ICIFSS, 8–10 January 2018, Erode, Tamil Nadu, India
Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 24, 2018, No. 1, 37–47
DOI: 10.7546/nifs.2018.24.1.37-47

The Cauchy problem for intuitionistic fuzzy differential equations

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Received: 20 October 2017 **Revised:** 4 December 2017 **Accepted:** 6 December 2017

Abstract: In this paper we discuss the existence and uniqueness theorem of a solution of the cauchy problem of intuitionistic fuzzy differential equation.
Keywords: Intuitionistic fuzzy differential equations, Approximate solutions, Existence, Uniqueness.
2010 Mathematics Subject Classification: 03E72.

1 Introduction

One of the generalizations of fuzzy sets theory [15] can be considered the proposed intuitionistic fuzzy sets(IFS). Later on Atanassov generalized the concept of fuzzy set and introduced the idea of intuitionistic fuzzy set [1–3]. They are very necessary and powerful tool in modeling imprecision, valuable applications of IFSs have been flourished in many different field [4, 6, 8, 9].

For intuitionistic fuzzy concepts, recently the authors [5, 11–13] established, the theory of metric space of intuitionistic fuzzy sets, intuitionistic fuzzy differential equations, intuitionistic fuzzy fractional equation and the Cauchy problem for complex intuitionistic fuzzy differential equations. They proved the existence and uniqueness of the intuitionistic fuzzy solution for these intuitionistic fuzzy differential equations using different concepts.

This paper is to investigate the existence and uniqueness theorem of intuitionistic fuzzy solutions for the follwing intuitionistic fuzzy differential equations:

$$(\langle u,v\rangle'(t)=f(t,(\langle u,v\rangle(t)),\qquad \langle u,v\rangle(t_0)=\langle u_0,v_0\rangle \tag{1.1}$$

when $\langle u_0, v_0\rangle$ is an intuitionistic fuzzy quantity and f satisfies the generalized Lipschitz condition.

The paper is organized as follows. In Section 2, we collect the fundamental notions and facts which will be used in the rest of the article and we list several comparison propositions on classical ordinary differential equations in [7]. In Section 3 we show the relation between a solution and its approximate solution to the Cauchy problem of the intuitionistic fuzzy differential equation, and furthermore, in Section 4, we prove the existence and uniqueness theorem for a solution to the Cauchy problem of the intuitionistic fuzzy differential equation.

2 Preliminaries

Throughout this paper, $(\mathbb{R}^n, B(\mathbb{R}^n), \mu)$ denotes a complete finite measure space.
Let us $P_k(\mathbb{R}^n)$ the set of all non empty compact convex subsets of $\mathbb{R}^n$.
we denote by

$$\mathbb{F}_n=\mathrm{IF}(\mathbb{R}^n)=\left\{\langle u,v\rangle\ :\ \mathbb{R}^n\to[0,1]^2,|\forall\, x\in\mathbb{R}^n\ 0\le u(x)+v(x)\le 1\right\}$$

An element $\langle u, v\rangle$ of $\mathbb{F}_n$ is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v\rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}^n$ such that $u(x_0) = 1$ and $v(x_1) = 1$.

(ii) u is fuzzy convex and v is fuzzy concave.

(iii) u is upper semi-continuous and v is lower semi-continuous

(iv) $supp\,\langle u, v\rangle = \mathrm{cl}\{x \in \mathbb{R}^n\ :\ |\ v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy numbers by $\mathbb{F}_n$
On the space $\mathbb{F}_n$ we will consider the following metric,

$$\begin{aligned}d_\infty^n\Big(\langle u,v\rangle,\langle z,w\rangle\Big)&=\frac{1}{4}\sup_{0<\alpha\le1}\Big\|\Big[\langle u,v\rangle\Big]_r^+(\alpha)-\Big[\langle z,w\rangle\Big]_r^+(\alpha)\Big\|\\&+\frac{1}{4}\sup_{0<\alpha\le1}\Big\|\Big[\langle u,v\rangle\Big]_l^+(\alpha)-\Big[\langle z,w\rangle\Big]_l^+(\alpha)\Big\|+\frac{1}{4}\sup_{0<\alpha\le1}\Big\|\Big[\langle u,v\rangle\Big]_r^-(\alpha)-\Big[\langle z,w\rangle\Big]_r^-(\alpha)\Big\|\\&+\frac{1}{4}\sup_{0<\alpha\le1}\Big\|\Big[\langle u,v\rangle\Big]_l^-(\alpha)-\Big[\langle z,w\rangle\Big]_l^-(\alpha)\Big\|\end{aligned}$$

where $\|\ .\ \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Theorem 2.1 ([12]). *d_∞^n define a metric on IF_n.*

Theorem 2.2 ([12]). *The metric space (IF_n, d_∞^n) is complete.*

The norm $\| \quad \|$ of an intuitionistic fuzzy number $\langle u, v\rangle \in IF_n$ is defined by

$$\| \langle u, v\rangle \| = d_\infty^n(0_{(1,0)}, \langle u, v\rangle) = \| [\langle u, v\rangle]_0 \| = \frac{1}{2} \sup_{a \in [\langle u,v\rangle]_0} |a| + \frac{1}{2} \inf_{b \in [\langle u,v\rangle]^0} |b|$$

Definition 2.1. *An intuitionistic fuzzy set $\langle u, v\rangle$ is called convex intuitionistic fuzzy set if and only if u is convex fuzzy set and v is concave fuzzy set.*

The question that arises, is what IF_n with addition and multiplication by a scalar is a vector

Theorem 2.3 ([13])**.** *There exists a normed space X and a function $j : IF_n \longrightarrow X$ with properties:*

1. *j is an isometry i.e.* $\| j(\langle u, v\rangle) - j(\langle u', v'\rangle) \| = d_\infty^n(\langle u, v\rangle, \langle u', v'\rangle)$
2. $j(\langle u, v\rangle \oplus \langle u', v'\rangle) = j(\langle u, v\rangle) + j(\langle u', v'\rangle)$
3. $j(\lambda\langle u, v\rangle) = \lambda j(\langle u, v\rangle) \quad \lambda \geq 0$

Remark 2.1. *if $\langle u, v\rangle(t) : \ T \longrightarrow IF_n$ is differentiable at $t_0 \in T$, then*

$$(j\langle u, v\rangle)(t) = j(\langle u, v\rangle(t)) \quad T \longrightarrow X$$

is Frechet differentiable at t_0 and $(j\langle u, v\rangle)'(t_0) = j(\langle u, v\rangle'(t_0))$, where j is the embedding in Theorem 2.3.

In the following we list several comparison propositions on classical ordinary differential equations following [7]

Proposition 2.1. *Let $G \subset \mathbb{R}^2$ be an open set and $g \in C[G, \mathbb{R}]$, $(t_0, x_0) \in G$. Suppose $r(t)$ is the maximum solution to the initial value problem*

$$x' = g(t, x), \qquad x(t_0) = x_0 \tag{2.1}$$

and its largest interval of existence of right solution is $[t_0, t_0 + a)$. If $[t_0, t_1] \subset [t_0, t_0 + a)$, then there exists an $\varepsilon_0 > 0$ such that the maximum solution $r(t, \varepsilon)$ to the initial value problem

$$x' = g(t, x) + \varepsilon, \qquad x(t_0) = x_0 + \varepsilon$$

exists on $[t_0, t_1]$ whenever $0 < \varepsilon < \varepsilon_0$, and $r(t, \varepsilon)$ uniformly converges to $r(t)$ on $[t_0, t_1]$ as $\varepsilon \longrightarrow 0^+$

Proposition 2.2. *Let $G \subset \mathbb{R}^2$ be an open set, $g \in C[G, \mathbb{R}]$, $(t, x_0) \in G$. Suppose that the maximum solution to the initial value problem* (2.1) *is $r(t)$ and its largest interval of existence of right solution is $[t_0, t_0 + a)$. If $m(t) \in C[[t_0, t_0 + a), \mathbb{R}]$, satisfies $(t, m(t)) \in G$ for all $t \in [t_0, t_0 + a)$, $m(t_0) \leq x_0$, and*

$$Dm(t) \leq g(t, m(t)), \quad \forall t \in [t_0, t_0 + a) \backslash \Gamma$$

where D is one of the four Dini derivatives (see [7]), G at most is a countable set on t. Then we must have

$$m(t) \leq r(t), \quad \forall t \in [t_0, t_0 + a)$$

3 The relation between a solution and its approximate solution to intuitionistic fuzzy differential equations

Assume that $f : T \times W \longrightarrow IF_n$ is continuous (it is denoted by $f \in C[T \times W, IF_n]$). Consider the initial value problem

$$\langle u,v\rangle'(t) = f\big(t, (\langle u,v\rangle(t)\big), \qquad \langle u,v\rangle(t_0) = \langle u,v\rangle_0 \tag{3.1}$$

where $W \subset IF_n$, $\langle u,v\rangle(t_0) \in W$.

In the following we give the relation between a solution and its approximate solutions. We denote $R_0 = [t_0, t_0 + p] \times B\big(\langle u,v\rangle_0, q\big)$ where $p > 0$, $q > 0$, $\langle u,v\rangle_0 \in IF_n$, $B\big(\langle u,v\rangle_0, q\big) = \{\langle u,v\rangle \in IF_n \setminus d^n_\infty\Big(\langle u,v\rangle, \langle u,v\rangle_0\Big) \le q\}$

Theorem 3.1. *Let $f \in C[R_0, IF_n]$, $r \in (0,p)$, $\langle u,v\rangle_n \in C^1[[t_0, t_0 + r], B(\langle u,v\rangle_0, q)]$ such that*

$$\begin{aligned} j\langle u,v\rangle'_n(t) = jf\big(t, \langle u,v\rangle_n(t)\big) + B_n(t), \ \ \langle u,v\rangle(t_0) = \langle u,v\rangle_0, \\ \parallel B_n(t) \parallel \le \varepsilon_n \ \forall t \in [t_0, t_0 + r](n = 0, 1, 2, ...), \end{aligned} \tag{3.2}$$

where $\varepsilon_n > 0$, $\varepsilon_n \longrightarrow 0$, $B_n(t) \in C[[t_0, t_0 + r], X]$ and j is the isometric embedding from (IF_n, d^n_∞) onto its range in the Banach space X. For each $t \in [t_0, t_0 + r]$ there exist an $\delta(t) > 0$ such that H-differences $\langle u,v\rangle_n(t + h) \ominus \langle u,v\rangle_n(t)$ and $\langle u,v\rangle_n(t) \ominus \langle u,v\rangle_n(t - h)$ exist for all $0 \le h < \delta(t)$ and $n = 1, 2, ...$
if we have

$$d^n_\infty\Big(\langle u,v\rangle_n(t), \langle u,v\rangle(t)\Big) \longrightarrow 0 \quad \textit{u.c} \ \ \forall t \in [t_0, t_0 + r](n \longrightarrow \infty) \tag{3.3}$$

(u.c. denotes the uniform convergence), then $\langle u,v\rangle \in C^1\big[[t_0, t_0 + r], B(\langle u,v\rangle_0, q)\big]$

$$\langle u,v\rangle'(t) = f\big(t, \langle u,v\rangle(t)\big), \qquad \langle u,v\rangle(t_0) = \langle u_0, v_0\rangle, \quad t \in [t_0, t_0 + r]. \tag{3.4}$$

Proof 1. From (3.3) we know that $\langle u,v\rangle \in C[[t_0, t_0 + r], B(x_0, q)]$. For fixed $t_1 \in [t_0, t_0 + r]$ and any $t \in [t_0, t_0 + r]$, $t > t_1$, denote

$$F(t,n) = \frac{j\langle u,v\rangle_n(t) - j\langle u,v\rangle_n(t_1)}{t - t_1} - jf(t_1, \langle u,v\rangle_n(t_1)) - B_n(t_1)$$

It is well known that

$$\lim_{n \longrightarrow \infty} F(t,n) = \frac{j\langle u,v\rangle(t) - j\langle u,v\rangle(t_1)}{t - t_1} - jf\big(t_1, \langle u,v\rangle(t_1)\big) \tag{3.5}$$

From $f \in C^1[R_0, IF_n]$ is known that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$d^n_\infty\Big(f\big(t, \langle z,w\rangle(t)\big), f\big(t_1, \langle u,v\rangle(t_1)\big)\Big) < \frac{\varepsilon}{4} \tag{3.6}$$

whenever $t_1 < t < t_1 + \delta_1$ and $d^n_\infty\Big(\langle z,w\rangle(t), \langle u,v\rangle(t_1)\Big) < \delta_1$ with $\langle z,w\rangle \in B\big(\langle u,v\rangle_0, q\big)]$.
Take natural number $N > 0$ such hat

$$\varepsilon_n < \frac{\varepsilon}{4}, \quad d^n_\infty\Big(\langle u,v\rangle_n(t), \langle u,v\rangle(t)\Big) < \frac{\delta_1}{2} \quad \text{for any} \quad n > N, t \in [t_0, t_0 + r]. \tag{3.7}$$

Take $\delta > 0$ such that $\delta < \delta_1$ and

$$d_\infty^n\Big(\langle u,v\rangle(t), \langle u,v\rangle(t_1)\Big) < \frac{\delta_1}{2} \quad \text{whenever} \quad t_1 < t < t_1 + \delta. \tag{3.8}$$

By the definition of $F(t,n)$ and (3.2), we have

$$j\langle u,v\rangle_n(t) - j\langle u,v\rangle_n(t_1) - (t-t_1)j\langle u,v\rangle_n'(t_1) = (t-t_1)F(t,n) \tag{3.9}$$

We choose $\varphi \in X^*$ such that $\| \varphi \| = 1$ and

$$\begin{aligned}&\varphi(j\langle u,v\rangle_n(t) - j\langle u,v\rangle_n(t_1) - (t-t_1)j\langle u,v\rangle_n'(t_1)) \\ &\qquad = \| j\langle u,v\rangle_n(t) - j\langle u,v\rangle_n(t_1) - (t-t_1)j\langle u,v\rangle_n'(t_1) \|\end{aligned}$$

Let $\psi(t) = \varphi(j\langle u,v\rangle_n(t)) - (t-t_1)\varphi(j\langle u,v\rangle_n'(t_1))$, consequently,

$$\psi'(t) = \varphi(j\langle u,v\rangle_n(t)) - \varphi(j\langle u,v\rangle_n'(t_1))$$

hence

$$\begin{aligned} & \| j\langle u,v\rangle_n(t) - j\langle u,v\rangle_n(t_1) - (t-t_1)j\langle u,v\rangle_n'(t_1) \| = \psi(t) - \psi(t_1) = \psi'(\bar{t})(t-t_1) \\ = \;& \varphi(j\langle u,v\rangle_n'(\bar{t}) - j\langle u,v\rangle_n'(t_1))(t-t_1) \leq \| \varphi \| \,.\, \| j\langle u,v\rangle_n'(\bar{t}) - j\langle u,v\rangle_n'(t_1) \| \,.(t-t_1) \\ = \;& \| j\langle u,v\rangle_n'(\bar{t}) - j\langle u,v\rangle_n'(t_1) \| \,.(t-t_1) \end{aligned}$$

where $t_1 \leq \bar{t} \leq t$. In view of (3.9), we have

$$\| F(t,n) \| \leq \| j\langle u,v\rangle_n'(\bar{t}) - j\langle u,v\rangle_n'(t_1) \|, \quad t_1 \leq \bar{t} \leq t \tag{3.10}$$

From (3.7) and (3.8) we know that

$$d_\infty^n\Big(\langle u,v\rangle(\bar{t}), \langle u,v\rangle(t_1)\Big) < \frac{\delta_1}{2}$$

$$\begin{aligned} d_\infty^n\Big(\langle u,v\rangle_n(\bar{t}), \langle u,v\rangle(t_1)\Big) &\leq d_\infty^n\Big(\langle u,v\rangle_n(\bar{t}), \langle u,v\rangle(\bar{t})\Big) + d_\infty^n\Big(\langle u,v\rangle(\bar{t}), \langle u,v\rangle(t_1)\Big) \\ &< \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1 \end{aligned}$$

Hence by (3.6) and (3.10) we have

$$\begin{aligned} \| F(t,n) \| &\leq \| j\langle u,v\rangle_n'(\bar{t}) - j\langle u,v\rangle_n'(t_1) \| \\ &= \| jf\big(\bar{t}, \langle u,v\rangle_n(\bar{t})\big) + B_n(\bar{t}) - jf\big(t_1, \langle u,v\rangle_n(t_1)\big) - B_n(t_1) \| \\ &\leq \| jf\big(\bar{t}, \langle u,v\rangle_n(\bar{t})\big) - jf\big(t_1, \langle u,v\rangle(t_1)\big) \| \\ &\quad + \| jf\big(t_1, \langle u,v\rangle(t_1)\big) - jf\big(t_1, \langle u,v\rangle_n(t_1)\big) \| + 2\varepsilon_n \\ &= d_\infty^n\Big(f\big(\bar{t}, \langle u,v\rangle_n(\bar{t})\big), f\big(t_1, \langle u,v\rangle(t_1)\big)\Big) \\ &\quad + d_\infty^n\Big(f\big(t_1, \langle u,v\rangle(t_1)\big), f\big(t_1, \langle u,v\rangle_n(t_1)\big)\Big) + 2\varepsilon_n \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2\varepsilon_n < \varepsilon \end{aligned}$$

whenever $n > N$ and $t_1 < t < t_1 + \delta$.
Now let $n \longrightarrow \infty$, and applying Eq. (3.5), we have

$$\left\| \frac{j\langle u, v\rangle(t) - j\langle u, v\rangle(t_1)}{t - t_1} - jf(t_1, \langle u, v\rangle(t_1)) \right\| \leq \varepsilon, \quad t_1 < t < t_1 + \delta. \tag{3.11}$$

On the other hand, from the assumption of Theorem 3.1, there exists an $\delta(t_1) \in (0, \delta)$ such that the H-differences

$$\langle u, v\rangle_n(t) \ominus \langle u, v\rangle_n(t_1)$$

exist for all $t \in [t_1, t_1 + \delta(t_1)]$ and $n = 1, 2, ...$
Let $\langle z, w\rangle_n(t) = \langle u, v\rangle_n(t) \ominus \langle u, v\rangle_n(t_1)$. We verify that the intuitionistic fuzzy number-valued sequence $\{\langle z, w\rangle_n(t)\}$ uniformly converges on $[t_1, t_1 + \delta(t_1)]$
In fact, from the assumption $d_\infty^n(\langle u, v\rangle_n(t), \langle u, v\rangle(t)) \longrightarrow 0$, u.c $\forall t \in [t_0, t_0 + r]$, we know

$$\begin{aligned}
d_\infty^n\Big(\langle z, w\rangle_n(t), \langle z, w\rangle_m(t)\Big) &= d_\infty^n\Big(\langle z, w\rangle_n(t) + \langle u, v\rangle_n(t_1), \langle z, w\rangle_m(t) + \langle u, v\rangle_n(t_1)\Big) \\
&\leq d_\infty^n\Big(\langle u, v\rangle_n(t), \langle u, v\rangle_m(t)\Big) \\
&+ d_\infty^n\Big(\langle u, v\rangle_m(t), \langle z, w\rangle_m(t) + \langle u, v\rangle_n(t_1)\Big) \\
&= d_\infty^n\Big(\langle u, v\rangle_n(t), \langle u, v\rangle_m(t)\Big) \\
&+ d_\infty^n\Big(\langle z, w\rangle_m(t) + \langle u, v\rangle_m(t_1), \langle z, w\rangle_m(t) + \langle u, v\rangle_n(t_1)\Big) \\
&= d_\infty^n\Big(\langle u, v\rangle_n(t), \langle u, v\rangle_m(t)\Big) + d_\infty^n\Big(\langle u, v\rangle_m(t_1), \langle u, v\rangle_n(t_1)\Big) \\
&\longrightarrow 0 u.c \qquad \forall t \in [t_1, t_1 + \delta(t_1)](n, m \longrightarrow \infty)
\end{aligned}$$

Since (IF_n, d_∞^n) is complete, there exists an intuitionistic fuzzy number-valued mapping such that $\{\langle z, w\rangle_n(t)\}$ uniformly converges to $\langle z, w\rangle(t)$ on $[t_1, t_1 + \delta(t_1)]$ as $n \longrightarrow \infty$

In addition, we have

$$\begin{aligned}
d_\infty^n\Big(\langle u, v\rangle(t_1) + \langle z, w\rangle(t), \langle u, v\rangle(t)\Big) &\leq d_\infty^n\Big(\langle u, v\rangle(t_1) + \langle z, w\rangle(t), \langle u, v\rangle_n(t_1) + \langle z, w\rangle_n(t)\Big) \\
&+ d_\infty^n\Big(\langle u, v\rangle_n(t_1) + \langle z, w\rangle_n(t), \langle u, v\rangle(t)\Big) \\
&\leq d_\infty^n\Big(\langle u, v\rangle(t_1) + \langle z, w\rangle(t), \langle u, v\rangle(t_1) + \langle z, w\rangle_n(t)\Big) \\
&+ d_\infty^n\Big(\langle u, v\rangle(t_1) + \langle z, w\rangle_n(t), \langle u, v\rangle_n(t_1) + \langle z, w\rangle_n(t)\Big) \\
&+ d_\infty^n\Big(\langle u, v\rangle_n(t_1), \langle u, v\rangle(t)\Big) \\
&= d_\infty^n\Big(\langle z, w\rangle_n(t), \langle z, w\rangle(t)\Big) + d_\infty^n\Big(\langle u, v\rangle_n(t_1), \langle u, v\rangle(t_1)\Big) \\
&+ d_\infty^n\Big(\langle u, v\rangle_n(t), \langle u, v\rangle(t)\Big)
\end{aligned}$$

$\forall t \in [t_1, t_1 + \delta(t_1)]$
Let $n \longrightarrow \infty$. It follows that

$$\langle u, v\rangle(t_1) \oplus \langle z, w\rangle(t) \equiv \langle u, v\rangle(t) \quad \text{for all} \quad t \in [t_1, t_1 + \delta(t_1)].$$

Hence the H-differences $\langle u,v\rangle(t)\ominus\langle u,v\rangle(t_1)$ exist for all $t_1\in[t_1,t_1+\delta(t_1)]$
Thus from (3.11) we have

$$d_\infty^n\Big(\frac{\langle u,v\rangle(t)\ominus\langle u,v\rangle(t_1)}{t-t_1},f\big(t_1,\langle u,v\rangle(t_1)\big)\Big)\leq\varepsilon,\quad t_1<t\leq t_1+\delta(t_1)$$

So $\lim\limits_{t\to t_1^+}\frac{\langle u,v\rangle(t)\ominus\langle u,v\rangle(t_1)}{t-t_1}=f\big(t_1,\langle u,v\rangle(t_1)\big)$. Similarly, we have

$$\lim_{t\to t_1^-}\frac{\langle u,v\rangle(t)\ominus\langle u,v\rangle(t_1)}{t-t_1}=f\big(t_1,\langle u,v\rangle(t_1)\big)$$

Hence $\langle u,v\rangle'(t_1)$ exists and

$$\langle u,v\rangle'(t_1)=f\big(t_1,\langle u,v\rangle(t_1)\big)$$

From $t_1\in[t_0,t_0+r]$ is arbitrary, we know that Eq. (3.4) holds true and

$$\langle u,v\rangle\in C^1[[t_0,t_0+r],B(\langle u,v\rangle_0,q)].$$

Thus, we conclude the proof. □

Corollary 3.1. *If we replace condition* (3.2) *by*

$$\begin{aligned}j\langle u,v\rangle_{n+1}'(t)=jf\big(t,\langle u,v\rangle_n(t)\big)+B_n(t),\ \ \langle u,v\rangle(t_0)=\langle u,v\rangle_0,\\ \parallel B_n(t)\parallel\leq\varepsilon_n\ \ \forall t\in[t_0,t_0+r](n=0,1,...)\end{aligned}\tag{3.12}$$

and retain other assumptions, then the conclusions also hold true.

Proof 2. This is completely similar to the proof of Theorem 3.1. □

4 Existence and uniqueness theorem for a solution

Theorem 4.1. *Let*

(a) $f\in C[R_0,IF_n]$ *and* $d_\infty^n\Big(f(t,\langle u,v\rangle),0_{(1,0)}\Big)\leq M$ *for all* $(t,\langle u,v\rangle)\in R_0$.

(b) $g\in C[[t_0,t_0+p]\times[0,q],R]$, $g(t,0)\equiv 0$ *and* $g(t,x)\leq M_1$, *for all* $t\in[t_0,t_0+p]$, $0\leq x\leq q$ *such that* $g(t,x)$ *is nondecreasing on* x *(i.e,*$t_0\leq t\leq t_0+p$, $0\leq x_1\leq x_2\leq q\implies g(t,x_1)\leq g(t,x_2)$*), the initial value problem*

$$x'(t)=g(t,x(t)),\qquad x(t_0)=0\tag{4.1}$$

has only the solution $x(t)\equiv 0$ *on* $[t_0,t_0+p]$.

(c) $d_\infty^n\Big(f\big(t,\langle u,v\rangle\big),f\big(t,\langle u',v'\rangle\big)\Big)\leq g\Big(t,d_\infty^n\big(\langle u,v\rangle,\langle u',v'\rangle\big)\Big)$, *for all* $(t,\langle u,v\rangle),(t,\langle u',v'\rangle)\in R_0$, *and* $d_\infty^n\Big(\langle u,v\rangle,\langle u',v'\rangle\Big)\leq q$.

Then the Cauchy problem (3.4) *has a unique solution* $\langle u,v\rangle \in C^1[[t_0,t_0+r],B(x_0,q)]$ *on* $[t_0,t_0+r]$*, where* $r=\min\{p,q/M,q/M_1\}$*, and the successive iterations*

$$\langle u,v\rangle_0(t)=\langle u,v\rangle_0,\ \ \langle u,v\rangle_{n+1}(t)=\langle u,v\rangle_0\oplus\int_{t_0}^{t} f(s,\langle u,v\rangle_n(s))ds \qquad (n=0,1,2,...) \quad (4.2)$$

uniformly converge to $\langle u,v\rangle(t)$ *on* $[t_0,t_0+r]$.

Proof 3. from (4.2) and the assumption (a), by the inductive method we know

$$\begin{aligned} d_\infty^n\Big(\langle u,v\rangle_{n+1}(t),\langle u,v\rangle_0\Big) &\le \int_{t_0}^{t} d_\infty^n\Big(f\big(s,\langle u,v\rangle_n(s)\big),0_{(1,0)}\Big)ds \\ &\le q \ \ \forall t\in[t_0,t_0+r] \quad n=0,1,2,... \end{aligned} \quad (4.3)$$

Hence $\langle u,v\rangle_{n+1}\in C^1[[t_0,t_0+r],B(x_0,q)]$ and

$$\langle u,v\rangle_{n+1}'(t)=f\big(t,(\langle u,v\rangle_n(t)\big), \qquad \langle u,v\rangle_n(t_0)=\langle u,v\rangle_0 \ \ (n=0,1,2,...) \quad (4.4)$$

Let $M_2=\max\{M,M_1\}$. Then $r=\min\{p,q/M_2\}$. and we get the successive iterations as

$$\begin{cases} x_0(t)= & M_2(t-t_0) \quad t_0\le t\le t_0+r \\ x_{n+1}(t)= & \int_{t_0}^{t} g\big(s,x_n(s)\big), \quad t_0\le t\le t_0+r \ \ (n=0,1,2,...) \end{cases} \quad (4.5)$$

It is immediate that

$$x_1(t)=\int_{t_0}^{t} g(s,x_0(s))\le M_1(t-t_0)\le x_0(t)\le q, \quad \forall t\in[t_0,t_0+r] \quad (4.6)$$

So, by the inductive method and in view that $g(t,x)$ is nondecreasing on x, we have

$$0\le x_{n+1}(t)\le x_n(t)\le q, \quad \forall t\in[t_0,t_0+r] \ \ (n=0,1,2,...) \quad (4.7)$$

As $|x_{n+1}'(t)| = |g(t,x_n(t))| \le M_1$, from the Ascoli-Arzela theorem and (4.7) we know that $\{x_n(t)\}$ uniformly converges to some continuous function $x(t)$ on $[t_0,t_0+r]$ and

$$x(t)=\int_{t_0}^{t} g(s,x(s))ds.$$

Thus $x\in C^1[[t_0,t_0+r],[0,q]]$ and x is the solution the initial value problem (4.1). From assumption (b) we get $x(t)\equiv 0$. In addition, we have

$$\begin{aligned} & d_\infty^n\Big(\langle u,v\rangle_1(t),\langle u,v\rangle_0\Big)=d_\infty^n\Big(\int_{t_0}^{t} f\big(s,\langle u,v\rangle_0(s)\big)ds,0_{(1,0)}\Big) \\ & \le \int_{t_0}^{t} d_\infty^n\Big(f\big(s,\langle u,v\rangle_0(s)\big),0_{(1,0)}\Big)ds\le M(t-t_0)\le x_0(t) \end{aligned}$$

Suppose $d_\infty^n\Big(\langle u,v\rangle_k(t),\langle u,v\rangle_{k-1}\Big)\le x_{k-1}(t)$, then by the assumption (c) , we have

$$\begin{aligned} & d_\infty^n\Big(\langle u,v\rangle_{k+1}(t),\langle u,v\rangle_k(t)\Big)=d_\infty^n\Big(\int_{t_0}^{t} f\big(s,\langle u,v\rangle_k(s)\big)ds,\int_{t_0}^{t} f\big(s,\langle u,v\rangle_{k-1}(s)\big)ds\Big) \\ & \le \int_{t_0}^{t} d_\infty^n\Big(f\big(s,\langle u,v\rangle_k(s)\big),f\big(s,\langle u,v\rangle_{k-1}(s)\big)\Big)ds\le \int_{t_0}^{t} g\Big(s,d_\infty^n\big(\langle u,v\rangle_k(s),\langle u,v\rangle_{k-1}(s)\big)\Big)ds \\ & \le \int_{t_0}^{t} g\Big(s,x_{k-1}(s)\Big)ds=x_k(t) \end{aligned}$$

Thus by the inductive method we know

$$d_\infty^n\Big(\langle u,v\rangle_{n+1}(t),\langle u,v\rangle_n(t)\Big) \ \le\ x_n(t) \quad t_0\le t\le t_0+r\ \ (n=0,1,2,...). \tag{4.8}$$

So, we have

$$\begin{aligned} d_\infty^n\Big(\langle u,v\rangle'_{n+1}(t),\langle u,v\rangle'_n(t)\Big) &= d_\infty^n\Big(f\big(t,\langle u,v\rangle_n(t)\big),f\big(t,\langle u,v\rangle_{n-1}(t)\big)\Big) \\ &\le g\Big(t,d_\infty^n\big(\langle u,v\rangle_n(t),\langle u,v\rangle_{n-1}(t)\big)\Big) \\ &\le g\big(t,x_{n-1}(t)\big) \end{aligned} \tag{4.9}$$

Assume $m\ge n$, and in view of (4.9) and (4.7) we get

$$\begin{aligned} d_\infty^n\Big(\langle u,v\rangle'_n(t),\langle u,v\rangle'_m(t)\Big) &\le d_\infty^n\Big(f\big(t,\langle u,v\rangle_{n-1}(t)\big),f\big(t,\langle u,v\rangle_n(t)\big)\Big) \\ &+ d_\infty^n\Big(f\big(t,\langle u,v\rangle_n(t)\big),f\big(t,\langle u,v\rangle_m(t)\big)\Big) \\ &+ d_\infty^n\Big(f\big(t,\langle u,v\rangle_m(t)\big),f\big(t,\langle u,v\rangle_{m-1}(t)\big)\Big) \\ &\le g\big(t,x_{n-1}(t)\big)+g\big(t,d_\infty^n\big(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t)\big)\big)+g\big(t,x_{m-1}(t)\big) \\ &\le 2g\big(t,x_{n-1}(t)\big)+g\big(t,d_\infty^n\big(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t)\big)\big) \end{aligned}$$

Furthermore, from

$$\begin{aligned} & d_\infty^n\Big(\langle u,v\rangle_n(t+h),\langle u,v\rangle_m(t+h))\Big) \\ &\le d_\infty^n\Big(\langle u,v\rangle_n(t+h),\langle u,v\rangle_m(t+h)-\langle u,v\rangle_m(t)+\langle u,v\rangle_n(t))\Big) \\ &+ d_\infty^n\Big(\langle u,v\rangle_m(t+h)\ominus\langle u,v\rangle_m(t)+\langle u,v\rangle_n(t)),\langle u,v\rangle_m(t+h))\Big) \\ &= d_\infty^n\Big(\langle u,v\rangle_n(t+h)\ominus\langle u,v\rangle_n(t),\langle u,v\rangle_m(t+h)\ominus\langle u,v\rangle_m(t))\Big) \\ &+ d_\infty^n\Big(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t))\Big) \end{aligned}$$

we deduce that

$$\begin{aligned} & D^+d_\infty^n\Big(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t))\Big) \\ &= \overline{\lim_{h\to0^+}}\frac{d_\infty^n(\langle u,v\rangle_n(t+h),\langle u,v\rangle_m(t+h)))-d_\infty^n(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t))}{h} \\ &\le \overline{\lim_{h\to0^+}}\frac{d_\infty^n\Big(\langle u,v\rangle_n(t+h)\ominus\langle u,v\rangle_n(t),\langle u,v\rangle_m(t+h))\ominus\langle u,v\rangle_m(t)\Big)}{h} \\ &= d_\infty^n\Big(\langle u,v\rangle'_n(t),\langle u,v\rangle'_m(t))\Big) < 2g\big(t,x_{n-1}(t)\big)+g\big(t,d_\infty^n(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t))\big) \end{aligned}$$

Since $g\big(t,x_{n-1}(t)\big)$ uniformly converges to 0, then for arbitrary $\varepsilon>0$ there exists a natural number N such that

$$D^+d_\infty^n\Big(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t))\Big) < g\big(t,d_\infty^n(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t))\big)+\varepsilon \qquad \forall m\ge n>N$$

Here D^+ is the Dini derivative (see [7]). From the fact that $d_\infty^n\Big(\langle u,v\rangle_n(t_0),\langle u,v\rangle_m(t_0)\Big)=0<\varepsilon$ and by proposition 2.1, we have

$$d_\infty^n\Big(\langle u,v\rangle_n(t),\langle u,v\rangle_m(t)\Big)\le w(t,\varepsilon)\quad \forall\in[t_0,t_0+r]\quad \forall m\ge n>N \tag{4.10}$$

where $w(t,\varepsilon)$ is the maximum solution to the initial value problem

$$x'(t)=g(t,x(t))+\varepsilon,\qquad x(t_0)=\varepsilon \tag{4.11}$$

By proposition 2.2 we know that $w(t,\varepsilon)$ uniformly converges to the maximum solution $x(t)\equiv 0$ of problem (4.1) on $t_0\le t\le t_0+r$ as $\varepsilon\to 0$.
Thus, according to (4.10) and that (IF_n,d_∞^n) is complete, we know that there exists an intuitionistic fuzzy set-valued mapping $\langle u,v\rangle : T\longrightarrow IF_n$ such that $d_\infty^n\Big(\langle u,v\rangle_n(t),\langle u,v\rangle(t)\Big)$ uniformly converges to 0 as $n\longrightarrow\infty$. Applying (4.4) and Corollary (3.1) we have $\langle u,v\rangle\in C^1[[t_0,t_0+r],B(\langle u,v\rangle_0,q)]$ and $\langle u,v\rangle(t)$ is the solution of the initial value problem (3.4).
Finally, we prove the uniqueness. Suppose $\langle z,w\rangle(t)$ is another solution of initial value problem (3.4). Let

$$m(t)=d_\infty^n\Big(\langle u,v\rangle(t),\langle z,w\rangle(t)\Big)$$

Then $m(t_0)\equiv 0$

$$D^+m(t)\le d_\infty^n\Big(\langle u,v\rangle'(t),\langle z,w\rangle'(t)\Big)=d_\infty^n(f\big(t,\langle u,v\rangle(t)\big),f\big(t,\langle z,w\rangle(t)\big))\le g(t,m(t)).$$

Hence from proposition 2.2 we know

$$d_\infty^n\Big(\langle u,v\rangle(t),\langle z,w\rangle(t)\Big)\le x(t)\equiv 0,\quad \forall\in[t_0,t_0+r]$$

where $x(t)\equiv 0$ is the maximum solution of problem (4.1) on $[t_0,t_0+r]$.
Therefore $\langle u,v\rangle(t)=\langle z,w\rangle(t)$. □

Corollary 4.1. *Let $f\in C[R_0,IF_n]$ such that $d_\infty^n\Big(f\big(t,\langle u,v\rangle\big),0\Big)\le M$ for all $(t,\langle u,v\rangle)\in R_0$ and f satisfies the Lipschitz condition*

$$d_\infty^n\Big(f\big(t,\langle u,v\rangle\big),f\big(t,\langle u',v'\rangle\big)\Big)\le Ld_\infty^n\Big(\langle u,v\rangle,\langle u',v'\rangle\Big),\qquad \forall(t,\langle u,v\rangle),(t,\langle u,v\rangle)\in R_0$$

where L is a constant. Then the Cauchy problem (3.4) *has an unique solution $\langle u,v\rangle\in C^1[[t_0,t_0+r],B(x_0,q)]$ on $[t_0,t_0+r]$, where $r=\min\{p,q/M,1/L\}$, and the successive iterations* (4.2) *uniformly converge to $\langle u,v\rangle(t)$ on $[t_0,t_0+r]$.*

Proof 4. In the proof of Theorem (4.1), taking $g(t,x)=L.x$ we then obtain the proof of Corollary 4.1, where $M_1=L.q$, hence $r=\min\{p,q/M,1/L\}$. □

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