Notes on Intuitionistic Fuzzy Sets
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On the degree of intuitionistic fuzzy functions and its various classification degrees

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Abstract: In this paper, we explore the concept of degree of the intuitionistic fuzzy functions. In [4], Demirci studied gradations of fuzzy functionhood. There, for a fuzzy relation f on $X \times Y$, considering the fuzzy equalities E_X on X and E_Y on Y the degree of f of being a fuzzy fuction, being surjective, being injective and being bijective is defined. We extend this study to intuitionistic fuzzy functions. In this paper, we use intuitionistic fuzzy functions and their types defined by Lim, Choi and Hur [7] by using intuitionistic fuzzy equalities. Since an intuitionistic fuzzy function is a $(\mu_A(x, y), \nu_A(x, y))$ ordered pair, we define its degree of being (α) and the degree of non-being (β) by using $(\alpha, \beta) \in L_*$. For an intuitionistic fuzzy relation f from $X \times Y$ to I^2, considering the intuitionistic fuzzy equalities E_X on X and E_Y on Y, we define the degree to which f is an intuitionistic fuzzy function, the degree of it being surjective, injective and bijective, respectively. We especially analyze the degrees of some types of intuitionistic fuzzy functions. We prove some theorems using these definitions.

Keywords: Intuitionistic fuzzy equality, Intuitionistic fuzzy function, Gradation of intuitionistic fuzzy functions.
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1 Introduction

The concept of fuzzy sets was introduced by Zadeh [9] in 1965. Then, Rosenfeld [8] extended this definition to fuzzy groups in 1971. For crisp sets X and Y, considering two fuzzy equalities E_X, E_Y on X and Y, respectively, the concept of fuzzy function and its several types were studied by Demirci [3]. As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Intuitionistic fuzzy relation structures were created by Bustince and Burillo [2].

For crisp sets X and Y, Demirci [4] defined several notions of the degree for fuzzy functions based on fuzzy relations f on $X \times Y$. Intuitionistic fuzzy equalities on X and Y are given in Lim, Choi and Hur's sense [7]. Kang, Lee and Hur [5] studied intuitionistic fuzzy mappings and intuitionistic fuzzy equivalence relations.

Many researchers extended the topics of fuzzy algebra to intuitionistic algebra. In this paper, we study intuitionistic fuzzy functions, define the gradation of the intuitionistic fuzzy function and the gradation of the surjective function, the gradation of the injective function, and the gradation of the bijective function. We used the gradations $\alpha, \beta \in [0, 1]$ as α-degree of being and β-degree of non-being conditions, the condition $0 \leq \alpha + \beta \leq 1$ is satisfied. We prove some propositions and theorems with the definitions we stated.

2 Basic definitions and theorems

In this section, we list some basic notions and results which are needed in the later sections. In this paper, X and Y always stand for two crisp non-empty sets.

Definition 2.1. [3] A relation $E_X : X \times X \longrightarrow I$ is called a fuzzy equality on X if and only if the following conditions are satisfied: $\forall x, \forall y, \forall z \in X$

(E.1) $E_X(x, y) = 1 \Leftrightarrow x = y$;

(E.2) $E_X(x, y) = E_X(y, x)$;

(E.3) $E_X(x, y) \wedge E_X(y, z) \leq E_X(x, z)$.

Let E be a fuzzy equality on X and let $x, y \in X$. Then, we interpret the value $E_X(x, y)$ as the grade of "x and y are nearly equal". We will denote the set of all fuzzy equalities on X as $E(X)$.

Definition 2.2. [3] Let E_X and E_Y be two fuzzy equalities on X and Y,respectively. For a fuzzy relation f on $X \times Y$, to introduce the definition of various types of fuzzy function is defined.

(F1) $\forall x \in X, \exists y \in Y$ such that $\mu_f(x, y) > 0$,

(F2) $\forall x, \forall y \in X, \forall z, \forall w \in Y$, $\mu_f(x, z) \wedge \mu_f(y, w) \wedge E_X(x, y) \leq E_Y(z, w)$,

(SF) $\forall y \in Y, \exists x \in X$ such that $\mu_f(x, y) > 0$,

(SF1) $\forall x \in X, \exists y \in Y$ such that $\mu_f(x, y) = 1$,

(SSF) $\forall y \in Y, \exists x \in X$ such that $\mu_f(x, y) = 1$,

(IF) $\forall x, \forall y \in X, \forall z, \forall w \in Y$, $\mu_f(x, z) \wedge \mu_f(y, w) \wedge E_Y(z, w) \leq E_X(x, y)$.

Definition 2.3. [4] Let E_X and E_Y be two fuzzy equalities on X and Y, respectively. Then, for a fuzzy subset f of $X \times Y$,

(i) f is a fuzzy function if and only if conditions (F1) and (F2) are satisfied,

(ii) f is a strong fuzzy function if and only if f holds (SF1) and (F2),

(iii) f is surjective fuzzy function if and only if f holds (SF),

(iv) f is strong surjective fuzzy function if and only if f holds (SSF),

(v) f is injective fuzzy function if and only if f holds (IF),

(vi) f is bijective fuzzy function if and only if f is surjective and injective,

(vii) f is strong bijective fuzzy function if and only if f is strong surjective and injective.

Definition 2.4. [4] Let f be a fuzzy relation on $X \times Y$, E_X and E_Y fuzzy equalities on X and Y, respectively. Let us define the following crisp sets and numbers:

(a) $F2(f) = \{\beta \in I : (\forall x, \forall z \in X)(\forall y, \forall w \in Y)$
$(\mu_f(x, y) \wedge \mu_f(z, w) \wedge E_X(x, z) \geq \beta \Rightarrow E_Y(y, w) \geq \beta)\}$,

(b) $f1(f) = \bigwedge_{x \in X} \bigvee_{y \in Y} \mu_f(x, y)$,

(c) $f2(f) = \bigvee F2(f)$,

(d) $F(f) = f1(f) \wedge f2(f)$,

(e) $I(f) = \{\beta \in I : (\forall x, \forall z \in X)(\forall y, \forall w \in Y)$
$(\mu_f(x, y) \wedge \mu_f(z, w) \wedge E_Y(y, w) \geq \beta \Rightarrow E_X(x, z) \geq \beta)\}$,

(f) $i(f) = \bigvee I(f)$,

(g) $s(f) = \bigwedge_{y \in Y} \bigvee_{x \in X} \mu_f(x, y)$,

(h) $b(f) = i(f) \wedge s(f)$.

We call the real numbers $F(f)$, $i(f)$, $s(f)$ and $b(f)$ "degrees of f of being a fuzzy function, injectively degree of f, surjectively degree of f and bijectively degree of f", respectively.

Definition 2.5. [1] A complex mapping $A : X \to I^2$ is called an intuitionistic fuzzy set (in short, IFS) in X if for each $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ define the degree of membership and the degree of non-membership of each $x \in X$ to A, respectively.

Definition 2.6. [1] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$, be two IFSs on a set X. Then:

(i) $A \subseteq B$ if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$, for all $x \in X$,

(ii) $A = B$ if $\mu_A = \mu_B$ and $\nu_A = \nu_B$, for all $x \in X$,

(iii) $A \cap B = \{\langle x, \mu_A \wedge \mu_B, \nu_A \vee \nu_B \rangle : x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A \vee \mu_B, \nu_A \wedge \nu_B \rangle : x \in X\}$,

(v) $A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle : x \in X\}$,

(vi) $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X\}$,

(vii) $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle : x \in X\}$.

Definition 2.7. [2] A complex mapping $R = (\mu_R, \nu_R) : X \times Y \to I \times I$ is called an intuitionistic fuzzy relation (in short, IFR) from X to Y if $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for each $(x, y) \in X \times Y$, i.e., $R \in IFS(X \times Y)$.

Definition 2.8. [2] Let $R \in IFR(X \times Y)$ and $S \in IFR(Y \times Z)$.

(a) R^{-1} is called the inverse of R if $R^{-1}(y, x) = R(x, y), \forall x \in X, \forall y \in Y$.

(b) The sub-min composition of R and S, denoted by $S \circ R$, is an intuitionistic fuzzy relation from X to Z defined by: For each $(x, z) \in X \times Y$,

$$\mu_{R \circ S}(x, z) = \bigvee_{y \in Y} [\mu_R(x, y) \wedge \mu_S(y, z)]$$

and

$$\nu_{R \circ S}(x, z) = \bigwedge_{y \in Y} [\nu_R(x, y) \vee \nu_S(y, z)].$$

Definition 2.9. [3] Let X be a nonempty set and let $IE_X = (\mu_{IE_X}, \nu_{IE_X}) \in IFR(X)$. Then IE_X is called an intuitionistic fuzzy equality (in short IE) on X if it satisfies the following conditions: $\forall x, y, z \in X$

(IE.1) $IE_X(x, y) = (1, 0) \Leftrightarrow x = y, \forall x, y \in X$,

(IE.2) $IE_X(x, y) = IE_X(y, x), \forall x, y \in X$,

(IE.3) $\mu_{IE_X}(x, y) \wedge \mu_{IE_X}(y, z) \leq \mu_{IE_X}(x, z)$ and $\nu_{IE_X}(x, y) \vee \nu_{IE_X}(y, z) \geq \nu_{IE_X}(x, z)$.

We will denote the set of all intuitionistic fuzzy equalities on X as $IE(X)$.

Definition 2.10. [3] For any two nonempty sets X and Y, let IE_X and IE_Y be two intuitionistic fuzzy equalities on X and Y, respectively. Let $f \in IFS(X \times Y)$. Then, f is called an intuitionistic fuzzy mapping (in short, IFM) from X to Y with respect to. $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$, denoted by $f : X \to Y$, if it satisfies the following conditions:

(IF1*) $\forall x \in X, \exists y \in Y$ such that $\mu_f(x, y) > 0$ and $\nu_f(x, y) < 1$,

(IF2*) $\forall x, y \in X, \forall z, w \in Y$, $\mu_f(x, z) \wedge \mu_f(y, w) \wedge \mu_{IE_X}(x, y) \leq \mu_{IE_Y}(z, w)$ and
$\nu_f(x, z) \vee \nu_f(y, w) \vee \nu_{IE_X}(x, y) \geq \nu_{IE_Y}(z, w)$.

Definition 2.11. [2] For sets X and Y, let $f : X \to Y$ be an intuitionistic fuzzy mapping from X to Y with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then f is said to be:

(a) strong if $\forall x \in X, \exists y \in Y$ such that $f(x, y) = (1, 0)$,

(b) surjective if $\forall y \in Y, \exists x \in X$ such that $\mu_f(x, y) > 0$ and $\nu_f(x, y) < 1$,

(c) strong surjective if $\forall y \in Y, \exists x \in X$ such that $f(x, y) = (1, 0)$,

(d) injective if $\mu_f(x, z) \wedge \mu_f(y, w) \wedge \mu_{IE_Y}(z, w) \leq \mu_{IE_X}(x, y)$ and

$$\nu_f(x, z) \vee \nu_f(y, w) \vee \nu_{IE_Y}(z, w) \geq \nu_{IE_X}(x, y), \forall x, y \in X, \forall z, w \in Y,$$

(e) bijective if it is surjective and injective,

(f) strong bijective if it is strong surjective and injective.

Definition 2.12. [7] Let $f : X \to Y$ be an IFM with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then, f is said to be invertible if the intuitionistic fuzzy relation f^{-1} on $Y \times X$ is an IFM $f^{-1} : Y \to X$ with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$.

Theorem 2.13. [3] *Let $f : X \to Y$ be an IFM with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then, f is invertible if and only if f is bijective.*

Definition 2.14. [6] Let $L_* = \{(\alpha, \beta) :$ for arbitrary $\alpha, \beta \in [0, 1], \alpha + \beta \leq 1\}$. For any (α_1, β_1), $(\alpha_2, \beta_2) \in L_*$, the orders $\leq$ and $<$ on L_* are defined as:

i) $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2) \Leftrightarrow \alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$;

ii) $(\alpha_1, \beta_1) < (\alpha_2, \beta_2) \Leftrightarrow (\alpha_1, \beta_1) \leq (\alpha_2, \beta_2) \Leftrightarrow (\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ and $\alpha_1 \leq \alpha_2$ or $\beta_1 > \beta_2$.

3 Gradation of an intuitionistic fuzzy function

Definition 3.1. For crisp sets X and Y, let $f : X \times Y \to I^2$ be an intuitionistic fuzzy mapping from with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then f is said to be:

(a) $(IF1)$ $\forall x \in X, \exists y \in Y$ such that $\mu_f(x, y) \geq 0$ and $\forall y \in Y, \exists x \in X$ such that $\nu_f(x, y) \leq 1$,

(b) $(IF2)$ $\forall x, y \in X, \forall z, w \in Y$, $\mu_f(x, z) \wedge \mu_f(y, w) \wedge \mu_{IE_X}(x, y) \leq \mu_{IE_Y}(z, w)$ and $\nu_f(x, z) \vee \nu_f(y, w) \vee \nu_{IE_X}(x, y) \geq \nu_{IE_Y}(z, w)$,

(c) (SIF) surjective if $\forall y \in Y, \exists x \in X$ such that $\mu_f(x, y) \geq 0$ and $\forall x \in X, \exists y \in Y$ such that $\nu_f(x, y) \leq 1$,

(d) $(SIF1)$ strong if $\forall x \in X, \exists y \in Y$ such that $\mu_f(x, y) = 1$ and $\forall y \in Y, \exists x \in X$ such that $\nu_f(x, y) = 0$,

(e) $(SSIF)$ strong surjective if $\forall y \in Y, \exists x \in X$ such that $\mu_f(x, y) = 1$ and $\forall x \in X, \exists y \in Y$ such that $\nu_f(x, y) = 0$,

(f) (IIF) injective if for $\forall x, y \in X, \forall z, w \in Y$, $\mu_f(x, z) \wedge \mu_f(y, w) \wedge \mu_{IE_Y}(z, w) \leq \mu_{IE_X}(x, y)$ and $\nu_f(x, z) \vee \nu_f(y, w) \vee \nu_{IE_Y}(z, w) \geq \nu_{IE_X}(x, y)$.

Definition 3.2. Let f is an intuitionistic fuzzy relation, $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then f is said to be:

(i) f is an intuitionistic fuzzy function if and only if conditions $(IF1)$ and $(IF2)$ are satisfied,

(ii) f is a strong intuitionistic fuzzy function if and only if f satisfies $(SIF1)$ and $(IF2)$,

(iii) f is surjective intuitionistic fuzzy function if and only if f satisfies (SIF),

(iv) f is strong surjective intuitionistic fuzzy function if and only if f satisfies $(SSIF)$,

(v) f is injective intuitionistic fuzzy function if and only if f satisfies (IIF),

(vi) f is bijective intuitionistic fuzzy function if and only if f is surjective and injective,

(vii) f is strong bijective intuitionistic fuzzy function if and only if f is strong surjective and injective.

Definition 3.3. Let f be an intutionistic fuzzy relation on $X \times Y$, E_X and E_Y intuitionistic fuzzy equalities on X and Y, respectively. Let us define the following crisp sets and numbers:

(a) $F2(f) = \{(\alpha, \beta) \in L_* : (\forall x, \forall z \in X)(\forall y, \forall w \in Y)$
$(\mu_f(x,y) \wedge \mu_f(z,w) \wedge \mu_{IE_X}(x,z) \geq \alpha \Rightarrow \mu_{IE_Y}(y,w) \geq \alpha)$ and
$(\nu_f(x,y) \vee \nu_f(z,w) \vee \nu_{IE_X}(x,z) \leq \beta \Rightarrow \nu_{IE_Y}(y,w) \leq \beta)\}$.

(b) $f1(f) = (\bigwedge_{x\in X}\bigvee_{y\in Y} \mu_f(x,y), \bigvee_{x\in X}\bigwedge_{y\in Y} \nu_f(x,y))$,

(c) $f2(f) = (\bigvee F2(f)(\alpha), \bigwedge F2(f)(\beta))$,

(d) $F(f) = ((\bigwedge_{x\in X}\bigvee_{y\in Y} \mu_f(x,y)) \wedge \vee F2(f)(\alpha), (\bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x,y)) \vee \wedge F2(f)(\beta))$,

(e) $I(f) = \{(\alpha, \beta) \in L_* : (\forall x, \forall z \in X)(\forall y, \forall w \in Y)$
$(\mu_f(x,y) \wedge \mu_f(z,w) \wedge \mu_{IE_Y}(y,w) \geq \alpha \Rightarrow E_X(x,z) \geq \alpha)$ and
$(\nu_f(x,y) \vee \nu_f(z,w) \vee \nu_{IE_Y}(y,w) \leq \beta \Rightarrow \nu_{IE_X}(x,z) \leq \beta)\}$,

(f) $i(f) = (\bigvee I(f)(\alpha), \bigwedge I(f)(\beta))$,

(g) $s(f) = (\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_f(x,y), \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x,y))$,

(h) $b(f) = (\vee I(f)(\alpha) \wedge (\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_f(x,y)), \wedge I(f)(\beta) \vee (\bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x,y)))$.

We call the real numbers $F(f)$, $i(f)$, $s(f)$ and $b(f)$ "degree of f of fuzzy function, injectively degree of f, surjectively degree of f and bijectively degree of f", respectively.

Theorem 3.4. *Let f be an intuitionistic fuzzy relation on $X \times Y \longrightarrow I^2$, IE_X and IE_Y intuitionistic fuzzy equalities on X and Y, respectively. The following statements are true:*

(i) If $f1(f) = (1, 0)$, then f satisfies condition $(IF1)$,

(ii) *If f satisfies condition $(SIF1)$, then $f1(f) = (1, 0)$,*

(iii) *f satisfies condition $(IF2)$ if and only if $f2(f) = (1, 0)$,*

(iv) *f satisfies condition (IIF) if and only if $i(f) = (1, 0)$,*

(v) *f satisfies condition $(SSIF)$ if and only if $s(f) = (1, 0)$.*

Proof.

(i) Some equalities from Definition 3.1 and Definition 3.3 are used.
Let $f1(f) = (\bigwedge_{x\in X}\bigvee_{y\in Y}\mu_f(x,y), \bigvee_{y\in Y}\bigwedge_{x\in X}\nu_f(x,y)) = (1,0)$.
$\bigwedge_{x\in X}\bigvee_{y\in Y}\mu_f(x,y) = 1$ and $\bigvee_{y\in Y}\bigwedge_{x\in X}\nu_f(x,y) = 0$
for $\forall x \in X$, $\bigvee_{y\in Y}\mu_f(x,y) = 1$ and $\exists y \in Y$ such that $\mu_f(x,y) = 1$
for $\forall y \in Y$, $\bigwedge_{x\in X}\nu_f(x,y) = 0$ and $\exists x \in X$ such that $\nu_f(x,y) = 0$.
Since $\mu_f(x,y) = 1 > 0$ and $\nu_f(x,y) = 0 < 1$, f satisfies $(IF1)$.

(ii) If $\forall x \in X, \exists y \in Y$ such that $\mu_f(x,y) = 1 \Rightarrow \bigwedge_{x\in X}\bigvee_{y\in Y}\mu_f(x,y) = 1$ and if $\forall y \in Y, \exists x \in X$ such that $\nu_f(x,y) = 0 \Rightarrow \bigvee_{y\in Y}\bigwedge_{x\in X}\nu_f(x,y)) = 0$,then, $f1(f) = (1,0)$.

(iii) <u>$(\Rightarrow)$</u>: Let f satisfies condition $(IF2)$: $\forall x, y \in X, \forall z, w \in Y$ $\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \leq \mu_{IE_Y}(z,w)$ and $\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y) \geq \nu_{IE_Y}(z,w)$.
For each $(\alpha,\beta) \in L_*, \alpha \leq \mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \Rightarrow \alpha \leq \mu_{IE_Y}(z,w)$ and $\beta \geq \nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y) \Rightarrow \beta \geq \nu_{IE_Y}(z,w)$. If for $z, w \in Y$ be $z = w$, then $\alpha \leq 1$ and $\beta \geq 0$. Similarly, if for $x, y \in X$ be $x = y$, then $\alpha \leq 1$ and $\beta \geq 0$. So $(1,0) \in F2(f)$ and $(\alpha_i, \beta_i) \in F2(f)(i = 1, 2, \ldots, n)$, then

$$f2(f) = (1 \vee \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n, 0 \wedge \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n) = (1,0).$$

<u>$(\Leftarrow)$</u>: Let $f2(f) = (1,0)$. If $(1,0) \in F2(f), \mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \geq 1$ and $\mu_{IE_Y}(z,w) \geq 1$, then $\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \geq \mu_{IE_Y}(z,w), \nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,w) \leq 0$ and $\nu_{IE_Y}(z,w) \leq 0$, then $\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,w) \leq \nu_{IE_Y}(z,w)$.
Therefore, f satisfies condition $(IF2)$.

(iv) <u>$(\Rightarrow)$</u>: If for $\forall x, y \in X, \forall z, w \in Y$, $\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_Y}(z,w) \leq \mu_{IE_X}(x,y)$ and $\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_Y}(z,w) \geq \nu_{IE_X}(x,y)$.
For each $(\alpha,\beta) \in L_*$, $\alpha \leq \mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_Y}(z,w) \leq \mu_{IE_X}(x,y)$ and $\beta \geq \nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_Y}(z,w) \geq \nu_{IE_X}(x,y)$.
$I(f) = (\alpha,\beta)$ exists. If for $x, y \in X$ be $x = y$, then $\alpha \leq 1$ and $\beta \geq 0$. Similarly if for $z, w \in Y$ be $z = w$, then $\alpha \leq 1$ and $\beta \geq 0$. If $(\alpha_i, \beta_i) \in I(f), i = \{0, 1, 2, \ldots, n\}$ and $(1,0) \in I(f)$, then $i(f) = (1 \vee \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n, 0 \wedge \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n) = (1,0)$.

<u>$(\Leftarrow)$</u>: Let $i(f) = (1,0)$. If $\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_Y}(z,w) \geq 1$ and $\mu_{IE_X}(x,y) \geq 1$, then $\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_Y}(z,w) = \mu_{IE_X}(x,y) = 1$. If $\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_Y}(z,w) \leq 0$ ve $\nu_{IE_X}(x,y) \leq 0$, then $\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_Y}(z,w) = \nu_{IE_X}(x,y) = 0$. f satisfies condition (IIF).

(v) Let $s(f) = (\bigwedge_{y\in Y}\bigvee_{x\in X}\mu_f(x,y), \bigvee_{y\in Y}\bigwedge_{x\in X}\nu_f(x,y)) = (1,0)$

($\Leftrightarrow$) $\bigwedge_{y\in Y}\bigvee_{x\in X}\mu_f(x,y) = 1$ and $\bigvee_{y\in Y}\bigwedge_{x\in X}\nu_f(x,y) = 0$

($\Leftrightarrow$) If $\forall y \in Y$, $\bigvee_{x\in X}\mu_f(x,y) = 1$, then $\exists x \in X, \mu_f(x,y) = 1$ and

If $\forall x \in X$, $\bigwedge_{x\in X}\nu_f(x,y) = 0$, then $\exists y \in Y$, $\nu_f(x,y) = 0$

($\Leftrightarrow$) f satisfies condition $(SSIF)$.

This completes the proof. □

Theorem 3.5. *Let f be an intuitionistic fuzzy relation on $X \times Y \longrightarrow I^2$, IE_X and IE_Y are intuitionistic fuzzy equalities on X and Y, respectively. The following assertions hold true:*

(i) f is an IFM $\Rightarrow F(f) = f1(f)$,

(ii) f is a strong IFM $\Rightarrow F(f) = (1,0)$,

(iii) A strong IFM f is injective $\Rightarrow F(f) \wedge i(f) = (1,0)$,

(iv) A strong IFM f is strong surjective $\Rightarrow F(f) \wedge s(f) = (1,0)$,

(v) $\bigvee_{x,y\in X,z,w\in Y}\{\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y)\} \leq 1$ *and*
$\bigwedge_{x,y\in X,z,w\in Y}\{\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y)\} \geq 0 \Rightarrow F(f) = f1(f)$,

(vi) $\bigvee_{x,y\in X,z,w\in Y}\{\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_Y}(z,w)\} \leq 1$ *and*
$\bigwedge_{x,y\in X,z,w\in Y}\{\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_Y}(z,w)\} \geq 0 \Rightarrow i(f) = (1,0)$,

(vii) f is a strong bijective IFM $\Rightarrow b(f) = (1,0)$.

Proof. (i) Since f is an intuitionistic fuzzy function, conditions $(IF1)$ and $(IF2)$ are satisfied.
Let $(\alpha,\beta) \in L_*$, $F(f) = (\alpha,\beta)$,
$\alpha \leq \mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \leq \mu_{IE_Y}(z,w)$ and
$\beta \geq \nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y) \geq \nu_{IE_Y}(z,w)$,
$\alpha \leq \mu_f(x,z)$ and $\nu_f(x,z) \geq \beta$,
$\alpha \leq \bigvee_{y\in Y}\mu_f(x,z)$ and $\beta \geq \bigwedge_{x\in X}\nu_f(x,z)$,
$\bigwedge_{x\in X}\bigvee_{y\in Y}\mu_f(x,y) = \alpha$ and $\bigvee_{y\in Y}\bigwedge_{x\in X}\nu_f(x,y) = \beta$,
$f1(f) = (\alpha,\beta) = F(f)$.

(ii) Since f is a strong IFM, f satisfies conditions $(IF2)$ and $(SIF1)$. From Theorem 3.6 (iii), $f2(f) = (\bigvee F2(f)(\alpha), \bigwedge F2(f)(\beta)) = (1,0)$. f satisfies condition $(SIF1) \Rightarrow f1(f) = (1,0)$. Then $F(f) = (1 \wedge 1, 0 \vee 0) = (1,0)$.

(iii) f satisfies the conditions $(IF2)$, (IIF) and $(SIF1)$. If f satisfies condition $(IF2) \Rightarrow f2(f) = (1,0)$ and f satisfies condition $(SIF1) \Rightarrow f1(f) = (1,0)$, then $F(f) = (1,0)$. f satisfies condition $(IIF) \Rightarrow i(f) = (1,0)$. Then $F(f) \wedge i(f) = (1,0)$.

(iv) f satisfies conditions $(IF2)$, $(SSIF)$ and $(SIF1)$. From (ii) with $F(f) = (1,0)$ and if f satisfies condition $(SSIF)$, then $s(f) = (1,0)$.
Therefore $F(f) \wedge s(f) = (1 \wedge 1, 0 \vee 0) = (1,0)$.

(v) $\forall x,y \in X, \forall z,w \in Y$, $\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \leq 1$ and $\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y) \geq 0$. Since $\mu_f(x,z) > 0$ and $\nu_f(x,z) < 1$, is holds $(IF1)$ $\mu_{IE_Y}(z,w) \leq 1$ and $\nu_{IE_Y}(z,w) \geq 0 \Rightarrow (1,0) \in F2(f) \Rightarrow f2(f) = (1,0) \Rightarrow$ is holds $(IF2)$. Therefore f is an IFM $\Rightarrow$ from (i) with $F(f) = f1(f)$.

(vi) $\forall x,y \in X, \forall z,w \in Y$, $\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_Y}(z,w) \leq 1$ and $\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_Y}(z,w) \geq 0$. $\mu_{IE_X}(x,y) \leq 1$ and $\nu_{IE_X}(x,y) \geq 0 \Rightarrow (1,0) \in I(f) \Rightarrow i(f) = (1,0)$.

(vii) Since f is a strong bijective IFM, then $SSIF$ and IIF are satisfied. From (iv) with $i(f) = (1,0)$ and from (v) with $s(f) = (1,0) \Rightarrow b(f) = (1 \wedge 1, 0 \vee 0) = (1,0)$. □

Example 3.6. Let $X = \{0,1,2,3\}$ and let $IE_X = (\mu_{IE_X}, \nu_{IE_X})$ be an intuitionistic fuzzy equality in X defined by: for each $x,y \in X$,

$$IE_X(x,y) = \begin{cases} (1,0), & \text{if } x = y, \\ \left(\frac{x+y}{5}, 1 - \frac{x+y}{5}\right), & \text{if } x \neq y \end{cases}$$

Let $Y = \{0,1\}$ and let $IE_Y = (\mu_{IE_Y}, \nu_{IE_Y})$ be an intuitionistic fuzzy equality in Y defined by: for each $x,y \in Y$,

$$IE_Y(x,y) = \begin{cases} (1,0), & \text{if } x = y, \\ (0,1), & \text{if } x \neq y \end{cases}$$

Let $f : X \times Y \to I^2$, $f(x,z) = (\mu_f(x,z), \nu_f(x,z))$ be an intuitionistic fuzzy function defined by: for any $x \in X$ and $y \in Y$, $\mu_f(x,z) = 1 - \frac{x.y}{10}$ and $\nu_f(x,z) = \frac{x.y}{10}$.

If f is a surjective intuitionistic fuzzy mapping, then $F(f) = (1,0)$ and $s(f) = (0.7, 0.3$ by the following calculations.

With this example, we can verify the Theorem 3.6, item(i) and Theorem 3.6, items (i), (ii) and (iii).

Theorem 3.7. *Let f be a strong (or surjective, strong surjective, injective, bijective, strong bijective) IFM with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. Then $\Box f$ and $\Diamond f$ are strong [surjective, strong surjective, injective, bijective, strong bijective] IFMs with respect to the intuitionistic fuzzy equalities $\Box IE_X$ and $\Box IE_Y$, $\Diamond IE_X$ and $\Diamond IE_Y$ on X and Y, respectively. Then:*

(i) $F(f) = (\alpha,\beta) \Rightarrow F(\Box f) = (\alpha, 1-\alpha)$ *and* $F(\Diamond f) = (1-\beta, \beta)$,

(ii) $i(f) = (\alpha,\beta) \Rightarrow i(\Box f) = (\alpha, 1-\alpha)$ *and* $i(\Diamond f) = (1-\beta, \beta)$,

(iii) $s(f) = (\alpha,\beta) \Rightarrow s(\Box f) = (\alpha, 1-\alpha)$ *and* $s(\Diamond f) = (1-\beta, \beta)$,

(iv) $b(f) = (\alpha,\beta) \Rightarrow b(\Box f) = (\alpha, 1-\alpha)$ *and* $b(\Diamond f) = (1-\beta, \beta)$.

Proof.

(iii) Let $s(f) = (\alpha, \beta) = (\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_f(x, y), \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x, y))$.

$$s(\Box f) = (\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_{\Box f}(x, y), \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_{\Box f}(x, y)) = (\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_f(x, y), \bigvee_{y\in Y}\bigwedge_{x\in X} (1 - \mu_f(x, y)))$$

$\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_f(x, y) = \alpha$ is obvious.

$$\bigvee_{y\in Y}\bigwedge_{x\in X} (1 - \mu_f(x, y)) = \bigvee_{y\in Y} [1 - \bigvee_{x\in X} \mu_f(x, y)] = 1 - \bigwedge_{y\in Y}\bigvee_{x\in X} \mu_f(x, y) = 1 - \alpha.$$

$$s(\Diamond f) = (\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_{\Diamond f}(x, y), \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_{\Diamond f}(x, y)) = (\bigwedge_{y\in Y}\bigvee_{x\in X} (1 - \nu_f(x, y)), \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x, y))$$

$\bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x, y) = \beta$ is obvious.

$$\bigwedge_{y\in Y}\bigvee_{x\in X} (1 - \nu_f(x, y) = \bigwedge_{y\in Y} [1 - \bigwedge_{x\in X} \nu_f(x, y)] = 1 - \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x, y) = 1 - \beta.$$

Therefore, $s(\Box f) = (\alpha, 1 - \alpha)$ and $s(\Diamond f) = (1 - \beta, \beta)$.

The rest are proved similarly. □

Theorem 3.8. *Let $f = (\mu_f, \nu_f)$ be an IFM with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. $f^* = (\mu_{f^*}, \nu_{f^*}) = (1 - \nu_f, 1 - \mu_f)$ IFM with respect to intuitionistic fuzzy equalities IE^*_X and IE^*_Y on X and Y, respectively. Then:*

(i) $F(f) = (\alpha, \beta) \Rightarrow F(f^*) = (1 - \beta, 1 - \alpha)$,

(ii) $i(f) = (\alpha, \beta) \Rightarrow i(f^*) = (1 - \beta, 1 - \alpha)$,

(iii) $s(f) = (\alpha, \beta) \Rightarrow s(f^*) = (1 - \beta, 1 - \alpha)$,

(iv) $b(f) = (\alpha, \beta) \Rightarrow b(f^*) = (1 - \beta, 1 - \alpha)$.

Proof.

(iii)
$$\begin{aligned} s(f^*) &= (\bigwedge_{y\in Y}\bigvee_{x\in X} \mu_{f^*}(x, y), \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_{f^*}(x, y)) \\ &= (\bigwedge_{y\in Y}\bigvee_{x\in X} \{1 - \nu_f(x, y)\}, \bigvee_{y\in Y}\bigwedge_{x\in X} \{1 - \mu_f(x, y)\}) \\ &= (1 - \bigvee_{y\in Y}\bigwedge_{x\in X} \nu_f(x, y), 1 - \bigwedge_{y\in Y}\bigvee_{x\in X} \mu_f(x, y)) \\ &= (1 - \beta, 1 - \alpha). \end{aligned}$$

The rest are proved similarly. □

Theorem 3.9. *Let $f = (\mu_f, \nu_f)$ is be an IFM with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. $f^c = (\mu_{f^c}, \nu_{f^c}) = (\nu_f, \mu_f)$ is an IFM with respect to intuitionistic fuzzy equalities IE^c_X and IE^c_Y on X and Y, respectively. If (α, β) is unique such that $f2(f) = (\alpha, \beta) \in L_*$, then $f2(f^c) = (\beta, \alpha)$. Similarly, $f1(f^c)$, $i(f^c)$, $s(f^c)$, $b(f^c)$ and $F(f^c)$ are obtained.*

Proof. Let $f2(f) = (\alpha, \beta)$ and $f2(f^c) = (\gamma, \delta)$. If f is an IFM, then

$$\alpha \leq \mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \leq \mu_{IE_Y}(z,w) \leq \mu_f(x,z) \vee \mu_f(y,w) \vee \mu_{IE_X}(x,y)$$

and

$$\beta \geq \nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y) \geq \nu_{IE_Y}(z,w) \geq \nu_f(x,z) \wedge \nu_f(y,w) \wedge \nu_{IE_X}(x,y).$$

$\alpha \leq \mu_f(x,z) \vee \mu_f(y,w) \vee \mu_{IE_X}(x,y) \leq \gamma$ and $\alpha \leq \mu_{IE_Y}(z,w) \leq \gamma$,
$\alpha \leq \nu_{f^c}(x,z) \vee \nu_{f^c}(y,w) \vee \nu_{IE_{X^c}}(x,y) \leq \gamma$ and $\alpha \leq \nu_{IE_{Y^c}}(z,w) \leq \gamma$,
$\beta \geq \nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y) \geq \delta$ and $\beta \geq \nu_{IE_Y}(z,w) \geq \delta$,
$\beta \geq \mu_{f^c}(x,z) \vee \mu_{f^c}(y,w) \vee \mu_{IE_{X^c}}(x,y) \geq \delta$ and $\beta \geq \mu_{IE_{Y^c}}(z,w) \geq \delta$.
$\alpha \leq \gamma$ and $\beta \geq \delta$, then $(\alpha, \beta) \leq (\gamma, \delta)$.

$$f2(f^c) = (\bigvee F2(f^c)(\gamma), \bigwedge F2(f^c)(\delta)) = (\bigvee F2(f)(\delta), \bigwedge F2(f)(\gamma)) = (\beta, \alpha).$$

Therefore $\alpha = \delta$ and $\beta = \gamma$, i.e., $f2(f) = (\alpha, \beta)$ and $f2(f^c) = (\beta, \alpha)$.
The rest are obtained similarly. □

Theorem 3.10. *Let $f = (\mu_f, \nu_f)$ be a strong bijective or bijective IFM with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. $f^{-1} : Y \times X \longrightarrow I^2$, $f^{-1} = (\mu_{f^{-1}}, \nu_{f^{-1}})$, $\mu_{f^{-1}}(y,x) = \mu_f(x,y)$ and $\nu_{f^{-1}}(y,x) = \nu_f(x,y)$ a strong bijective or bijective IFM with respect to $IE_X \in IE(X)$ and $IE_Y \in IE(Y)$. f^{-1} is called the inverse of f intuitionistic fuzzy function. f and f^{-1} have equal degrees. Then,*

(i) $F(f) = F(f^{-1})$,

(ii) $i(f) = i(f^{-1})$,

(iii) $s(f) = s(f^{-1})$,

(iv) $b(f) = b(f^{-1})$.

Proof.

(iii) $s(f) = (\alpha, \beta) = (\bigwedge_{y \in Y} \bigvee_{x \in X} \mu_f(x,y), \bigvee_{y \in Y} \bigwedge_{x \in X} \nu_f(x,y))$ and

$s(f^{-1}) = (\bigwedge_{x \in X} \bigvee_{y \in Y} \mu_{f^{-1}}(y,x), \bigvee_{x \in X} \bigwedge_{y \in Y} \nu_{f^{-1}}(y,x))$.
If $\mu_f(x,y) = \mu_{f^{-1}}(y,x)$ and $\nu_f(x,y) = \nu_{f^{-1}}(y,x)$, then

$$(\bigwedge_{x \in X} \bigvee_{y \in Y} \mu_{f^{-1}}(y,x), \bigvee_{x \in X} \bigwedge_{y \in Y} \nu_{f^{-1}}(y,x)) = (\bigwedge_{y \in Y} \bigvee_{x \in X} \mu_f(x,y), \bigvee_{y \in Y} \bigwedge_{x \in X} \nu_f(x,y)),$$

i.e., $s(f) = s(f^{-1})$.

Similarly to (iii), items (i), (ii) and (iv) are proved by the definitions of $F(f)$, $i(f)$ and $b(f)$, respectively. □

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