Notes on Intuitionistic Fuzzy Sets
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Morphological operations on temporal intuitionistic fuzzy sets

R. Parvathi[1] and C. Yuvapriya[2,*]

[1] Department of Mathematics, Vellalar College for Women
Erode, Tamilnadu, India
e-mail: paarvathis@rediffmail.com

[2] PG Department of Mathematics, Vellalar College for Women
Erode, Tamilnadu, India
e-mail: yuvapriya.c@gmail.com

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Abstract: This paper is devoted to develop the theory of temporal intuitionistic fuzzy sets. The matrix representation of a TIFS is also introduced for easy symbolization. In addition to a few basic operations, length of a TIFS and its properties are discussed. Morphological operations on temporal intuitionistic fuzzy sets are defined using (i) mathematical operations, (ii) structuring element, (iii) inclusion indicators, and (iv) temporal intuitionistic fuzzy divergence and verified with suitable examples.
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1 Introduction

Fuzzy sets (FSs) introduced by Lotfi A. Zadeh in 1965 [26] are a generalization of crisp sets. Krassimir T. Atanassov introduced the concept of intuitionistic fuzzy sets (IFSs) in 1983 [3, 4] as an extension of FSs. These sets include the membership and non-membership of the element in the set along with a degree of hesitancy. Krassimir T. Atanassov also extended the concept

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of IFSs into temporal intuitionistic fuzzy sets (TIFSs) [6]. TIFSs give a possibility to trace the changes of the object considered for all the time moments from a time scale and permit more detailed estimations of the real time processes flowing in time.

Fuzzy sets and intuitionistic fuzzy sets handle uncertainty and vagueness which Cantorian sets could not handle. Temporal intuitionistic fuzzy set with a time domain is an extension of intuitionistic fuzzy set and is useful in dealing with uncertainty and vagueness present in the time dependent real environment. In [20], a new type of intuitionistic fuzzy set called multiparameter temporal intuitionistic fuzzy set is proposed and its operations are defined. Further, fuzzification tools like extended triangular membership and non-membership functions for TIFSs and multiparameter TIFSs are defined. Geometric interpretation of the extended triangular intuitionistic fuzzification functions of a TIFS is also dealt with a suitable example.

Defuzzification functions in intuitionistic fuzzy environment such as triangular, trapezoidal, L-trapezoidal, R-trapezoidal, Gaussian, S-shaped, Z-shaped functions are defined in [24]. Procedures for de-i-fuzzification of intuitionistic fuzzy sets are described in [11,12]. Defuzzification methods interpret the fuzzy sets in the form of a precise crisp value needed by the designer and the corresponding crisp values of the fuzzy system are calculated in [23]. Defuzzification methods of TIFSs are essential in the development of temporal intuitionistic fuzzy systems. There are several defuzzification methods like maxima methods, centroid methods and weighted average methods available in literature for fuzzy sets and also for IFSs. In [23], some standard crispification methods for TIFSs are defined which are useful to apply TIFSs for temporal data with uncertainty. The proposed methods play a major role in dealing with the most common dynamic systems occuring in nature.

The temporal logic is one of the basic areas of mathematical logic developing through last century. The first intuitionistic fuzzy interpretations of the temporal logic were discussed in [5], where the temporal logic operators "always" and "once" are studied. The two other temporal operators "sometimes" and "at the currently" are defined in [7]. Some new operators are introduced in [9] in which intuitionistic fuzzy interpretations are discussed. In [19], some properties of TIFSs are discussed. In [25], $N_B(A)$ and $N_B{*}(A)$ operators on TIFSs with extension to new universal are introduced and some properties of these operators are examined.

In [17], distance measures, similarity measures, entropy and inclusion measure for TIFSs are investigated and some properties of these measures are proposed by Fatih Kutlu et al. They defined these concepts in two different ways, namely temporal and overall and examined the relationship between these definitions. Also they gave numerical examples for TIFS and compared these measures defined with two and three parameters in terms of reliability and applicability. In [22], entropy measure of TIFSs is introduced along with its axiomatic definition. The relationship among entropy, similarity and distance measures of TIFSs are studied.

The first method for aggregating infinite sequences of IFSs in the literature is proposed in [2]. This new tool allows us to aggregate an infinite list of intuitionistic fuzzy sets over time as a particular case of temporal intuitionistic fuzzy sets into a traditional intuitionistic fuzzy set. As an application, scores and accuracy degrees of temporal intuitionistic fuzzy elements is defined. Then the authors used these tools to solve the decision making problems where data are available in the form of intuitionistic fuzzy sets along with an indefinitely long number of periods [2].

A method to define fuzzy morphological operators (erosion and dilation) has been proposed in [14]. A family of fuzzy implication operators and the inclusion grade are the basis for this method.

After making a thorough study/survey on TIFSs, it is observed that model the real life problems in mathematics which is time variant, temporal intuitionistic fuzzy set theory is needed. Therefore, it is motivated to develop the temporal IFS theory.

In Section 2, the basic definitions of TIFSs are given. In Section 3, matrix representation and length of a TIFS are defined and its properties are discussed. Section 4 is devoted to morphological operations on TIFSs. Section 5 concludes the paper.

2 Preliminaries

Definition 2.1 ([8]). *Let E be the universe and T be a non-empty set of time moments. Then, a temporal intuitionistic fuzzy set (TIFS) is an object having the form*

$$A(T) = \{\langle x, \mu_A(x,t), \nu_A(x,t)\rangle : (x,t) \in E \times T\}$$

where:

(i) $A \subset E$ *is a fixed set;*

(ii) $\mu_A(x,t)$ *and* $\nu_A(x,t)$ *denote the degrees of membership and non-membership respectively of the element* (x,t) *such that* $0 \leq \mu_A(x,t) + \nu_A(x,t) \leq 1$ *for all* $(x,t) \in E \times T$.

Note. It is obvious that $\mu_A : E \times T \to [0,1]$ and $\nu_A : E \times T \to [0,1]$.

Definition 2.2 ([8]). For every two TIFSs,

$$A(T') = \{\langle x, \mu_A(x,t), \nu_A(x,t)\rangle : \langle x,t\rangle \in E \times T'\}$$
$$B(T'') = \{\langle x, \mu_B(x,t), \nu_B(x,t)\rangle : \langle x,t\rangle \in E \times T''\}$$

the following basic operations are defined:

$A(T') \cap B(T'') = \Big\{\langle x, \min(\overline{\mu}_A(x,t), \overline{\mu}_B(x,t)), \max(\overline{\nu}_A(x,t), \overline{\nu}_B(x,t))\rangle : \langle x,t\rangle \in E \times (T' \cup T'')\Big\}$

$A(T') \cup B(T'') = \Big\{\langle x, \max(\overline{\mu}_A(x,t), \overline{\mu}_B(x,t)), \min(\overline{\nu}_A(x,t), \overline{\nu}_B(x,t))\rangle : \langle x,t\rangle \in E \times (T' \cup T'')\Big\}$

$\overline{A}(T') = \{\langle x, \overline{\nu}(x,t), \overline{\mu}(x,t)\rangle : \langle x,t\rangle \in E \times T'\}$,

where

$$\overline{\mu}_A(x,t) = \begin{cases} \mu_A(x,t), & t \in T' \\ 0, & t \in T'' - T' \end{cases}$$

$$\overline{\mu}_B(x,t) = \begin{cases} \mu_B(x,t), & t \in T'' \\ 0, & t \in T' - T'' \end{cases}$$

$$\overline{\nu}_A(x,t) = \begin{cases} \nu_A(x,t), & t \in T' \\ 1, & t \in T'' - T' \end{cases}$$

$$\overline{\nu}_B(x,t) = \begin{cases} \nu_B(x,t), & t \in T'' \\ 1, & t \in T' - T'' \end{cases}$$

Definition 2.3 ([21]). *For every two TIFSs,*

$$A(T') = \{\langle (x,t), \mu_A(x,t), \nu_A(x,t)\rangle : (x,t) \in E \times T'\}$$
$$\textit{and } B(T'') = \{\langle (x,t), \mu_B(x,t), \nu_B(x,t)\rangle : (x,t) \in E \times T''\}$$

Then $A(T') + B(T'')$ is defined as

$$A(T') + B(T'') = \left\{ \left\langle (x,t), \mu_{A(T')+B(T'')}(x,t), \nu_{A(T')+B(T'')}(x,t) \right\rangle : (x,t) \in E \times (T' \cup T'') \right\},$$

where

$$\left\langle (x,t), \mu_{A(T')+B(T'')}(x,t), \nu_{A(T')+B(T'')}(x,t) \right\rangle$$
$$= \begin{cases} \langle (x,t), \mu_A(x,t'), \nu_A(x,t')\rangle, & \textit{if } t = t' \in T' - T'' \\ \langle (x,t), \mu_B(x,t''), \nu_B(x,t'')\rangle, & \textit{if } t = t'' \in T'' - T' \\ \langle (x,t), \mu_A(x,t') + \mu_B(x,t'') - \mu_A(x,t').\mu_B(x,t''), \nu_A(x,t').\nu_B(x,t'')\rangle, & \textit{if } t = t' = t'' \in T' \cap T'' \\ \langle (x,t), 0, 1\rangle, & \textit{otherwise} \end{cases}$$

3 Matrix representation and length of a TIFS

Definition 3.1. *Let $E = \{x_1, x_2, x_3, ..., x_m\}$ be the universe and $T = \{t_1, t_2, t_3, ..., t_n\}$ be the set of time moments. Then the matrix representation of the TIFS $A(T)$ is as follows:*

$$\begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & \cdots & t_n \end{array} \\ \begin{pmatrix} \langle \mu_A(x_1,t_1), \nu_A(x_1,t_1)\rangle & \langle \mu_A(x_1,t_2), \nu_A(x_1,t_2)\rangle & \cdots & \langle \mu_A(x_1,t_n), \nu_A(x_1,t_n)\rangle \\ \langle \mu_A(x_2,t_1), \nu_A(x_2,t_1)\rangle & \langle \mu_A(x_2,t_2), \nu_A(x_2,t_2)\rangle & \cdots & \langle \mu_A(x_2,t_n), \nu_A(x_2,t_n)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu_A(x_m,t_1), \nu_A(x_m,t_1)\rangle & \langle \mu_A(x_m,t_2), \nu_A(x_m,t_2)\rangle & \cdots & \langle \mu_A(x_m,t_n), \nu_A(x_m,t_n)\rangle \end{pmatrix} \end{array}$$

where $\langle \mu_A(x_i,t_j), \nu_A(x_i,t_j)\rangle$ is the membership and non-membership values of the element x_i in the TIFS $A(T)$ at the time moment $t_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

Definition 3.2. *Let $A(T)$ be a TIFS. The length of $A(T)$ is defined as the closed interval*

$$\left[\sum_i \sum_j \mu_{A(T)}(x_i,t_j), \sum_i \sum_j 1 - \nu_{A(T)}(x_i,t_j) \right],$$

denoted by $|A(T)|$. That is,

$$|A(T)| = \left[\sum_i \sum_j \mu_{A(T)}(x_i,t_j), \sum_i \sum_j 1 - \nu_{A(T)}(x_i,t_j) \right],$$

Theorem 3.3. *For any two TIFSs $A(T')$ and $B(T'')$ on E,*

(i) $|A(T') \cup B(T'')| + |A(T') \cap B(T'')| = |A(T')| + |B(T'')|$

(ii) $|A(T') + B(T'')| + |A(T') \cdot B(T'')| = |A(T')| + |B(T'')|$

Proof. $|A(T') \cup B(T'')| + |A(T') \cap B(T'')|$

$$\begin{aligned}
&= \left[\sum_i \sum_j \max(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j), \sum_i \sum_j 1 - \min(\overline{\nu}_A(x_i,t_j), \overline{\nu}_B(x_i,t_j))\right] \\
&\quad + \left[\sum_i \sum_j \min(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j)), \sum_i \sum_j 1 - \max(\overline{\nu}_A(x_i,t_j), \overline{\nu}_B(x_i,t_j))\right] \\
&= \left[\sum_i \sum_j \max(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j)), \sum_i \sum_j \max(1-\overline{\nu}_A(x_i,t_j), 1-\overline{\nu}_B(x_i,t_j))\right] \\
&\quad + \left[\sum_i \sum_j \min(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j)), \sum_i \sum_j \min(1-\overline{\nu}_A(x_i,t_j), 1-\overline{\nu}_B(x_i,t_j))\right] \\
&= \left[\sum_i \sum_j (\overline{\mu}_A(x_i,t_j) + \overline{\mu}_B(x_i,t_j)), \sum_i \sum_j (1-\overline{\nu}_A(x_i,t_j) + 1-\overline{\nu}_B(x_i,t_j))\right] \\
&= \left[\sum_i \sum_j (\overline{\mu}_A(x_i,t_j), \sum_i \sum_j (1-\overline{\nu}_A(x_i,t_j)\right] \\
&\quad + \left[\sum_i \sum_j \overline{\mu}_B(x_i,t_j)) \sum_i \sum_j 1-\overline{\nu}_B(x_i,t_j))\right] \\
&= |A(T')| + |B(T'')|,
\end{aligned}$$

where

$$\overline{\mu}_A(x,t) = \begin{cases} \mu_{A(T')}(x_i,t_j), & t \in T' \\ 0, & t \in T'' - T' \end{cases}$$

$$\overline{\mu}_B(x_i,t_j) = \begin{cases} \mu_{B(T'')}(x_i,t_j), & t \in T'' \\ 0, & t \in T' - T'' \end{cases}$$

$$\overline{\nu}_A(x_i,t_j) = \begin{cases} \nu_{A(T')}(x_i,t_j), & t \in T' \\ 1, & t \in T'' - T' \end{cases}$$

$$\overline{\nu}_B(x_i,t_j) = \begin{cases} \nu_{B(T'')}(x_i,t_j), & t \in T'' \\ 1, & t \in T' - T'' \end{cases}$$

and $i = 1, 2, ..., m$, $j = 1, 2, \ldots, n$. Similarly (ii) can also be proved. □

Theorem 3.4. *For any TIFSs $A(T')$ and $B(T'')$ on E,*

(i) $\left|\overline{A(T') \cup B(T'')}\right| + \left|\overline{A(T') \cap B(T'')}\right| = \left|\overline{A(T')}\right| + \left|\overline{B(T'')}\right|$

(ii) $\left|\overline{A(T') + B(T'')}\right| + \left|\overline{A(T') \cdot B(T'')}\right| = \left|\overline{A(T')}\right| + \left|\overline{B(T'')}\right|$

Proof. $\left|\overline{A(T') \cup B(T'')}\right| + \left|\overline{A(T') \cap B(T'')}\right|$

$$
\begin{aligned}
= \quad & \left[\sum_i \sum_j 1 - \min(\overline{\nu}_A(x_i,t_j), \overline{\nu}_B(x_i,t_j)), \sum_i \sum_j \max(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j))\right] \\
& + \left[\sum_i \sum_j 1 - \max(\overline{\nu}_A(x_i,t_j), \overline{\nu}_B(x_i,t_j)), \sum_i \sum_j \min(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j))\right] \\
= \quad & \left[\sum_i \sum_j \min(1 - \overline{\nu}_A(x_i,t_j), 1 - \overline{\nu}_B(x_i,t_j)), \sum_i \sum_j \max(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j))\right] \\
& + \left[\sum_i \sum_j \min(1 - \overline{\nu}_A(x_i,t_j), 1 - \overline{\nu}_B(x_i,t_j)), \sum_i \sum_j \min(\overline{\mu}_A(x_i,t_j), \overline{\mu}_B(x_i,t_j))\right] \\
= \quad & \left[\sum_i \sum_j (1 - \overline{\nu}_A(x_i,t_j) + 1 - \overline{\nu}_B(x_i,t_j)), \sum_i \sum_j (\overline{\mu}_A(x_i,t_j) + \overline{\mu}_B(x_i,t_j))\right] \\
= \quad & \left[\sum_i \sum_j (1 - \overline{\nu}_A(x_i,t_j), \sum_i \sum_j (\overline{\mu}_A(x_i,t_j)\right] \\
& + \left[\sum_i \sum_j \overline{\mu}_B(x_i,t_j)), \sum_i \sum_j 1 - \overline{\nu}_B(x_i,t_j))\right] \\
= \quad & \left|\overline{A(T')}\right| + \left|\overline{B(T'')}\right|,
\end{aligned}
$$

where

$$
\overline{\mu}_A(x_i,t_j) = \begin{cases} \mu_{A(T')}(x_i,t_j), & t \in T' \\ 0, & t \in T'' - T' \end{cases}
$$

$$
\overline{\mu}_B(x_i,t_j) = \begin{cases} \mu_{B(T'')}(x,t), & t \in T'' \\ 0, & t \in T' - T'' \end{cases}
$$

$$
\overline{\nu}_A(x_i,t_j) = \begin{cases} \nu_{A(T')}(x_i,t_j), & t \in T' \\ 1, & t \in T'' - T' \end{cases}
$$

$$
\overline{\nu}_B(x_i,t_j) = \begin{cases} \nu_{B(T'')}(x_i,t_j), & t \in T'' \\ 1, & t \in T' - T'' \end{cases}
$$

and $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Similarly (ii) can also be proved. □

4 Temporal intuitionistic fuzzy morphological operations

4.1 Mathematical operations

In this section, the basic concepts of the mathematical morphology is extended to temporal intuitionistic fuzzy set theory.

Definition 4.1. *The erosion operation on a TIFS $A(T)$ of the universe E, denoted by $ERO(A(T))$, is defined as*

$$ERO(A(T)) = \{\langle (x,t), \mu_{ERO}(x,t), \nu_{ERO}(x,t)\rangle : (x,t) \in E \times T\},$$

where

$$\mu_{ERO}(x,t) = \left[\mu_{A(T)}(x,t)\right]^2$$

and

$$\nu_{ERO}(x,t) = 1 - \left[1 - \nu_{A(T)}(x,t)\right]^2.$$

That is, the operation of erosion is defined as $ERO(A(T)) = (A(T))^2$.

Definition 4.2. *The dilation operation on a TIFS $A(T)$ of the universe E, denoted by $DIL(A(T))$, is defined as*

$$DIL(A(T)) = \{\langle (x,t), \mu_{DIL}(x,t), \nu_{DIL}(x,t)\rangle : (x,t) \in E \times T\},$$

where

$$\mu_{DIL}(x,t) = \left[\mu_{A(T)}(x,t)\right]^{\frac{1}{2}}$$

and

$$\nu_{DIL}(x,t) = 1 - \left[1 - \nu_{A(T)}(x,t)\right]^{\frac{1}{2}}.$$

That is, the operation of *dilation* is defined as $DIL(A(T)) = (A(T))^{\frac{1}{2}}$.

Definition 4.3. The *contrast intensification operation* on a TIFS $A(T)$ of the universe E, denoted by $INTEN(A(T))$, is defined as

$$INTEN(A(T)) = \left\{\langle (x,t), \mu_{INTEN}(x,t), \nu_{INTEN)}(x,t)\rangle : (x,t) \in E \times T\right\}$$

where

$$\mu_{INTEN}(x,t) = 1 - (1 - \mu_{A(T)}(x,t)^2)^2$$

and

$$\nu_{INTEN}(x,t) = [1 - (1 - \nu_{A(T)}(x,t))^2]^2.$$

Definition 4.4. *The normalization of a TIFS $A(T)$ of the universe E, denoted by $NORM(A(T))$, is defined as*

$$NORM(A(T)) = \{\langle (x,t), \mu_{NORM}(x,t), \nu_{NORM}(x,t)\rangle : (x,t) \in E \times T\},$$

where

$$\mu_{NORM}(x,t) = \frac{\mu_{A(T')}(x,t)}{\sup(\mu_{A(T')}(x,t))}$$

and

$$\nu_{NORM}(x,t) = \frac{\nu_{A(T')}(x,t) - \inf(\nu_{A(T')}(x,t))}{1 - \inf(\nu_{A(T')}(x,t))}.$$

These four mathematical operations are graphically represented on the following Figure 1.

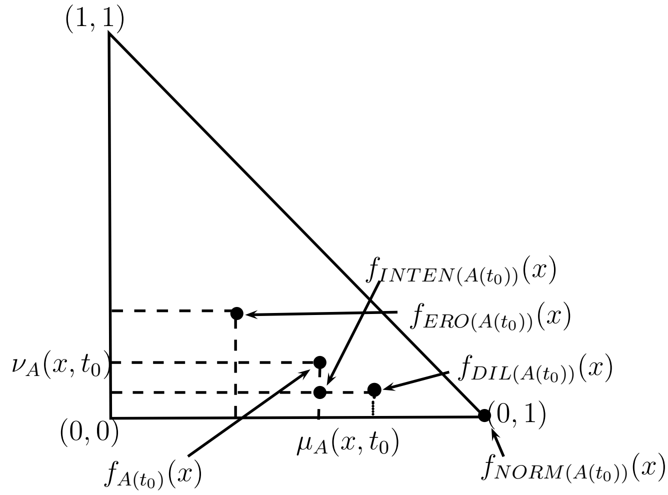

Figure 1. Geometric interpretation of $ERO(A(t_0))$, $DIL(A(t_0))$, $INTEN(A(t_0))$ and $NORM(A(t_0))$

4.2 Using a structuring element

Another interesting way of perfoming morphology is by using a structuring element. In this section, temporal intuitionistic fuzzy morphological operations are defined using a structuring element.

Definition 4.5. *Let E be the universe, let $A(T)$ be a TIFS and let B be an IFS on E. Then the erosion of $A(T)$ by B, denoted by $ERO(A,B)$, is defined as:*

$$ERO(A,B)) = \{\langle (x,t), \mu_{EROAB}(x,t), \nu_{EROAB}(x,t)\rangle : (x,t) \in E \times T\}$$

where

$$\mu_{EROAB}(x,t) = \min(\mu_{A(T)}(x,t)^2, \mu_B(x)^2)$$

and

$$\nu_{EROAB}(x,t) = \max(1 - \left[1 - \nu_{A(T)}(x,t)\right]^2, 1 - [\nu_B(x)]^4).$$

Definition 4.6. *Let E be the universe, $A(T)$ be a TIFS and let B be an IFS on E. Then the dilation of $A(T)$ by B, denoted by $DIL(A,B)$, is defined as:*

$$DIL(A,B) = \{\langle (x,t), \mu_{DILAB}(x,t), \nu_{DILAB}(x,t)\rangle : (x,t) \in E \times T\}$$

where

$$\mu_{DILAB}(x,t) = \max\{\mu_{A(T)}(x,t), \mu_B(x)\}$$

and

$$\nu_{DILAB}(x,t) = \min\{1 - \sqrt{1 - \nu_{A(T)}(x,t)^2}, 1 - \sqrt{1 - \nu_B(x)^2}\}.$$

Definition 4.7. *The opening and closing of a TIFS $A(T)$ and an IFS B, denoted by $O(A(T),B)$ and $C(A(T),B)$, respectively, are defined by:*

$$O(A(T),B) = DIL(ERO(A(T),B),B)$$
$$C(A(T),B) = ERO(DIL(A(T),B),B).$$

The next Figure 2 presents graphically the defined temporal intuitionistic fuzzy morphological operations.

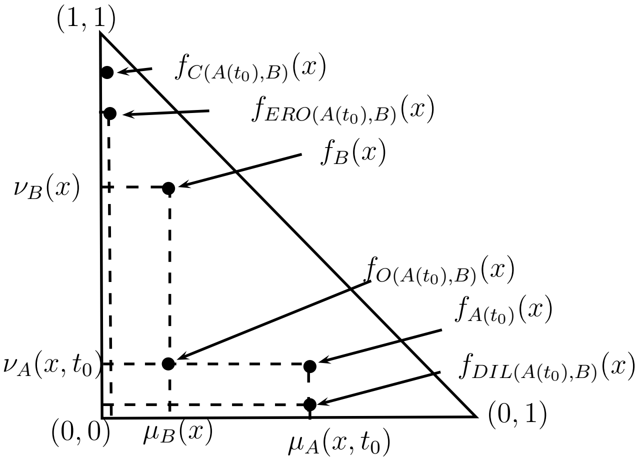

Figure 2. Geometric interpretation of $ERO(A(t_0),B)$, $DIL(A(t_0),B)$, $O(A(t_0),B)$ and $C(A(t_0),B)$

4.3 Using inclusion indicators

In this section, temporal intuitionistic fuzzy morphological operations are defined by using inclusion indicators. Also, translate, reflection, scalar addition, bold union and bold intersection of a TIFS are defined to extend the morphological operations in temporal intuitionistic fuzzy set theory.

Definition 4.8. *Let E be the universe and let $A(T)$ be a TIFS. Then the translate of a TIFS $A(T)$ by a vector v, denoted by $\tau(A(T),v)$, is defined in terms of vector subtraction as*

$$\tau(A(T),v)=\left\{\langle(x,t),\mu_{\tau_{(A(T),v)}}(x,t),\nu_{\tau_{(A(T),v)}}(x,t)\rangle:(x,t)\in E\times T\right\}$$

where

$$\mu_{\tau_{(A(T),v)}}(x,t)=\mu_{A(T)}(x-v,t),\nu_{\tau_{(A(T),v)}}(x,t)=\nu_{A(T)}(x-v,t).$$

Definition 4.9. *Let E be the universe and let $A(T)$ be a TIFS. Then the reflection of a TIFS $A(T)$, denoted by $-A(T)$, is defined as:*

$$-A(T)=\left\{\langle(x,t),\mu_{(-A(T))}(x,t),\nu_{(-A(T))}(x,t)\rangle:(x,t)\in E\times T\right\}$$

where

$$\mu_{(-A(T))}(x,t)=\mu_{A(T)}(-x,t),\nu_{(-A(T))}(x,t)=\nu_{A(T)}(-x,t).$$

Definition 4.10. *Let E be the universe and let $A(T)$ be a TIFS. Then the scalar addition of a TIFS $A(T)$ and constant α, denoted by $A(T)+\alpha$, is defined as:*

$$A(T)+\alpha=\left\{\langle(x,t),\mu_{A(T)+\alpha}(x,t),\nu_{A(T)+\alpha}(x,t)\rangle:(x,t)\in E\times T\right\},$$

where

$$\mu_{A(T)+\alpha}(x,t) = \min(1, \max[0, \mu_{A(T)}(x,t) + \alpha])$$

and

$$\nu_{A(T)+\alpha}(x,t) = \max(0, \min[1, \nu_{A(T)}(x,t) - \alpha]).$$

Definition 4.11 ([10])**.** *Let E be the universe and let $A(T')$ and $B(T'')$ be two TIFSs. Then the bold union of two TIFS $A(T')$ and $B(T'')$, denoted by $A(T') \bigcup B(T'')$, is defined as:*

$$A(T') \bigcup B(T'') = \left\{ \langle (x,t), \mu_{A(T') \bigcup B(T'')}(x,t), \nu_{A(T') \bigcup B(T'')}(x,t) \rangle : (x,t) \in E \times (T' \cup T'') \right\}$$

where

$$\mu_{A(T') \bigcup B(T'')}(x,t) = \min(1, \overline{\mu}_{A(T')}(x,t) + \overline{\mu}_{B(T'')}(x,t))$$

and

$$\nu_{A(T') \bigcup B(T'')}(x,t) = \max(0, \overline{\nu}_{A(T')}(x,t) - \overline{\nu}_{B(T'')}(x,t)),$$

where

$$\overline{\mu}_{A(T')}(x,t) = \begin{cases} \mu_{A(T')}(x,t), & t \in T^{'} \\ 0, & t \in T^{''} - T^{'} \end{cases}$$

$$\overline{\mu}_{B(T'')}(x,t) = \begin{cases} \mu_{B(T'')}(x,t), & t \in T^{''} \\ 0, & t \in T^{'} - T^{''} \end{cases}$$

$$\overline{\nu}_{A(T')}(x,t) = \begin{cases} \nu_{A(T')}(x,t), & t \in T^{'} \\ 1, & t \in T^{''} - T^{'} \end{cases}$$

$$\overline{\nu}_{B(T'')}(x,t) = \begin{cases} \nu_{B(T'')}(x,t), & t \in T^{''} \\ 1, & t \in T^{'} - T^{''} \end{cases}$$

Definition 4.12. *Let E be the universe and let $A(T')$ and $B(T'')$ be two TIFSs. Then the bold intersection of two TIFS $A(T')$ and $B(T'')$, denoted by $A(T') \bigcap B(T'')$, is defined as:*

$$A(T') \bigcap B(T'') = \left\{ \langle (x,t), \mu_{A(T') \bigcap B(T'')}(x,t), \nu_{A(T') \bigcap B(T'')}(x,t) \rangle : (x,t) \in E \times (T' \cup T'') \right\}$$

where

$$\mu_{A(T') \bigcap B(T'')}(x,t) = \max(0, \overline{\mu}_{A(T')}(x,t) + \overline{\mu}_{B(T'')}(x,t)))$$

and

$$\nu_{A(T') \bigcap B(T'')}(x,t) = \min(1, \overline{\nu}_{A(T')}(x,t) - \overline{\nu}_{B(T'')}(x,t) + 1)$$

where

$$\overline{\mu}_{A(T')}(x,t) = \begin{cases} \mu_{A(T')}(x,t), & t \in T^{'} \\ 0, & t \in T^{''} - T^{'} \end{cases}$$

$$\overline{\mu}_{B(T'')}(x,t) = \begin{cases} \mu_{B(T'')}(x,t), & t \in T^{''} \\ 0, & t \in T^{'} - T^{''} \end{cases}$$

$$\overline{\nu}_{A(T')}(x,t) = \begin{cases} \nu_{A(T')}(x,t), & t \in T' \\ 1, & t \in T'' - T' \end{cases}$$

$$\overline{\nu}_{B(T'')}(x,t) = \begin{cases} \nu_{B(T'')}(x,t), & t \in T'' \\ 1, & t \in T' - T'' \end{cases}$$

Definition 4.13. *Let E be the universe, T be the set of time moments and W be the family of TIFSs on E. Let $A(T'), B(T'') \in E$. An inclusion indicator is a function $I : W \times W \to V$ where:*

(i) $V = \{\langle \mu_I(x,t), \nu_I(x,t)\rangle\}$ *for all* $(x,t) \in E \times T$ *such that* $0 \leq \mu_I(x,t) + \nu_I(x,t) \leq 1$*;*

(ii)

$$\mu_I(x,t) = \inf_{x \in E}(\mu_{\overline{A(T')} \bigcup B(T'')}(x,t))$$
$$\nu_I(x,t) = \inf_{x \in E}(\nu_{\overline{A(T')} \bigcup B(T'')}(x,t)).$$

Properties of Inclusion Indicator (Extended from [16]) *An inclusion indicator is satisfying the following nine properties:*

(i) $I(A(T'), B(T'')) = \langle 1, 0\rangle$ *if and only if* $A(T') \subseteq B(T'')$

(ii) $I(A(T'), B(T'')) = \langle 0, 1\rangle$ *if and only if* $\{(x,t) : \mu_{A(T')}(x,t) = 1 \,\&\, \nu_{A(T')}(x,t) = 0\} \cap \{(x,t) : \mu_{B(T'')}(x,t) = 0 \,\&\, \nu_{B(T'')}(x,t) = 1\} \neq \phi$

(iii) If $B(T'') \subseteq C(T''')$*, then* $I(A(T'), B(T'')) \subseteq I(A(T'), C(T'''))$

(iv) If $B(T'') \subseteq C(T''')$*, then* $I(C(T'''), A(T')) \subseteq I(B(T''), A(T'))$

(v) $I(A(T'), B(T'')) = I(\tau(A(T'), v), \tau(B(T''), v))$ *for all* v

(vi) $I(A(T'), B(T'')) = I(\overline{B(T'')}, \overline{A(T')})$

(vii) $I(\bigcup_i B(T'')_i, A(T')) = inf_i I(B(T'')_i, A(T'))$

(viii) $I(A(T'), \bigcap_i B(T'')_i) = inf_i(A(T'), B(T'')_i)$

(ix) $I(A(T'), \bigcup_i B(T'')_i) \geq sup_i I(A(T'), B(T'')_i)$*.*

Definition 4.14. *Let E be the universe, let $A(T')$ be a TIFS and $B(T'')$ be an IFS on E. An inclusion indicator I is given. Then the erosion of $A(T)$ by B, denoted by $ERO(A,B)$, is defined as:*

$$ERO(A,B)) = \{\langle (x,t), \mu_{EROAB}(x,t), \nu_{EROAB}(x,t)\rangle : (x,t) \in E \times T\}$$

where

$$\mu_{EROAB}(x,t) = \mu_{I(\tau(B(T''),x),A(T'))}$$

and

$$\nu_{EROAB}(x,t) = \nu_{I(\tau(B(T''),x),A(T'))}$$

Note. $$DIL(A,B) = ERO(\overline{A(T')}, -\overline{B(T'')}).$$

Example 4.1. *Consider the TIFSs* $A(T) = \{\langle 50, 0.1, 0.3\rangle, \langle 150, 0.2, 0.5\rangle, \langle 100, 0.6, 0.1\rangle\}$ *and* $B(T) = \{\langle 50, 0.1, 0.2\rangle, \langle 150, 0.6, 0.1\rangle, \langle 100, 0.1, 0.6\rangle\}$ *for a fixed time* $T = t_0$*. then*

(i) $\tau(A(t_0), 50) = \{\langle 50, 0, 1\rangle, \langle 150, 0.6, 0.1\rangle, \langle 100, 0.1, 0.3\rangle\}$

(ii) $-A(t_0) = \{\langle 50, 0.3, 0.1\rangle, \langle 150, 0.5, 0.2\rangle, \langle 100, 0.1, 0.6\rangle\}$

(iii) *If* $\alpha = 0.1$*, then* $A(t_0) + 0.1 = \{\langle 50, 0.2, 0.2\rangle, \langle 150, 0.3, 0.3\rangle, \langle 100, 0.7, 0\rangle\}$
If $\alpha = 0.7$*, then* $A(t_0) + 0.7 = \{\langle 50, 0.8, 0\rangle, \langle 150, 1, 0\rangle, \langle 100, 1, 0\rangle\}$

(iv) $(A(t_0) \bigcup B(t_0)) = \{\langle 50, 0.2, 0\rangle, \langle 150, 0.4, 0\rangle, \langle 100, 1, 0\rangle\}$

(v) $(A(t_0) \bigcap B(t_0)) = \{\langle 50, 0, 1\rangle, \langle 150, 0, 1\rangle, \langle 100, 1, 0\rangle\}$

4.4 Using temporal intuitionistic fuzzy divergence

In this section, temporal intuitionistic fuzzy morphology is defined based on temporal intuitionistic fuzzy divergence. Also, temporal intuitionistic fuzzy generator is introduced to construct temporal intuitionistic fuzzy sets where membership values are used to compute non-membership values at each time moment. Temporal intuitionistic fuzzy divergence (TIFD) is introduced to detemine howfar two TIFSs are different from one to another. By comparing this TIFD value of the pixels in image/frame which is represented as temporal intuitionstic fuzzy sets and the threshold value, the temporal intuitionistic fuzzy morphology method is proposed.

Temporal intuitionistic fuzzy generator

In this section, temporal intuitionistic fuzzy sets are constructed by using fuzzy sets at each time moment. Here, an attempt is made to define temporal intuitionistic fuzzy generator.

Definition 4.15. *Let E be the universe and T be the set of time moments. Then the temporal intuitionistic fuzzy generator is defined as a function*

$$\psi : [0, 1] \times T \to [0, 1] \times T$$

satisfying $\psi(y, t) = (1 - y, t)$ *for every* $y \in [0, 1]$ *and* $t \in T$*.*
Therefore, $\psi(0, t) \leq (1, t)$ *and* $\psi(1, t) = (0, t)$*.*

The temporal fuzzy complement functional is defined as

$$M(\mu(x, t)) = h^{-1}(h(1) - h(\mu(x, t))),$$

where $h : [0, 1] \to [0, 1]$ *is an increasing function. Then* h *can be generated as*

$$h(\mu(x, t)) = \frac{1}{\lambda} \log(1 + \lambda\mu(x, t))$$

and the temporal intuitionistic fuzzy generator is given as

$$M(\mu(x, t)) = \frac{1 - \mu(x, t)}{1 + \lambda\mu(x, t)}, \lambda > 0,$$

where $M(0) = 1$ *and* $M(1) = 0$ *and* $t \in T$*.*

Note. *The temporal intuitionistic fuzzy set $A(T)$ is given as*

$$A(T)_\lambda = \left\{ \left\langle (x,t), \mu_{A(T)}(x,t), \frac{1-\mu_{A(T)}(x,t)}{1+\lambda\mu_{A(T)}(x,t)} \right\rangle : (x,t) \in E \times T \right\}.$$

Sugeno based temporal intuitionistic fuzzy generator is used in this work to create a temporal intuitionistic fuzzy image/frame which ultimately helps to enhance a color video.

Temporal intuitionistic fuzzy divergence

In this section, the temporal intuitionistic fuzzy divergence (TIFD) between two temporal intuitionistic fuzzy sets is defined. TIFD is introduced to detemine howfar two TIFSs are different from one to another. TIFD value is needed to perform temporal intuitionistic fuzzy morphology on videos.

The Atanassov's intuitionistic fuzzy index value associated with a pixel has the value zero when the expert is absolutely sure that the pixel belongs either to the background or to the object. However, the ignorance/intuition should have the least possible influence on the choice of the membership degree. Let

$$A(T') = \{\langle (x,t), \mu_A(x,t), \nu_A(x,t)\rangle : \langle x,t\rangle \in E \times T'\}$$

and

$$B(T'') = \{\langle (x,t), \mu_B(x,t), \nu_B(x,t)\rangle : \langle x,t\rangle \in E \times T''\}$$

be two temporal intuitionistic fuzzy sets. It is obvious that, the range for membership function of the above two temporal intuitionistic fuzzy sets can be represented by the closed intervals

$$\left[\mu_{A(T')}(x,t), \mu_{A(T')}(x,t) + \pi_{A(T')}(x,t)\right]$$

and

$$\left[\mu_{B(T'')}(x,t), \mu_{B(T'')}(x,t) + \pi_{B(T'')}(x,t)\right],$$

respectively, where $\mu_{A(T')}(x,t)$ and $\mu_{B(T'')}(x,t)$ are the membership values and $\pi_{A(T')}(x,t)$ and $\pi_{B(T'')}(x,t)$ are hesistancy indices. The interval is due to the hesitation or lack of knowledge in the assignment of the membership values.

Definition 4.16 (Extended from [13])**.** *The temporal intuitionistic fuzzy divergence (TIFD), between two temporal intuitionistic fuzzy sets $A(T')$ and $B(T'')$ is defined as:*
$TIFD(A(T'), B(T''))$

$$\begin{aligned} &= \sum_i \sum_j \{2 - [1-\mu_{AB}(x_i,t_j)]e^{\mu_{AB}(x_i,t_j)} - [1+\mu_{AB}(x_i,t_j)]e^{-\mu_{AB}(x_i,t_j)} + 2 \\ &\qquad - [1-\mu_{AB}(x_i,t_j) - \pi_{AB}(x_i,t_j]e^{\mu_{AB}(x_i,t_j)-\pi_{AB}(x_i,t_j)} \\ &\qquad - [1+\mu_{AB}(x_i,t_j) + \pi_{AB}(x_i,t_j]e^{-\mu_{AB}(x_i,t_j)-\pi_{AB}(x_i,t_j)}\}, \end{aligned}$$

where, $\mu_{AB}(x_i,t_j) = \mu_{A(T')}(x_i,t_j) - \mu_{B(T'')}(x_i,t_j)$ and $\pi_{AB}(x_i,t_j) = \pi_{A(T')}(x_i,t_j) - \pi_{B(T'')}(x_i,t_j)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

Example 4.2. *Consider the TIFSs $A(T) = \{\langle 50, 0.1, 0.3\rangle, \langle 100, 0.6, 0.1\rangle, \langle 150, 0.2, 0.5\rangle\}$ and $B(T) = \{\langle 50, 0.1, 0.2\rangle, \langle 100, 0.1, 0.6\rangle, \langle 150, 0.6, 0.1\rangle\}$ for a fixed time $T = t_0$. Then*

$$TIFD(A(t_0), B(t_0)) = 0.67.$$

5 Algorithm

The following are the steps involved in the proposed method using temporal intuitionistic fuzzy divergence (TIFD):

(i) Select an image from the video at the particular time moment and read the image.

(ii) Fuzzify the pixel values using temporal intuitionistic fuzzy generator. If the image is with size $M \times N$, then:

(a) the first coordinate represents the degree $x_{ij} \in (0^0, 360^0), 0 \leq i \leq M, 0 \leq j \leq N$.

(b) the second coordinate is the normalized value of x. The membership function $\mu_{A(T)}(x_{ij}, t) = \frac{(x_{ij}, t)}{360^0}$.

(c) the non-membership value is obtaind using temporal intuitionistic fuzzy generator as follows:

$$\nu_{A(T)}(x_{ij}, t) = \frac{1 - \mu_{A(T)}(x_{ij}, t)}{1 + \lambda \mu_{A(T)}(x_{ij}, t)}.$$

(iii) Select a pixel from the image (p) as reference. This pixel is represented by its intuitionistic fuzzy coordinates and its hesitance degree as $p = \left\langle p, \mu_A(p), \frac{1-\mu_A(p)}{1+\lambda\mu_A(p)} \right\rangle$.

(iv) Compute the temporal intuitionistic fuzzy divergence between each element and the reference pixel. Selection of reference pixel is either at random or any existing algorithms like fuzzy mathematical morphology [13]. TIFD is a matrix of size $M \times N$ in which each number represents the similarity between (x_{ij}, t) and the fixed pixel p.

(v) If the TIFD for a pixel is less than or equal to a threshold value θ, the pixel is retained, otherwise discarded. Proceed to steps (i) to (iv) until eroded/dilated image output is obtained.

- If $TIFD((x_{ij}, t), p) \leq \theta \Rightarrow output(i, j) = (x_{ij}, t)$.
- If $TIFD((x_{ij}, t), p) > \theta \Rightarrow output(i, j) = (0, t)$,

where $output(i, j)$ represents the result image to pixel (i, j).

6 Conclusion

Matrix representation and length of a TIFS and its properties have been discussed. Morphological operations on temporal intuitionistic fuzzy sets have been defined using (i) mathematical operations, (ii) structuring element, (iii) inclusion indicators and (iv) temporal intuitionistic fuzzy divergence and have been verified.

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