,m,d_\alpha}((abc)) = \frac{e^{-3m\alpha}}{\Phi(m)} \text{ and } p_9 = \mathcal{P}_{\pi_0,m,d_\alpha}(cba) = \frac{e^{-3m}}{\Phi(m)}. \end{cases} \tag{2}$$

We end this subsection by recalling the generalized Plackett–Luce probability distribution on P_n (simply called P_n-P.L. distribution) as suggested by Ngibasona *et al.* [5].

A complete pre-order on X is considered as a complete order made up of a finite number of blocks of indifferent elements. Formally speaking, for $\pi \in P_n$, there exist $l \in \{1, \ldots, n\}$, $\pi = \pi(l) = \pi_{[1]}\pi_{[2]} \cdots \pi_{[l]}$ when $\pi_{[i]}$ is the i-th block in the l-order π (see Andjiga *et al.* [1]). If $l = 1$ (respectively, $l = n$), then all elements of π are indifferent (respectively, π is a complete linear order). Thus, π belongs to the set $S_{\pi(l)}$ of permutations of its blocks (set of pre-orders having the same blocks). Moreover, $S_{\pi(l)}$ is an equivalence class of the equivalence relation $\simeq$ defining the equivalence of two complete pre-orders if and only if they have the same blocks. π belongs to the subset $S_{\pi(l)}$ of P_n made up of complete pre-orders having l blocks. Based on that new formulation of π, Ngibasona *et al.* [5] introduced the second parametric distribution on P_n as follows: for all $l \in \{1, \ldots, n\}$, for all $\pi \in S_{\pi(l)}$, the probability to obtain π is

$$\mathcal{P}_{\vec{v}}(\pi) = \beta_{S_{\pi(l)}} \prod_{1 \leq i \leq l} \frac{v_i^{\pi}}{v_i^{\pi} + \cdots + v_l^{\pi}}, \tag{3}$$

where $(\beta_{S_{\pi(l)}})_{\pi(l) \in P_n/_{\simeq}}$ is the family of parameters assigned to equivalence classes satisfying $\sum_{\pi(l) \in P_n/_{\simeq}} \beta_{S_{\pi(l)}} = 1$ and $\vec{v} = (v_i^{\pi(l)})_{i \in \{1,\ldots,l\}}$ is the family of parameters assigned to positions of the l blocks of the equivalence class $S_{\pi(l)}$.

Let us illustrate the P_3-PL probability distribution in the following Example.

Example 3. *(see Ngibasona et al [5]) Let $X = \{a, b, c\}$, we have five equivalent classes:*

$$P_3/_{\simeq} = \{S_{abc}, S_{(bc)a}, S_{(ab)c}, S_{(ac)b}, S_{(abc)}\}.$$

- *In the class S_{abc}, we have:*

$$\begin{aligned}
p_5 = \mathcal{P}_{\vec{v}}(abc) &= \beta_{S_{abc}} \frac{v_a v_b}{(v_a + v_b + v_c)(v_b + v_c)}, \\
p_3 = \mathcal{P}_{\vec{v}}(acb) &= \beta_{S_{abc}} \frac{v_a v_c}{(v_a + v_b + v_c)(v_c + v_b)}, \\
p_{13} = \mathcal{P}_{\vec{v}}(bac) &= \beta_{S_{abc}} \frac{v_b v_a}{(v_a + v_b + v_c)(v_a + v_c)}, \\
p_{11} = \mathcal{P}_{\vec{v}}(bca) &= \beta_{S_{abc}} \frac{v_b v_c}{(v_a + v_b + v_c)(v_c + v_a)}, \\
p_1 = \mathcal{P}_{\vec{v}}(cab) &= \beta_{S_{abc}} \frac{v_c v_a}{(v_a + v_b + v_c)(v_a + v_b)}, \\
p_9 = \mathcal{P}_{\vec{v}}(cba) &= \beta_{S_{abc}} \frac{v_c v_b}{(v_a + v_b + v_c)(v_b + v_a)}.
\end{aligned}$$

- *In the class $S_{(ab)c}$, we have:*
  $p_8 = \mathcal{P}_{\vec{v}}((ab)c) = \beta_{S_{(ab)c}} \dfrac{v_{(ab)}}{v_{(ab)} + v_c}$ *and* $p_6 = \mathcal{P}_{\vec{v}}(c(ab)) = \beta_{S_{(ab)c}} \dfrac{v_c}{v_c + v_{(ab)}}$.

- *In the class $S_{(bc)a}$, we have:*
  $p_{10} = \mathcal{P}_{\vec{v}}((bc)a) = \beta_{S_{a(bc)}} \dfrac{v_{(bc)}}{v_{(bc)} + v_a}$ *and* $p_4 = \mathcal{P}_{\vec{v}}(a(bc)) = \beta_{S_{a(bc)}} \dfrac{v_a}{v_a + v_{(bc)}}$.

- *In the class $S_{(ac)b}$, we have:*
  $p_{12} = \mathcal{P}_{\vec{v}}(b(ac)) = \beta_{S_{(ac)b}} \dfrac{v_b}{v_b + v_{(ac)}}$ *and* $p_2 = \mathcal{P}_{\vec{v}}((ac)b) = \beta_{S_{(ac)b}} \dfrac{v_{(ac)}}{v_{(ac)} + v_b}$.

- *And the probability to have (abc) is $p_7 = \mathcal{P}_{\vec{v}}((abc)) = \beta_{S_{(abc)}}$.*

When necessary, we set for $n = 3$, $\beta_1 = \beta_{S_{abc}}$, $\beta_2 = \beta_{S_{\pi(2)}}$ (with $\pi(2) \in \{a(bc), b(ac), c(ab)\}$) and $\beta_3 = \beta_{S_{(abc)}}$.

Throughout this paper, we introduce three IFRs on X generated by the previous probability distributions on P_n. We then study conditions, on probability distributions, under which the obtained relations satisfy the well-known and usual intuitionistic (min,max)-transitivity.

3 Intuitionistic fuzzy binary relations, induced by probability distributions on complete pre-orders

3.1 Definitions and examples

We define the IFR $R = (\mu_R, \nu_R)$ on X associated with $\mathcal{P}$ by using the sum operator as follows:

$$\forall a, b \in X, \mu_R(a, b) = \sum_{\pi \in P_n^{(a>b)}} \mathcal{P}(\pi) \text{ and } \nu_R(a, b) = \sum_{\pi \in P_n^{(b>a)}} \mathcal{P}(\pi). \quad (4)$$

Intuitively speaking, for each $a, b \in X$, $\mu_R(a, b)$ is the probability that "a is better than b" and $\nu_R(a, b)$ is the probability that "b is better than a" in a randomly generated ranking distributed according to $\mathcal{P}$.

Note that the restrictions of μ_R and ν_R on S_n become the reciprocal binary relations, that is, $(\forall a, b \in X)(\mu_R(a, b) + \mu_R(b, a) = \nu_R(a, b) + \nu_R(b, a) = 1.)$

Throughout this paper, we simply denote by R the sum-intuitionistic fuzzy binary relation (sum-IFR for short) on X due to the fact that $R = (\mu_R, \nu_R)$ is the IFR generated by a probability distribution on complete pre-orders by using the sum operator. In the case of the P_n-Mallows (respectively, P_n-P.L.) distribution, we simply say the sum-intuitionistic Mallows relation (respectively, the sum-intuitionistic Plackett–Luce relation) on X (we denote them sum-IFMR and sum-IFPR for short).

Let us illustrate the expressions of the sum-IFR for a universe with three elements in the following example.

Example 4. *Let $X = \{a, b, c\}$. The sum-IFR R associated with $\mathcal{P}$ on P_3 is given by:*

$$\begin{cases} \mu_R(x, x) = \nu_R(x, x) = 0, \forall x \in X, \\ \mu_R(a, b) = \nu_R(b, a) = p_1 + p_2 + p_3 + p_4 + p_5, \\ \mu_R(a, c) = \nu_R(c, a) = p_3 + p_4 + p_5 + p_8 + p_{13}, \\ \mu_R(b, a) = \nu_R(a, b) = p_9 + p_{10} + p_{11} + p_{12} + p_{13}, \\ \mu_R(b, c) = \nu_R(c, b) = p_5 + p_8 + p_{11} + p_{12} + p_{13}, \\ \mu_R(c, a) = \nu_R(a, c) = p_1 + p_6 + p_9 + p_{10} + p_{11}, \\ \mu_R(c, b) = \nu_R(b, c) = p_1 + p_2 + p_3 + p_6 + p_9. \end{cases} \quad (5)$$

By replacing the general probability distribution $\mathcal{P}$ for the P_n-Mallows and P_n-P.L. distributions, we obtain the sum-IFMR and the sum-IFPR as illutrated in the following example.

Example 5. *For $n = 3, X = \{a, b, c\}$ and the Mallows distribution with respect to the order $\pi_0 = abc$, the associated sum-IFMR is defined by:*

$$\begin{cases} \mu_R(x,x) = \nu_R(x,x) = 0, \forall x \in X, \\ \mu_R(a,b) = \nu_R(b,a) = \frac{e^{-m(1+\alpha)}+e^{-2m\alpha}+e^{-m\alpha}+e^{-\theta}+1}{\Phi(m)}, \\ \mu_R(a,c) = \nu_R(c,a) = \frac{2e^{-m}+2e^{-m\alpha}+1}{\Phi(m)}, \\ \mu_R(b,a) = \nu_R(a,b) = \frac{e^{-m(1+\alpha)}+e^{-2m}+e^{-m(2+\alpha)}+e^{-3m}+e^{-m}}{\Phi(m)}, \\ \mu_R(b,c) = \nu_R(c,b) = \frac{e^{-m(1+\alpha)}+e^{-2m}+e^{-m\alpha}+e^{-m}+1}{\Phi(m)}, \\ \mu_R(c,a) = \nu_R(a,c) = \frac{2e^{-m(2+\alpha)}+2e^{-2m}+e^{-3m}}{\Phi(m)}, \\ \mu_R(c,b) = \nu_R(b,c) = \frac{e^{-m(1+\alpha)}+e^{-2m}+e^{-m(2+\alpha)}+e^{-m}+e^{-3m}}{\Phi(m)}. \end{cases} \tag{6}$$

Example 6. *The sum-IFPR R for $X = \{a, b, c\}$ is defined by:*

$$\begin{cases} \mu_R(x,x) = \nu_R(x,x) = 0, \forall x \in X, \\ \mu_R(a,b) = \nu_R(b,a) = \frac{\beta_1 v_a}{v_a+v_b} + \frac{\beta_2 v_{(ac)}}{v_b+v_{(ac)}} + \frac{\beta_2 v_a}{v_a+v_{(bc)}}, \\ \mu_R(a,c) = \nu_R(c,a) = \frac{\beta_1 v_a}{v_a+v_c} + \frac{\beta_2 v_{(ab)}}{v_c+v_{(ab)}} + \frac{\beta_2 v_a}{v_a+v_{(bc)}}, \\ \mu_R(b,a) = \nu_R(a,b) = \frac{\beta_1 v_b}{v_b+v_a} + \frac{\beta_2 v_{(bc)}}{v_a+v_{(bc)}} + \frac{\beta_2 v_b}{v_b+v_{(ac)}}, \\ \mu_R(b,c) = \nu_R(c,b) = \frac{\beta_1 v_b}{v_b+v_c} + \frac{\beta_2 v_{(cb)}}{v_a+v_{(bc)}} + \frac{\beta_2 v_b}{v_b+v_{(ac)}}, \\ \mu_R(c,a) = \nu_R(a,c) = \frac{\beta_1 v_c}{v_c+v_a} + \frac{\beta_2 v_{(cb)}}{v_a+v_{(bc)}} + \frac{\beta_2 v_c}{v_c+v_{(ba)}}, \\ \mu_R(c,b) = \nu_R(b,c) = \frac{\beta_1 v_c}{v_c+v_b} + \frac{\beta_2 v_{(ac)}}{v_b+v_{(ac)}} + \frac{\beta_2 v_c}{v_c+v_{(ba)}}. \end{cases} \tag{7}$$

In what follows, we study the $(\min, \max)$-transitivity of the sum-IFR, and we deduce conditions on parameters under which the sum-IFPR and the sum-IFMR are transitive.

3.2 Transitivity of (μ_R, ν_R) associated with $\mathcal{P}$ on P_n

We start this section by recalling the definition of the (min,max)-transitivity of an IFR.

Definition 1. *Let $R = (\mu_R, \nu_R)$ be an IFR on X. R is $(\min, \max)$-transitive on X if for all $(x, y, z) \in X^3$,*

$$\begin{cases} \mu_R(x,z) \geq \min(\mu_R(x,y), \mu_R(y,z)), \\ \nu_R(x,z) \leq \max(\nu_R(x,y), \nu_R(y,z)). \end{cases} \tag{8}$$

The following first main result of our paper establishes necessary and sufficient conditions on the probability distribution under which the IFR is transitive.

Theorem 1. *Let R be the sum-IFR on X associated with a probability distribution $\mathcal{P}$ on P_n. R is $(\min, \max)$-transitive on X if and only if for all $(x, y, z) \in X^3$,*

$$\begin{cases} \mathcal{P}(x \succ z \succeq y) \geq \mathcal{P}(y \succ z \succeq x) \quad or \quad \mathcal{P}(y \succeq x \succ z) \geq \mathcal{P}(z \succeq x \succ y), \\ \mathcal{P}(x \succeq z \succ y) \geq \mathcal{P}(y \succeq z \succ x) \quad or \quad \mathcal{P}(y \succ x \succeq z) \geq \mathcal{P}(z \succ x \succeq y). \end{cases} \tag{9}$$

Proof. Let R be the sum-IFR on X. We know that R is $(\min, \max)$-transitive on X if and only if it is $(\min, \max)$-transitive on all the subsets $\{a, b, c\}$ of X. That is

$$(\forall (x,y,z) \in \{(a,b,c),(a,c,b),(b,a,c),(b,c,a),(c,b,a),(c,a,b)\})$$
$$\begin{cases} \mu_R(x,z) \geq \min(\mu_R(x,y),\mu_R(y,z)), \\ \nu_R(x,z) \leq \max(\nu_R(x,y),\nu_R(y,z)). \end{cases}$$

This means that the following six conditions are all satisfied:

$$\begin{cases} \mu_R(a,b) \geq \min(\mu_R(a,c),\mu_R(c,b)), \\ \nu_R(a,b) \leq \max(\nu_R(a,c),\nu_R(c,b)) \end{cases} \quad \textit{for } (x,y,z) = (a,c,b), \tag{10}$$

$$\begin{cases} \mu_R(b,a) \geq \min(\mu_R(b,c),\mu_R(c,a)), \\ \nu_R(b,a) \leq \max(\nu_R(b,c),\nu_R(c,a)) \end{cases} \quad \textit{for } (x,y,z) = (b,c,a), \tag{11}$$

$$\begin{cases} \mu_R(a,c) \geq \min(\mu_R(a,b),\mu_R(b,c)), \\ \nu_R(a,c) \leq \max(\nu_R(a,b),\nu_R(b,c)) \end{cases} \quad \textit{for } (x,y,z) = (a,b,c), \tag{12}$$

$$\begin{cases} \mu_R(c,a) \geq \min(\mu_R(c,b),\mu_R(b,a)), \\ \nu_R(c,a) \leq \max(\nu_R(c,b),\nu_R(b,a)) \end{cases} \quad \textit{for } (x,y,z) = (c,b,a), \tag{13}$$

$$\begin{cases} \mu_R(b,c) \geq \min(\mu_R(b,a),\mu_R(a,c)), \\ \nu_R(b,c) \leq \max(\nu_R(b,a),\nu_R(a,c)) \end{cases} \quad \textit{for } (x,y,z) = (b,a,c) \tag{14}$$

and

$$\begin{cases} \mu_R(c,b) \geq \min(\mu_R(c,a),\mu_R(a,b)), \\ \nu_R(c,b) \leq \max(\nu_R(c,a),\nu_R(a,b)) \end{cases} \quad \textit{for } (x,y,z) = (c,a,b). \tag{15}$$

The rest of the proof consists of showing that each of the six previous conditions is equivalent to Eq. (9). For that purpose, let us recall that

$$(\forall \alpha,\beta,\gamma \in \mathbb{R})\Big((\alpha \geq \min(\beta,\gamma) \Leftrightarrow \alpha \geq \beta \text{ or } \alpha \geq \gamma) \text{ and } (\alpha \leq \max(\beta,\gamma) \Leftrightarrow \alpha \leq \beta \text{ or } \alpha \leq \gamma)\Big).$$

Thus,

<u>Condition (10):</u> $\begin{cases} \mu_R(a,b) \geq \min(\mu_R(a,c),\mu_R(c,b)), \\ \nu_R(a,b) \leq \max(\nu_R(a,c),\nu_R(c,b)) \end{cases}$ *for* $(x,y,z) = (a,c,b)$ is equivalent to

$$\begin{cases} \mu_R(a,b) \geq \mu_R(a,c) \quad \text{or} \quad \mu_R(a,b) \geq \mu_R(c,b), \\ \nu_R(a,b) \leq \nu_R(a,c) \quad \text{or} \quad \nu_R(a,b) \leq \nu_R(c,b)), \end{cases}$$

that becomes by using equalities of (5) within the probabilities,

$$\begin{cases} \begin{cases} p_1+p_2+p_3+p_4+p_5 & \geq p_3+p_4+p_5+p_8+p_{13}, \\ & \text{or} \\ p_1+p_2+p_3+p_4+p_5 & \geq p_1+p_2+p_3+p_6+p_9, \end{cases} \\ \begin{cases} p_9+p_{10}+p_{11}+p_{12}+p_{13} & \leq p_1+p_6+p_9+p_{10}+p_{11}, \\ & \text{or} \\ p_9+p_{10}+p_{11}+p_{12}+p_{13} & \leq p_5+p_8+p_{11}+p_{12}+p_{13}, \end{cases} \end{cases}$$

and by simplifying probabilities, the condition (10) finally becomes:

$$\begin{cases} p_1 + p_2 \geq p_8 + p_{13} & \text{or} \quad p_4 + p_5 \geq p_6 + p_9, \\ p_{12} + p_{13} \leq p_1 + p_6 & \text{or} \quad p_9 + p_{10} \leq p_5 + p_8, \end{cases}$$

that is

$$\begin{cases} p_1 + p_2 \geq p_8 + p_{13} & \text{or} \quad p_4 + p_5 \geq p_6 + p_9, \\ p_1 + p_6 \geq p_{12} + p_{13} & \text{or} \quad p_5 + p_8 \geq p_9 + p_{10}, \end{cases}$$

that can still be written (by using Table 1),

$$\begin{cases} \mathcal{P}(cab) + \mathcal{P}((ca)b) \geq \mathcal{P}(bac) + \mathcal{P}((ba)c) & \text{or} \quad \mathcal{P}(abc) + \mathcal{P}(a(bc)) \geq \mathcal{P}(cba) + \mathcal{P}(c(ba)), \\ \mathcal{P}(cab) + \mathcal{P}(c(ab)) \geq \mathcal{P}(bac) + \mathcal{P}(b(ac)) & \text{or} \quad \mathcal{P}(abc) + \mathcal{P}((ab)c) \geq \mathcal{P}(cba) + \mathcal{P}((bc)a), \end{cases}$$

And with $(x, y, z) = (a, c, b)$, that is equivalent to:

$$\begin{cases} \mathcal{P}(x \succ z \succeq y) \geq \mathcal{P}(y \succ z \succeq x) & \text{or} \quad \mathcal{P}(y \succeq x \succ z) \geq \mathcal{P}(z \succeq x \succ y), \\ \mathcal{P}(x \succeq z \succ y) \geq \mathcal{P}(y \succeq z \succ x) & \text{or} \quad \mathcal{P}(y \succ x \succeq z) \geq \mathcal{P}(z \succ x \succeq y). \end{cases}$$

Hence the equivalence between Eq. (10) and Eq. (9).

Condition (11):

$$\begin{cases} \mu_R(b, a) \geq \min(\mu_R(b, c), \mu_R(c, a)), \\ \nu_R(b, a) \leq \max(\nu_R(b, c), \nu_R(c, a)) \end{cases} \quad \text{for} \quad (x, y, z) = (b, c, a),$$

is equivalent to

$$\begin{cases} \mu_R(b, a) \geq \mu_R(b, c) & \text{or} \quad \mu_R(b, a) \geq \mu_R(c, a), \\ \nu_R(b, a) \leq \nu_R(b, c) & \text{or} \quad \nu_R(b, a) \leq \nu_R(c, a)), \end{cases}$$

by using equalities of Eq. (5) and simplifying probabilities, we obtain

$$\begin{cases} p_9 + p_{10} \geq p_5 + p_8 & \text{or} \quad p_{12} + p_{13} \geq p_1 + p_6, \\ p_4 + p_5 \leq p_6 + p_9 & \text{or} \quad p_1 + p_2 \leq p_8 + p_{13}, \end{cases}$$

that is

$$\begin{cases} p_9 + p_{10} \geq p_5 + p_8 & \text{or} \quad p_{12} + p_{13} \geq p_1 + p_6, \\ p_6 + p_9 \geq p_4 + p_5 & \text{or} \quad p_8 + p_{13} \geq p_1 + p_2, \end{cases}$$

that can still be written (by using table1),

$$\begin{cases} \mathcal{P}(cba) + \mathcal{P}((cb)a) \geq \mathcal{P}(abc) + \mathcal{P}((ab)c) & \text{or} \quad \mathcal{P}(bac) + \mathcal{P}(b(ac)) \geq \mathcal{P}(cab) + \mathcal{P}(c(ab)), \\ \mathcal{P}(cba) + \mathcal{P}(c(ba)) \geq \mathcal{P}(abc) + \mathcal{P}(a(bc)) & \text{or} \quad \mathcal{P}(bac) + \mathcal{P}((ab)c) \geq \mathcal{P}(cab) + \mathcal{P}((ca)b), \end{cases}$$

that is equivalent to

$$\begin{cases} \mathcal{P}(bac) + \mathcal{P}(b(ac)) \geq \mathcal{P}(cab) + \mathcal{P}(c(ab)) & \text{or} \quad \mathcal{P}(cba) + \mathcal{P}((cb)a) \geq \mathcal{P}(abc) + \mathcal{P}((ab)c), \\ \mathcal{P}(bac) + \mathcal{P}((ab)c) \geq \mathcal{P}(cab) + \mathcal{P}((ca)b) & \text{or} \quad \mathcal{P}(cba) + \mathcal{P}(c(ba)) \geq \mathcal{P}(abc) + \mathcal{P}(a(bc)), \end{cases}$$

that can still exactly be written in this case (with $a = z, b = x$ and $c = y$)

$$\begin{cases} \mathcal{P}(x \succ z \succeq y) \geq \mathcal{P}(y \succ z \succeq x) & \text{or} \quad \mathcal{P}(y \succeq x \succ z) \geq \mathcal{P}(z \succeq x \succ y), \\ \mathcal{P}(x \succeq z \succ y) \geq \mathcal{P}(y \succeq z \succ x) & \text{or} \quad \mathcal{P}(y \succ x \succeq z) \geq \mathcal{P}(z \succ x \succeq y). \end{cases}$$

Hence the equivalence between Eq. (11) and Eq. (9).

We establish in the same way that each of the remain four other conditions (12), (13), (14) and (15) by respectively setting $(x, y, z) \in \{(b, a, c), (b, c, a), (c, b, a), (c, a, b)\}$ is equivalent to (9). Hence the result. □

Three particular cases of the previous result can be obviously deduced.

Corollary 1. *1. If $\mathcal{P}$ is the uniform distribution on P_n, then R is* $(\min, \max)$*-transitive on X.*

2. If $\mathcal{P}$ is probability distribution on S_n, then R is $(\min, \max)$*-transitive on X if and only if for all $(x, y, z) \in X^3$,*

$$\mathcal{P}(x \succ z \succ y) \geq \mathcal{P}(y \succ z \succ x) \quad \textit{or} \quad \mathcal{P}(y \succ x \succ z) \geq \mathcal{P}(z \succ x \succ y). \tag{16}$$

3. If $\mathcal{P}$ is a probability distribution on S_n, such that each order and its reverse have the same probability, then the sum-IFR R associated with $\mathcal{P}$ is $(\min, \max)$*-transitive.*

Proof. 1) If $\mathcal{P}$ is the uniform distribution on P_n, then the inequalities of Equation (9) become equalities and we have the result.

2) Let R be the sum-IFR associated with $\mathcal{P}$ on P_n. If $\mathcal{P}$ is a probability distribution on S_n, then ties do not occurred and the previous condition of Eq. (9): $\forall (x, y, z) \in X^3$, $(\mathcal{P}(xzy) + \mathcal{P}(x(yz)) \geq \mathcal{P}(yzx) + \mathcal{P}(y(zx))$ or $\mathcal{P}(yxz) + \mathcal{P}((yx)z) \geq \mathcal{P}(zxy) + \mathcal{P}((zx)y))$ and $(\mathcal{P}(xzy) + \mathcal{P}((xz)y) \geq \mathcal{P}(yzx) + \mathcal{P}((yz)x)$ or $\mathcal{P}(yxz) + \mathcal{P}(y(xz)) \geq \mathcal{P}(zxy) + \mathcal{P}(z(xy)))$ of the (min,max)-transitivity of R becomes,

$$\forall (x, y, z) \in X^3, \begin{cases} \mathcal{P}(xzy) \geq \mathcal{P}(yzx) & \text{or} \quad \mathcal{P}(yxz) \geq \mathcal{P}(zxy), \\ \mathcal{P}(xzy) \geq \mathcal{P}(yzx) & \text{or} \quad \mathcal{P}(yxz) \geq \mathcal{P}(zxy), \end{cases}$$

that is,

$$\forall (x, y, z) \in X^3, \mathcal{P}(x \succ z \succ y) \geq \mathcal{P}(y \succ z \succ x) \quad \text{or} \quad \mathcal{P}(y \succ x \succ z) \geq \mathcal{P}(z \succ x \succ y).$$

3) Let R be an sum-IFR associated with $\mathcal{P}$ on S_n such that each order and its reverse have the same probability. That is for all $(x, y, z) \in X^3, \mathcal{P}(x \succ y \succ z) = \mathcal{P}(z \succ y \succ x)$. Then we deduce that the condition for all $(x, y, z) \in X^3, \mathcal{P}(x \succ z \succ y) \geq \mathcal{P}(y \succ z \succ x)$ *or* $\mathcal{P}(y \succ x \succ z) \geq \mathcal{P}(z \succ x \succ y)$ is satisfied. □

In the remain of the paper, we study the (min,max)-transitivity of the sum-IFMR $R = (\mu_R, \nu_R)$ by establishing the second main result of our paper.

Proposition 1. *Let X be a finite set of n alternatives $x_1; x_2; \ldots; x_n$, $m \geq 0, \alpha \in [\frac{1}{2}; 1]$, $\pi_0 = x_1 x_2 \cdots x_n$ and R be the sum-IFMR associated with $\mathcal{P}_{\pi_0, m, d_\alpha}$. R is* $(\min, \max)$*-transitive on X if and only if $m = 0$.*

Proof. We know that R is $(\min, \max)$-transitive on X if and only if it is $(\min, \max)$-transitive on all the subsets $\{a, b, c\}$ of X, that is, Eq. (9) of Theorem 1 is satisfied on $\{a, b, c\}$.
Thus, by setting $(x, y, z) = (a, c, b)$, in Eq. (8), the Eq. (9) becomes

$$\begin{cases} \mathcal{P}(cab) + \mathcal{P}((ca)b) \geq \mathcal{P}(bac) + \mathcal{P}((ba)c) & \text{or} \quad \mathcal{P}(abc) + \mathcal{P}(a(bc)) \geq \mathcal{P}(cba) + \mathcal{P}(c(ba)), \\ \mathcal{P}(cab) + \mathcal{P}(c(ab)) \geq \mathcal{P}(bac) + \mathcal{P}(b(ac)) & \text{or} \quad \mathcal{P}(abc) + \mathcal{P}((ab)c) \geq \mathcal{P}(cba) + \mathcal{P}((bc)a), \end{cases} \tag{17}$$

Using the proof of Theorem. 1, the system of Eq. (17) corresponds to

$$\begin{cases} p_1 + p_2 \geq p_8 + p_{13} & \text{or} \quad p_4 + p_5 \geq p_6 + p_9, \\ p_1 + p_6 \geq p_{12} + p_{13} & \text{or} \quad p_5 + p_8 \geq p_9 + p_{10}, \end{cases}$$

that generates the following four systems

$$\begin{cases} p_1 + p_2 \geq p_8 + p_{13}, \\ p_1 + p_6 \geq p_{12} + p_{13}, \end{cases} \quad \text{or} \quad \begin{cases} p_1 + p_2 \geq p_8 + p_{13}, \\ p_5 + p_8 \geq p_9 + p_{10}, \end{cases}$$

or

$$\begin{cases} p_4 + p_5 \geq p_6 + p_9, \\ p_1 + p_6 \geq p_{12} + p_{13}, \end{cases} \quad \text{or} \quad \begin{cases} p_4 + p_5 \geq p_6 + p_9, \\ p_5 + p_8 \geq p_9 + p_{10}. \end{cases}.$$

Since $\mathcal{P} = \mathcal{P}_{\pi_0, m, d_\alpha}, \forall \alpha \in [\frac{1}{2}, 1]$,with $\pi_o = abc$, by using its expression defined by Eq. (2), the four previous systems are respectively become,

$$\begin{cases} e^{-m} + e^{-m\alpha} \leq e^{-2m} + e^{-(1+\alpha)m}, \\ e^{-(1+\alpha)m} + e^{-m} \leq e^{-(2+\alpha)m} + e^{-2m}, \end{cases} \tag{18}$$

$$\begin{cases} e^{-\alpha m} + e^{-m} \leq e^{-2m} + e^{-(1+\alpha)m}, \\ e^{-3m} + e^{-(2+\alpha)m} \leq 1 + e^{-m\alpha}, \end{cases} \tag{19}$$

$$\begin{cases} e^{-(2+\alpha)m} + e^{-3m} \leq e^{-m\alpha} + 1, \\ e^{-(1+\alpha)m} + e^{-m} \leq e^{-(2+\alpha)m} + e^{-2m}, \end{cases} \tag{20}$$

$$\begin{cases} e^{-(2+\alpha)m} + e^{-3m} \leq e^{-m\alpha} + 1, \\ e^{-3m} + e^{-(2+\alpha)m} \leq 1 + e^{-m\alpha}. \end{cases} \tag{21}$$

Thus, R is $(\min, \max)$-transitive on $\{a, b, c\} \subseteq X$ if and only if these four last systems are satisfied.

Let us show that if the four systems hold for any $\alpha \in [\frac{1}{2}, 1]$, then $m = 0$. Let us remark that the second inequality of Eq. (19), the first inequality of Eq. (20) and the system of Eq. (21) hold for any $\alpha \in [\frac{1}{2}, 1]$, and for any $m > 0$.

Let us assume that the Eq. (18) holds and $m > 0$. Then, for any $\alpha \in [\frac{1}{2}, 1]$ and $m > 0$, ones has

$$\begin{cases} e^{-\alpha m} + e^{-m} \leq e^{-2m} + e^{-(1+\alpha)m} \\ e^{-(1+\alpha)m} + e^{-m} \leq e^{-(2+\alpha)m} + e^{-2m}. \end{cases}$$

However, for any $\alpha \in [\frac{1}{2}, 1]$, and $m > 0$, ones has

$$\begin{cases} -\alpha m > -2m \quad \textit{and} \quad -m > -(1+\alpha)m, \\ -m > -2m \quad \textit{and} \quad -m(1+\alpha) > -(2+\alpha)m, \end{cases}$$

that implies that for any $\alpha \in [\frac{1}{2}, 1]$, and $m > 0$,

$$\begin{cases} e^{-\alpha m} + e^{-m} > e^{-2m} + e^{-(1+\alpha)m}, \\ e^{-(1+\alpha)m} + e^{-m} > e^{-(2+\alpha)m} + e^{-2m}, \end{cases}$$

(because the increasing of the exponential). That contradicts Eq. (18). Hence, as $m \geq 0$ if Eq. (18) holds, then $m = 0$.

We prove in the same way that if the first inequality of Eq. (19) and the second of Eq. (20) hold $\forall \alpha \in [\frac{1}{2}, 1]$, then $m = 0$.

Reciprocally, if $m = 0$, then $\mathcal{P}_{\pi_0, m, d_\alpha}$ becomes the uniform distribution on P_n. And the first result of Corollary 1 implies that R is $(\min, \max)$-transitive on X. □

We finally study the (min,max)-transitivity of the sum-IFPR R by assuming that the weights $\vec{v} = (v_i^{\pi(l)})_{i \in \{1,\ldots,l\}}$ of positions in pre-orders satisfy the following assumption:

$$(H_1): \ (\forall x, y \in X)(\forall E \subset X)\Big(v_{(\{x\}\cup E)} \geq v_x \ \text{ and } \ v_x \geq v_y \Leftrightarrow v_{(\{x\}\cup E)} \geq v_{(\{y\}\cup E)}\Big) \qquad (22)$$

where $\alpha_{(\{x\}\cup E)}$ is the weight of the block $(\{x\} \cup E)$. This assumption stipulates that adding elements to a given block of a pre-order increases the weight of this block, and that if the weight of an alternative x is greater than the one of y, then while adding to each one the same block of elements the weight of the block containing x remains greater.

In addition, we need the following condition on $\vec{v}$ which stipulates that all the weights on positions are either equals (if $A = X$) or either reduced to two (if $A \neq X$):

$$(H_2): \ (\exists A \neq \varnothing, A \subseteq X)(\forall x, y \in A)\Big(v_x = v_y \ \text{ and } \ \forall x, y \in X \setminus A, v_x = v_y.\Big) \qquad (23)$$

The following third main result of our paper establishes conditions on the weight vector $\vec{v}$ under which the sum-IFPR R on X is $(\min, \max)$-transitive.

Proposition 2. *Let X be the finite set of n alternatives $x_1, x_2, \ldots, x_n$ $(n > 3)$ with the weight vector $\vec{v}$ and R be the sum-IFPR. If $\vec{v}$ satisfies H_1, then R is $(\min, \max)$-transitive on X if and only if $\vec{v}$ satisfies H_2.*

Proof. Let us assume that the weights satisfy H_1.

$(\Rightarrow)$ Let us assume that R is $(\min, \max)$-transitive on X and let us show that $\vec{v}$ satisfies H_2. The transitivity of R implies that $\forall x, y, z \in \{a, b, c\}$, Eq. (9) of Theorem 1 is satisfied on $\{a, b, c\}$. Thus, we have the system defined by Eq. (17). Let us distinguish two steps.

i) Firstly, we show that at least two of the three weights v_a, v_b and v_c are equal.
Using $\mathcal{P} = \mathcal{P}_{\vec{v}}$ given by (3) on P_3, the system defined by Eq. (17) gives:

$$\frac{\beta_1 v_a v_b}{(v_a+v_b+v_c)(v_b+v_c)} + \frac{\beta_2 v_a}{v_a + v_{(bc)}} \geq \frac{\beta_1 v_c v_b}{(v_a+v_b+v_c)(v_b+v_a)} + \frac{\beta_2 v_c}{v_c+v_{(ba)}} \quad \textit{or}$$
$$\frac{\beta_1 v_c v_a}{(v_a+v_b+v_c)(v_a+v_b)} + \frac{\beta_2 v_{(ca)}}{v_b + v_{(ca)}} \geq \frac{\beta_1 v_b v_a}{(v_a+v_b+v_c)(v_a+v_c)} + \frac{\beta_2 v_{(ba)}}{v_c+v_{(ba)}}$$
$$\textit{and} \frac{\beta_1 v_a v_b}{(v_a+v_b+v_c)(v_b+v_c)} + \frac{\beta_2 v_{(ab)}}{v_c + v_{(ab)}} \geq \frac{\beta_1 v_c v_b}{(v_a+v_b+v_c)(v_b+v_a)} + \frac{\beta_2 v_{(cb)}}{v_a+v_{(bc)}}, \quad \textit{or}$$
$$\frac{\beta_1 v_c v_a}{(v_a+v_b+v_c)(v_a+v_b)} + \frac{\beta_2 v_c}{v_c + v_{(ab)}} \geq \frac{\beta_1 v_b v_a}{(v_a+v_b+v_c)(v_a+v_c} + \frac{\beta_2 v_b}{v_b+v_{(ac)}}.$$

After calculation, the two first and the two second inequalities of each disjunction are equal and become the following disjunction:

$$\frac{\beta_1 v_b(v_a - v_c)}{(v_a+v_b)(v_b+v_c)} + \frac{\beta_2(v_a v_{(ba)} - v_c v_{(bc)})}{(v_a+v_{(bc)})(v_c+v_{(ba)})} \geq 0 \quad \text{or}$$
$$\frac{\beta_1(v_c - v_b)}{(v_c+v_a)(v_a+v_b)} + \frac{\beta_2(v_c v_{(ca)} - v_b v_{(ba)})}{(v_b+v_{(ca)})(v_c+v_{(ba)})} \geq 0.$$

Since the weights satisfy assumption H_1, this disjunction of inequalities becomes $v_a \geq v_c$ *or* $v_c \geq v_b$. By permuting the roles of x, y and z to a, b and c, the last two inequalities generate the following system:

$$\begin{cases} v_a \geq v_c \ \text{ or } \ v_c \geq v_b, \\ v_b \geq v_c \ \text{ or } \ v_c \geq v_a, \\ v_c \geq v_a \ \text{ or } \ v_a \geq v_b, \\ v_a \geq v_b \ \text{ or } \ v_b \geq v_c, \\ v_b \geq v_a \ \text{ or } \ v_a \geq v_c, \\ v_c \geq v_b \ \text{ or } \ v_b \geq v_a. \end{cases} \tag{24}$$

Each combination of conjunctions of (24) provides an equality of at least two weights amount the three v_a, v_b and v_c.

ii) Let us show that the weight v satisfies H_2. Since at least two of each given three weights, on a finite numbers, are equal, then all the weights are reduced to two or they are equal. Thus, the weights satisfy H_2.

($\Leftarrow$) Assume that the weights satisfy H_2 on X and let us show that R is $(\min, \max)$-transitive on X. It suffices to show that for any $a, b, c \subset X$, R is $(\min, \max)$-transitive on $\{a, b, c\}$. Let $\{a, b, c\} \subset X$. Condition H_2 on X stipulates that there exists a non-empty subset A of X such that $\forall x, y \in A, v_x = v_y$ and $\forall x, y \in X \setminus A, v_x = v_y$. Then obviously at least two of weights v_a, v_b and v_c are equal. We distinguish two main cases:

(i) If the three weights v_a, v_b and v_c are equal, then the weight vector $\vec{v}$ satisfies H_1 on $\{a, b, c\}$. So the sum-IFPR R is constant on $\{a, b, c\}$, notably for all $(x, y) \in \{a, b, c\}^2$, $\mu_R(x, y) = \nu_R(y, x) = \frac{\beta_1}{2} + \beta_2$. Then R is $(\min, \max)$-transitive on $\{a, b, c\}$.

(ii) If two of the three weights are equal: for example $v_a = v_b \neq v_c$. Then, H_1 gives $v_{(ac)} = v_{(bc)}$ and the probability distribution P_v becomes

$$\begin{cases} p_1 = \mathcal{P}_v(cab) = p_9 = \mathcal{P}_v(cba), \\ p_2 = \mathcal{P}_v((ca)b) = p_{10} = \mathcal{P}_v((cb)a), \\ p_3 = \mathcal{P}_v(acb) = p_{11} = \mathcal{P}_v(bca), \\ p_4 = \mathcal{P}_v(a(bc)) = p_{12} = \mathcal{P}_v((ac)b). \end{cases} \tag{25}$$

With (25), the sum-IFPR R is defined by

$$\begin{cases} \mu_R(a,b) = \mu_R(b,a) = \frac{\beta_1}{2} + \beta_2, \\ \mu_R(a,c) = \mu_R(b,c), \\ \mu_R(c,a) = \mu_R(c,b). \end{cases} \tag{26}$$

Thus, the transitivity of R comes from the fact that the system of Eq. (24) is satisfied. □

4 Conclusion

This paper introduces a pairwise comparison method, through an intuitionitic fuzzy binary relation (IFR), of a finite number of elements derived from a probability distribution $\mathcal{P}$ giving the occurrence of each complete pre-order of these elements. Given a pair of element (a, b) of the universe, the method displays the probability that an alternative a is preferred to b (described by the membership function of IFR) and the probability that a is not preferred to b (described by the non-membership function of IFR) in the randomly generated ranking with possible ties distributed according to $\mathcal{P}$.

The study of the transitivity of the IFR provides three results.

The first one establishes that the IFR satisfies the well-known $(\min, \max)$-transitivity if and only if one can assign probabilities to complete pre-orders such that there is the same comparison between probabilities of each pre-order on a triplet and its reverse.

The second one stipulates that the only generalized parametric Mallows probability distribution on complete pre-orders which generates the $(\min, \max)$-transitivity of an IFR is the uniform distribution, that is, the one with the spread parameter equal zero.

The third result stipulates that the IFR derived from the generalized parametric Plackett–Luce probability distribution on complete pre-orders is $(\min, \max)$-transitive if and only if all the weights of elements are reduced to one or two.

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