Notes on Q-probabilities on intuitionistic fuzzy events

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Abstract— *Following [9] some properties of Q-probability and Q-states are studied. Representation theorem of Q-probabilities and Q-states, the existence of the joint observable and The central limit theorem are proved.*

Keywords— Q-probability, Q-state, representation theorem, intuitionistic fuzzy events, joint Q-observable, Central limit theorem.

1 Introduction

Although there are different opinions about intuitionistic fuzzy events, the following definitions are accepted generally ([1], [5]). Let $(\Omega, \mathcal{S})$ be a measurable space. By an intuitionistic fuzzy event ([5]) we mean any pair

$$\mathbf{A} = (\mu_A, \nu_A)$$

of $\mathcal{S}-$measurable functions, such that $\mu_A, \nu_A : \Omega \to [0,1]$ and $\mu_A + \nu_A \leq 1$.

The function μ_A is the membership function and the function ν_A is the non-membership function. The family $\mathcal{F}$ of all intuitionistic fuzzy events is ordered in the following way:

$$\mathbf{A} \leq \mathbf{B} \Leftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Evidently

$$\mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B),$$

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B).$$

It is easy to see that $\mathbf{A_n} \nearrow \mathbf{A}$ if and only if $\mu_{A_n} \nearrow \mu_A$ and $\nu_{A_n} \searrow \nu_A$.

The notion of intuitionistic fuzzy event is a natural generalization of the notion of a fuzzy event. Given a fuzzy event μ_A, the pair $(\mu_A, 1-\mu_A)$ is an intuitionistic fuzzy event, so intuitionistic fuzzy events can be seen as generalizations of fuzzy events. Hence we want to define probability on intuitionistic fuzzy events generalizing probability on fuzzy events. And actually, two constructions were proposed independently by Gregorzewski [5] and Gerstenkorn [4], both based on the Łukasiewicz operations

$$a \oplus b = \min(a+b, 1),$$

$$a \odot b = \max(a+b-1, 0).$$

Operations $\oplus, \odot$ on $[0,1]^2$ (not necessarily Łukasiewicz operations) can be naturally extended to intuitionistic fuzzy events in the following way

$$\mathbf{A} \oplus \mathbf{B} = (\mu_A \oplus \mu_B, \nu_A \odot \nu_B),$$

$$\mathbf{A} \odot \mathbf{B} = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B),$$

where $\mathbf{A} = (\mu_A, \nu_A)$ and $\mathbf{B} = (\mu_B, \nu_B)$.

If $\mu : \Omega \to [0,1]$ is a fuzzy set, then $(\mu, 1-\mu)$ is an IF set corresponding to this fuzzy set. Similarly as in the classical case, in the fuzzy case and in the quantum case, a probability (or state) has been introduced as a mapping $m : \mathcal{F} \to [0,1]$ being continuous, additive and satisfying some boundary conditions. Here the main difference is the additivity which is now of the following form

$$m(\mathbf{A}) + m(\mathbf{B}) = m(\mathbf{A} \oplus \mathbf{B}) + m(\mathbf{A} \odot \mathbf{B}).$$

There exists a general representation theorem for IF-probability. If $(\Omega, \mathcal{S}, P)$ is a probability space, then to any Łukasiewicz state $m : \mathcal{F} \to [0,1]$ there exists $\alpha \in [0,1]$ such that

$$m(\mathbf{A}) = (1-\alpha) \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$$

for any $\mathbf{A} \in \mathcal{F}$ (see [2]). Of course, the constructions (see [4] [5]) can be obtained as a very special case.

Generally, there are infinitely many possibilities how to define additivity

$$m(\mathbf{A}) + m(\mathbf{B}) = m(\mathcal{S}(\mathbf{A}, \mathbf{B})) + m(\mathcal{T}(\mathbf{A}, \mathbf{B}))$$

where

$$\mathcal{S}(\mathbf{A}, \mathbf{B}) = (S(\mu_A, \mu_B), T(\nu_A, \nu_B)),$$

$$\mathcal{T}(\mathbf{A}, \mathbf{B}) = (T(\mu_A, \mu_B), S(\nu_A, \nu_B))$$

$$S, T : [0,1]^2 \to [0,1]$$

being such binary operations (T is a t-norm and S is dual t-conorm [6]), that

$$S(u,v) + T(1-u, 1-v) \leq 1.$$

The Kolmogorov probability theory has 3 fundamental notions: probability, random variable and expectation. In our fuzzy case, an analogous situation occurs.

Throughout this paper we consider the following operations with intuitionistic fuzzy events

$$A \oplus_Q B = (({\mu_A}^n + {\mu_B}^n)^{\frac{1}{n}} \wedge 1; 1 - ((1-\nu_A)^n + (1-\nu_B)^n)^{\frac{1}{n}} \wedge 1),$$

$$A \odot B = (\mu_A, \nu_A) \odot (\mu_B, \nu_B) = ((\mu_A + \mu_B - 1) \vee 0; (\nu_A + \nu_B) \wedge 1).$$

Remark 1.1 *The operation $\oplus_Q$ was introduced by Yager [6], the operation $\odot$ is Łukasiewicz operation. This is a special case of operations studied in [9], where $\varphi(u) = u^n$, $n \in N$ is fixed for each $u \in [0,1]$. Special case $n = 2$ is studied in [2].*

We are not able to embed the family $\mathcal{F}$ with these operations into an MV-algebra. Of course, we are able to prove probability representation theorems, to construct the joint observable and prove such fundamental theorems as central limit theorem or laws of large numbers.

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2 Q-probability and Q-observables

Definition 2.1 *Let $\mathcal{F}$ be the family of all intuitionistic fuzzy events, $\mathcal{J}$ be the family of all compact subintervals of the unit interval $[0,1]$. Q-probability is any mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ satisfying the following conditions:*

(i) $\mathcal{P}((\boldsymbol{1},\boldsymbol{0})) = [1,1], \mathcal{P}((\boldsymbol{0},\boldsymbol{1})) = [0,0]$;

(ii) $\mathbf{A} \odot \mathbf{B} = (0,1) \Rightarrow \mathcal{P}(\mathbf{A} \oplus_Q \mathbf{B}) = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B})$;

(iii) $\mathbf{A_n} \nearrow \mathbf{A} \Rightarrow \mathcal{P}(\mathbf{A_n}) \nearrow \mathcal{P}(\mathbf{A})$.
(Here $[a_n, b_n] \nearrow [a,b]$, if $a_n \nearrow a, b_n \nearrow b$.)

Remark 2.2 *If A is a crisp set, then $\mu_A = \xi_A, \nu_A = \xi_B$, Theorem 2.7 implies that*

$$\mathcal{P}^{sharp}(A) = (1-\alpha)p(A) + \alpha r(A)$$

$$\mathcal{P}^{sharp}(A') = (1-\alpha)p(A') + \alpha r(A')$$

hence $\mathcal{P}^{sharp}(A) + \mathcal{P}^{sharp}(A') = (1-\alpha)p(\Omega) + \alpha r(\Omega) = 1$. It follows that there is a large class of examples extending the classical definition. It makes possible to construct different models describing some real processes.

Definition 2.3 *A mapping $m : \mathcal{F} \to [0,1]$ is called a Q-state, if the following conditions are satisfied:*

*(i) m((**1**,**0**))=1, m((**0**,**1**))=0;*

(ii) $\mathbf{A} \odot \mathbf{B} = (0,1) \Rightarrow m(\mathbf{A} \oplus_Q \mathbf{B}) = m(\mathbf{A}) + m(\mathbf{B})$;

(iii) $\mathbf{A_n} \nearrow \mathbf{A} \Rightarrow m(\mathbf{A_n}) \nearrow m(\mathbf{A})$.

Example 2.4 *Let $(\Omega, \mathcal{S}, p)$ be a probability space, then a natural example of Q-state is a function $m : \mathcal{F} \to [0,1]$ defined by the following*

$$m((\mu_A, \nu_A)) = \int_\Omega {\mu_A}^n dp,$$

where $n \in N$ is fixed natural number.

Let us suppose, that $\mathcal{P}$ maps $\mathcal{F}$ to $\mathcal{J}$. We will present this mapping with functions $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to [0,1]$ in the following manner $\mathcal{P}(\mathbf{A}) = [\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})], \mathbf{A} \in \mathcal{F}$. Shorter notation is used further on is $\mathcal{P} = [\mathcal{P}^\flat, \mathcal{P}^\sharp]$.

Theorem 2.5 *$\mathcal{P} : \mathcal{F} \to \mathcal{J}$, is a Q-probability if and only if $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to [0,1]$ are Q-states.*

Proof Let us suppose that $\mathcal{P}$ is an Q- probability, then since

$$[1,1] = \mathcal{P}((\mathbf{1},\mathbf{0})) = [\mathcal{P}^\flat((\mathbf{1},\mathbf{0})), \mathcal{P}^\sharp((\mathbf{1},\mathbf{0}))],$$

we have $1 = \mathcal{P}^\flat((\mathbf{1},\mathbf{0}))$ and $1 = \mathcal{P}^\sharp((\mathbf{1},\mathbf{0}))$.
Further let $\mathbf{A} \odot \mathbf{B} = (\mathbf{0},\mathbf{1})$. Then

$$[\mathcal{P}^\flat(\mathbf{A}) + \mathcal{P}^\flat(\mathbf{B}), \mathcal{P}^\sharp(\mathbf{A}) + \mathcal{P}^\sharp(\mathbf{B})] =$$
$$[\mathcal{P}^\flat(\mathbf{A}), \mathcal{P}^\sharp(\mathbf{A})] + [\mathcal{P}^\flat(\mathbf{B}), \mathcal{P}^\sharp(\mathbf{B})] = \mathcal{P}(\mathbf{A}) + \mathcal{P}(\mathbf{B}) =$$
$$\mathcal{P}(\mathbf{A} \oplus_Q \mathbf{B}) = [\mathcal{P}^\flat(\mathbf{A} \oplus_Q \mathbf{B}), \mathcal{P}^\sharp(\mathbf{A} \oplus_Q \mathbf{B})],$$

hence

$$\mathcal{P}^\flat(\mathbf{A}) + \mathcal{P}^\flat(\mathbf{B}) = \mathcal{P}^\flat(\mathbf{A} \oplus_Q \mathbf{B})$$

and

$$\mathcal{P}^\sharp(\mathbf{A}) + \mathcal{P}^\sharp(\mathbf{B}) = \mathcal{P}^\sharp(\mathbf{A} \oplus_Q \mathbf{B}).$$

Finally

$$\mathbf{A_n} \nearrow \mathbf{A} \text{ implies } [\mathcal{P}^\flat(\mathbf{A_n}), \mathcal{P}^\sharp(\mathbf{A_n})] = \mathcal{P}(\mathbf{A_n}) \nearrow \mathcal{P}(\mathbf{A}),$$

hence

$$\mathcal{P}^\flat(\mathbf{A_n}) \nearrow \mathcal{P}^\flat(\mathbf{A}) \text{ and } \mathcal{P}^\sharp(\mathbf{A_n}) \nearrow \mathcal{P}^\sharp(\mathbf{A}).$$

The opposite implication can be proved similarly. □

Let us find the representation theorems for Q-states and Q-probabilities. We are able to find this representation only for representable Q-probabilities.

Definition 2.6 *Q-probability $\mathcal{P}_0 = [\mathcal{P}_0^\flat, \mathcal{P}_0^\sharp] : \mathcal{F} \to \mathcal{J}$ is representable, if there exist functions $f, g : R^2 \to R$ and probabilities $p, r : \mathcal{S} \to [0,1]$, such that*

$$\mathcal{P}_0((\mu_A, \nu_A)) =$$

$$= [f(\int_\Omega (\mu_A)^n dp, \int_\Omega (1-\nu_A)^n dr), g(\int_\Omega (\mu_A)^n dp, \int_\Omega (1-\nu_A)^n dr)].$$

Definition 2.7 *Q-state $m_0 : \mathcal{F} \to [0,1]$ is representable, if there exist a function $f : R^2 \to R$ and the probabilities $p, r : \mathcal{S} \to [0,1]$ such that*

$$m_0((\mu_A, \nu_A)) = f\left(\int_\Omega (\mu_A)^n dp, \int_\Omega (1-\nu_A)^n dr\right).$$

Theorem 2.8 Representation Theorem
Let $m_0 : \mathcal{F} \to [0,1]$ be a representable Q-state. Then there exist $\alpha \in [0,1]$ and probabilities $p, r : \mathcal{F} \to [0,1]$ such that for each $A \in \mathcal{F}$

$$m_0((\mu_A, \nu_A)) = (1-\alpha)\int_\Omega {\mu_A}^n dp + \alpha(\int_\Omega (1-\nu_A)^n dr),$$

where $n \in N$ is fixed.

Firstly, let us prove the following lemma.

Lemma 2.9 *Let $f : [0,1]^2 \to R$ be an additive and continuous function, then f is linear.*

Proof Let $f : [0,1]^2 \to R$, we show, that for each $A \in [0,1]^2$ and each $\alpha \in R$ such that

$$f(\alpha A) = \alpha f(A).$$

Consider cases:

(I) $\alpha \in N$, then $f(\alpha A) = f(\underbrace{A + ... + A}_{\alpha}) = \alpha f(A)$.

(II) $\alpha \in Q^+$, so $\exists p, q \in Z, (p,q) = 1, \frac{p}{q} > 0$.
We have $f(A) = \underbrace{f\left(\frac{1}{q}A\right) + ... + f\left(\frac{1}{q}A\right)}_{q},$
where $A \in [0,1]^2$, so $\frac{1}{q}A \in [0,1]^2$.
Then $f(A) = qf\left(\frac{1}{q}A\right)$ and so $f\left(\frac{1}{q}A\right) = \frac{1}{q}f(A)$.
Let us take $A \in [0,1]^2$ such that $\frac{p}{q}A \in [0,1]^2$,
then $\frac{1}{q}A \in [0,1]^2$ and

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$$f(\tfrac{p}{q}A)=\underbrace{f\left(\frac{1}{q}A\right)+...+f\left(\frac{1}{q}A\right)}_{p}=$$
$$=pf\left(\tfrac{1}{q}A\right)=\tfrac{p}{q}f(A).$$

(III) Since f is continuous and $f(rA)=rf(A)$ for $r\in Q^{+}$, such that $A,rA\in[0,1]^2$, by approximating any real $x\in R$ by rational $(x_n)_{n=1}^{\infty}$ we get

$$f(xA)=f(\lim_{n\to\infty}x_nA)=\lim_{n\to\infty}f(x_nA)=$$
$$=\lim_{n\to\infty}x_nf(A)=xf(A).$$

Thus we proved that f is linear on [0,1]. □

Proof of Theorem 2.7
Let m_0 be a representable Q-state, we are looking for a formula for function f from Definition 2.6. From property (i) of Q-state (Definition 2.2) we get

$$m_0((\mathbf{1},\mathbf{0}))=f(\int_\Omega 1^n dp,\int_\Omega(1-0)^n dr)=f(1,1),$$

because of that
$$\int_\Omega 1dp=\int_\Omega 1dr=1$$

we get
$$m_0((\mathbf{1},\mathbf{0}))=f(1,1)=1.$$

Analogously $m_0((\mathbf{0},\mathbf{1}))=f(0,0)=0$.
What about additivity? Assume that $A\odot B=(\mathbf{0},\mathbf{1})$, so

$$\mu_A+\mu_B\leq 1,\nu_A+\nu_B\geq 1.$$

We get

$$m_0(A\oplus_Q B)=$$
$$=f(\int_\Omega(\sqrt[n]{((\mu_A)^n+(\mu_B)^n)})^n dp,$$
$$\int_\Omega(1-(1-\sqrt[n]{((1-\nu_A)^n+(1-\nu_B)^n)}))^n dr)=$$
$$=f(\int_\Omega(\mu_A)^n+(\mu_B)^n dp,\int_\Omega(1-\nu_A)^n+(1-\nu_B)^n dr).$$

Analogously

$$m_0(A)+m_0(B)=$$
$$f(\int_\Omega(\mu_A)^n dp,\int_\Omega(1-\nu_A)^n dr)+f(\int_\Omega(\mu_B)^n dp,\int_\Omega(1-\nu_B)^n dr).$$

Let us denote by

$$\int_\Omega\mu_A dp=u_1,\int_\Omega\mu_B dp=v_1,$$

$$\int_\Omega(1-\nu_A)dr=u_2,\int_\Omega(1-\nu_B)dr=v_2,$$

we get

$$m_0(A\oplus_Q B)=f(u_1+v_1,u_2+v_2),$$
$$m_0(A)+m_0(B)=f(u_1,u_2)+f(v_1,v_2).$$

Since the property (ii) holds, we have

$$f(u_1+v_1,u_2+v_2)=f(u_1,u_2)+f(v_1,v_2),$$

and that is why the function f is linear ($f(x+y)=f(x)+f(y)$ holds). This equality holds by the previous Lemma. Finaly f is continuous by the (iii) property of Definition 2.2. □

Theorem 2.10 Representation Theorem of Q-probabilities
If $\mathcal{P}_0$ is a representable Q-probability, then there exist real numbers $\alpha,\beta\in[0,1]$ and probability mesures $p,r_1,r_2:\mathcal{S}\to[0,1]$ such that $\alpha r_1\leq\beta r_2$ that for each $A=(\mu_A,\nu_A)\in\mathcal{F}$ there holds

$$\mathcal{P}_0((\mu_A,\nu_A))=[(1-\alpha)\int_\Omega(\mu_A)^n dp+\alpha\int_\Omega(1-\nu_A)^n dr_1,(1-\beta)\int_\Omega(\mu_A)^n dp+\beta\int_\Omega(1-\nu_A)^n dr_2].$$

Proof Let $\mathcal{P}_0(A)=[\mathcal{P}_0^\flat(A),\mathcal{P}_0^\sharp(A)]$ be representable Q-probability. Following Q-states $\mathcal{P}_0^\flat,\mathcal{P}_0^\sharp$ could be written by previous formulas. □

3 Q-observables and p-joint Q-observables

First, let us denote Borelian sets by $\mathcal{B}(R)$.

Definition 3.1 *A mapping $x:\mathcal{B}(R)\to\mathcal{F}$ is called a Q-observable, if the following conditions are satisfied:*

(i) $x(R)=(1,0),x(\emptyset)=(0,1)$;

(ii) *if* $A\cap B=\emptyset$ *then* $x(A)\odot x(B)=(\mathbf{0},\mathbf{1})$, *and* $x(A\cup B)=x(A)\oplus_Q x(B)$;

(iii) $A_n\nearrow A\Rightarrow x(A_n)\nearrow x(A)$.

Theorem 3.2 *Let $x:\mathcal{B}(R)\to\mathcal{F}$ be an Q-observable, $\mathcal{P}=[\mathcal{P}^\flat,\mathcal{P}^\sharp]:\mathcal{F}\to\mathcal{J}$ be an Q-probability. Then the functions $\mathcal{P}^\flat\circ x:\mathcal{B}(R)\to[0,1],\mathcal{P}^\sharp\circ x:\mathcal{B}(R)\to[0,1]$, are probability measures.*

Proof The proof is straightforward. □

Theorem 3.3 *Let $x:\mathcal{B}(R)\to\mathcal{F}$ be an Q-observable, $x(A)=(x^\flat(A),1-x^\sharp(A));\omega\in\Omega$. Then the functions $p_\omega^\flat,p_\omega^\sharp:\mathcal{B}(R)\to[0,1]$ defined by*

$$p_\omega^\flat(A)=(x^\flat(A)(\omega))^n;$$

$$p_\omega^\sharp(A)=(x^\sharp(A)(\omega))^n$$

are probability measures.

Proof Use instead of $\varphi(u)=u^n$ in Theorem 2.7 in [9]. □

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Definition 3.4 *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be Q-observables. By the p-joint Q-observable h of x, y we understand a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions*

(i) $h(R^2) = (\mathbf{1}, \mathbf{0}); h(\emptyset) = (\mathbf{0}, \mathbf{1})$;

(ii) if $A \cap B = \emptyset$ *then* $h(A) \odot h(B) = (\mathbf{0}, \mathbf{1})$ *and* $h(A \cup B) = h(A) \oplus_Q h(B)$;

(iii) $A_n \nearrow A \Rightarrow h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C).y(D)$ *for any* $C, D \in \mathcal{B}(R)$. *here* $(\mu_C, \nu_C).(\mu_D, \nu_D) = (\mu_C.\mu_D, 1 - (1 - \nu_C).(1 - \nu_D))$

Remark 3.5 *Analogously we can extend Definition 3.4 for finite collection of Q-observables.*

Theorem 3.6 *To any Q-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ there exists their p-joint Q-observable $h : \mathcal{B}(R^2) \to \mathcal{F}$.*

Proof Use instead of $\varphi(u) = u^n$ in Theorem 2.9 in [9]. □

4 Application of Q-observables

Let us mention one version of Central limit theorem: let $(\xi_i)_{i=1}^{\infty}$ be a sequence of independent, equally distributed, square integrable random variables,

$$E(\xi_i) = a, \sigma^2(\xi_i) = \sigma^2 \text{ for all } i \in N.$$

Then for any $t \in R$ there holds

$$\lim_{n \to \infty} p\left(\left\{\omega; \frac{\overline{\zeta_n}(\omega) - a}{\sigma}\sqrt{n} < t\right\}\right) = \Phi(t).$$

Here $\overline{\zeta_n} = \frac{1}{n}\sum_{i=1}^{n} \xi_i$ and $\Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}} du$.
Now we are going to formulate an analogous assertion for Q-observables.
First, we shall mention some useful definitions:

Definition 4.1 *For any Q-probability $\mathcal{P} = [\mathcal{P}^\flat, \mathcal{P}^\sharp] : \mathcal{F} \to \mathcal{J}$ and any Q-observable $x : \mathcal{B}(R) \to \mathcal{F}$ we define the expected values by*

$$E_\flat(x) = \int_R t d\mathcal{P}_x^\flat(t);$$

$$E_\sharp(x) = \int_R t d\mathcal{P}_x^\sharp(t)$$

and the variances by

$$\sigma_\flat^2(x) = \int_R (t - E_\flat(x))^2 d\mathcal{P}_x^\flat(t);$$

$$\sigma_\sharp^2(x) = \int_R (t - E_\sharp(x))^2 d\mathcal{P}_x^\flat(t),$$

where $\mathcal{P}_x^\flat = \mathcal{P}^\flat \circ x, \mathcal{P}_x^\sharp = \mathcal{P}^\sharp \circ x$, assuming that the integrals exist.

Assume $T = (\xi_1, ..., \xi_n) : \Omega^n \to R^n$ is a random vector and $g : R^n \to R$ is a Borel measurable function (e.g. $g(u_1, ..., u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i$). Then

$$g(\xi_1, ..., \xi_n) = g \circ T : \Omega^n \to R^n,$$

is a transformation of T. Hence we get the following formula

$$(g \circ T)^{-1}(A) = T^{-1}(g^{-1}(A))$$

for any $A \in \mathcal{B}(R)$. The formula justifies the following definition.

Definition 4.2 *Let $g_n : R^n \to R$ be a Borel function, $x_1, ..., x_n : \mathcal{B}(R) \to \mathcal{F}$ be Q-observables, $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ their joint observable. Then the g_n-transformation of h_n is a Q-observable $y_n : \mathcal{B}(R) \to \mathcal{F}$ given by $y_n(A) = h_n(g_n^{-1}(A))$ for any $A \in \mathcal{B}(R)$.*

Definition 4.3 *Let $(x_n)_{n=1}^{\infty}$ be a sequence of Q-observables, $(h_n)_{n=1}^{\infty}$ be a sequence of the joint Q-observables $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ of $x_1, x_2, ..., x_n$ (for $n \in N$), $m : \mathcal{F} \to [0, 1]$ be a Q-state. The sequence $(x_n)_{n=1}^{\infty}$ is independent (with respect to m), if for any $n \in N$ and any $C_1, C_2, ..., C_n \in \mathcal{B}(R)$ there holds*

$$m(h_n(C_1 \times C_2 \times ... \times C_n)) = m(x_1(C_1))...m(x_n(C_n)).$$

Definition 4.4 *A sequence $(x_n)_{n=1}^{\infty}$ of Q-observables is equally distributed, if $m(x_n(A)) = m(x_1(A))$ for any $n \in N$ and $A \in \mathcal{B}(R)$.*

Theorem 4.5 (Central limit theorem)
Let $(x_n)_{n=1}^{\infty}$ be a sequence of independent, equally distributed, square integrable Q-observables, where $E_\flat(x_n) = a^\flat, (E_\sharp(x_n) = a^\sharp)$ $\sigma_\flat^2(x_n) = \sigma_\flat^2$, $(\sigma_\sharp^2(x_n) = \sigma_\sharp^2)$ for each $n \in N$. Then for any $t \in R$ there the following holds

$$\lim_{n \to \infty} \mathcal{P}^\flat\left(\frac{x_1 + ... + x_n - na^\flat}{\sigma_\flat\sqrt{n}}((-\infty, t))\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}} du,$$

$$(\lim_{n \to \infty} \mathcal{P}^\sharp\left(\frac{x_1 + ... + x_n - na^\sharp}{\sigma_\sharp\sqrt{n}}((-\infty, t))\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{u^2}{2}} du).$$

Proof Use Theorem 4.1 in [9]. □

5 Conclusion

Generalizing some notions proposed in [2] we constructed a $Q-$probability theory. The theory includes some known results ($n = 1, n = 2$), and also it opens the door for some other applications. We have proved some representation theorems. As an open question and an inspiration for the future research remains the problem of conditional probabilities for this framework.

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