Notes on Intuitionistic Fuzzy Sets
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Cartesian product of intuitionistic fuzzy subgroups

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Abstract: Fuzzy sets have become fundamental tools for addressing uncertainty and ambiguity across a wide range of scientific disciplines. A significant development within fuzzy set theory is the emergence of fuzzy subgroups, which adapt fuzzy set principles to the context of group theory. This research introduces the Cartesian product of two (or more) intuitionistic fuzzy subgroups and analyzes the characterization of this Cartesian product within the framework of intuitionistic fuzzy subgroups. Furthermore, we investigated various properties of this concept, aiming to make a significant contribution to the understanding of the interplay between these concepts and thus paving the way for new applications in fuzzy algebra and related fields.
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1 Introduction

The concept of fuzzy sets, introduced by Zadeh [20] in 1965, has become a cornerstone in addressing uncertainty and ambiguity across various disciplines. A significant development stemming from this theory is the emergence of fuzzy subgroups, proposed by Rosenfeld [17] in 1971. This concept adapts the principles of fuzzy sets to the context of groups, enabling a more profound analysis of algebraic structures.

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Subsequently, in 1983, Atanassov [5,6] introduced intuitionistic fuzzy sets, which generalize the concept of fuzzy sets. Atanassov presented several operations, including the Cartesian product between two intuitionistic fuzzy sets. This concept provides greater flexibility in modelling uncertainty and expands the applicability of fuzzy set theory.

In 1990, Biswas [10] developed the concept of anti-fuzzy subgroups, offering a novel perspective by focusing on the degree of non-membership of elements within subgroups. This concept, along with the ideas introduced by Rosenfeld and Atanassov, led to the development of intuitionistic fuzzy subgroups by Biswas [11] in 1997. Intuitionistic fuzzy subgroups combine elements from fuzzy subgroups and intuitionistic fuzzy sets, creating a richer framework for analyzing group structures.

The Cartesian product in the context of intuitionistic fuzzy sets has attracted attention in several previous studies. Deschrijver and Kerre [12] discussed the Cartesian product in intuitionistic fuzzy sets in general, while Palanivelrajan *et al.* [16] examined the Cartesian product of two intuitionistic fuzzy number subgroups. However, the approaches used in these studies remain limited in terms of both scope and depth of analysis.

Other studies, such as those conducted by Abdullah *et al.* [1], Biswas [11], Li *et al.* [13], Oqla Massa'deh [15], and Yuan *et al.* [19], have made significant contributions to the development of intuitionistic fuzzy subgroup theory. Nevertheless, none have comprehensively addressed the Cartesian product of two intuitionistic fuzzy subgroups using a broader approach.

This study aims to address this gap by providing a novel perspective. It not only discusses the Cartesian product of two intuitionistic fuzzy subgroups but also examines the converse of relevant theorems and explores additional properties that have not been addressed in previous literature. Thus, this research provides a novel contribution to the understanding of the structure and characteristics of Cartesian products within the framework of intuitionistic fuzzy subgroups.

2 Preliminaries

In this section, we present the fundamental definitions and preliminary results that will be used throughout the subsequent sections.

Definition 2.1. [2] *A group $(G, \star)$ is a nonempty set G closed under a binary operation $\star$ such that the following axioms are satisfied:*

(1) ***(Associativity)*** *For every $a, z, w \in G$*

$$a \star (z \star w) = (a \star z) \star w.$$

(2) ***(Identity)*** *There exists an element $e \in G$, called the identity element of the group, such that*

$$z \star e = e \star z = z$$

for every $z \in G$.

(3) ***(Inverse)*** *For every element $z \in G$, there exists an element $z^{-1} \in G$, called the inverse of z, such that*

$$z \star z^{-1} = z^{-1} \star z = e.$$

From this point forward, the notation $z \star w$ will be abbreviated to zw.

Theorem 2.2. [2] *Let G be a group and let K be a subset of G. Then K is a subgroup of G if and only if K is nonempty and $zw^{-1} \in K$ for every $z, w \in K$.*

In the following, we present the Cartesian product of two groups as induced in Abdurrahman [3], which is then generalized to the Cartesian product of n groups.

Theorem 2.3. *Establishes the Cartesian product of two groups. $(G_1, \star_1)$ and $(G_2, \star_2)$ is itself a group under the component-wise operation defined by*

$$(a, z) \star (d, w) := (a \star_1 d, z \star_2 w)$$

for all $(a, z), (d, w) \in G_1 \times G_2$.

The conclusions of Theorem 2.3 can be extended to groups $(G_1, \star_1), (G_2, \star_2), \ldots, (G_n, \star_n)$, which leads to the result that $(G_1 \times G_2 \times \cdots \times G_n, \star)$ is a structure that forms a group.

Definition 2.4. [4, 10] *A fuzzy subset μ of a non-empty set G is defined as a function μ from G to the closed interval $[0, 1]$. The complement of μ, denoted by μ^c, is defined as $\mu^c(a) := 1 - \mu(a)$ for every $a \in G$.*

Definition 2.5. [17] *A fuzzy subset μ of a group G is called a fuzzy subgroup of G if and only if, for every $a, c \in G$, we have:*

$$\mu(ac) \geq \mu(a) \wedge \mu(c) \quad \textit{and} \quad \mu(a^{-1}) \geq \mu(a).$$

Theorem 2.6. [17] *Let μ be a fuzzy subgroup of G. Then $\mu(a^{-1}) = \mu(a)$ and $\mu(a) \leq \mu(e)$ for all $a \in G$, where e is the identity element of G.*

Definition 2.7. [10] *A fuzzy subset ν of a group G is called an anti-fuzzy subgroup of G if and only if, for every $a, c \in G$, the following conditions hold:*

$$\nu(ac) \leq \nu(a) \vee \nu(c) \quad \textit{and} \quad \nu(a^{-1}) \leq \nu(a).$$

Theorem 2.8. [18] *Suppose ν is an anti-fuzzy subgroup of the group G. Then for every element a in G, we have $\nu(a^{-1}) = \nu(a)$ and $\nu(a) \geq \nu(e)$, where e denotes the identity element of G.*

Theorem 2.9. [17] *A fuzzy subset μ of group G is a fuzzy subgroup of G if and only if, for every $a, c \in G$, the following conditions hold:*

$$\mu(ac^{-1}) \geq \mu(a) \wedge \mu(c).$$

Theorem 2.10. [10, 14] *A fuzzy subset ν of group G is an anti-fuzzy subgroup of G if and only if the condition*

$$\nu(ac^{-1}) \leq \nu(a) \vee \nu(c)$$

is satisfied for all $a, c \in G$.

Definition 2.11. [9,12] *Let G be a non-empty set. An intuitionistic fuzzy subset $\mathcal{A}$ of G is defined as a set of ordered triples*

$$\mathcal{A} = \{(a, \mu_{\mathcal{A}}(a), \nu_{\mathcal{A}}(a)) \mid a \in G\},$$

where $\mu_{\mathcal{A}} : G \longrightarrow [0,1]$ represents the degree of membership and $\nu_{\mathcal{A}} : G \longrightarrow [0,1]$ represents the degree of non-membership of the element $a \in G$, satisfying the condition

$$0 \leq \mu_{\mathcal{A}}(a) + \nu_{\mathcal{A}}(a) \leq 1, \quad \textit{for all } a \in G.$$

Thus, each intuitionistic fuzzy set consists of three components: the element a, its membership degree $\mu_{\mathcal{A}}(a)$, and its non-membership degree $\nu_{\mathcal{A}}(a)$. For simplicity, we denote this as $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$. Clearly, any standard fuzzy set can be expressed in the form:

$$\{(a, \mu_{\mathcal{A}}(a), 1 - \mu_{\mathcal{A}}(a)) \mid a \in G\}.$$

3 Main result

In the following, we present the definition of an intuitionistic fuzzy subgroup of the group G, as referred to in [11,16].

Definition 3.1. *An intuitionistic fuzzy subset $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ of group G is an intuitionistic fuzzy subgroup of G if and only if for all $a, c \in G$:*

(1) $\mu_{\mathcal{A}}(ac) \geq \mu_{\mathcal{A}}(a) \wedge \mu_{\mathcal{A}}(c)$,

(2) $\mu_{\mathcal{A}}(a^{-1}) \geq \mu_{\mathcal{A}}(a)$,

(3) $\nu_{\mathcal{A}}(ac) \leq \nu_{\mathcal{A}}(a) \vee \nu_{\mathcal{A}}(c)$, *and*

(4) $\nu_{\mathcal{A}}(a^{-1}) \leq \nu_{\mathcal{A}}(a)$.

Through the integration of Definition 3.1 with Theorems 2.9 and 2.10, we deduce that an intuitionistic fuzzy subset $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ of a group G qualifies as an intuitionistic fuzzy subgroup of G if and only if $\mu_{\mathcal{A}}$ is a fuzzy subgroup of G and $\nu_{\mathcal{A}}$ is an anti-fuzzy subgroup of G. The following theorem encapsulates this derived condition.

Theorem 3.2. *An intuitionistic fuzzy subset $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ of group G is an intuitionistic fuzzy subgroup of G if and only if for all $a, c \in G$:*

$$\mu_{\mathcal{A}}(ac^{-1}) \geq \mu_{\mathcal{A}}(a) \wedge \mu_{\mathcal{A}}(c) \quad \textit{and} \quad \nu_{\mathcal{A}}(ac^{-1}) \leq \nu_{\mathcal{A}}(a) \vee \nu_{\mathcal{A}}(c).$$

Proof. The proof of Theorem 3.2 is straightforward by connecting the conditions outlined in Theorem 2.9, Theorem 2.10, and Definition 3.1. □

Remark 1. *It follows from Theorem 3.2, 2.9, and 2.10 that $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ is an intuitionistic fuzzy subgroup of group G if and only if $\mu_{\mathcal{A}}$ is a fuzzy subgroup of group G and $\nu_{\mathcal{A}}$ is an anti-fuzzy subgroup of group G.*

Theorem 3.3. *Suppose that $\mathcal{A} = (\mu_\mathcal{A}, \mu^c_\mathcal{A})$ is an intuitionistic fuzzy subset of a group G. If $\mu_\mathcal{A}$ is a fuzzy subgroup of G, then $\mathcal{A}$ is an intuitionistic fuzzy subgroup of G.*

Proof. Assume that $\mu_\mathcal{A}$ is a fuzzy subgroup of G. Thus, for all $a, c \in G$:

$$\begin{aligned}\mu^c_\mathcal{A}(ac^{-1}) &= 1 - \mu_\mathcal{A}(ac^{-1})\\ &\le 1 - (\mu_\mathcal{A}(a) \wedge \mu_\mathcal{A}(c))\\ &= (1 - \mu_\mathcal{A}(a)) \vee (1 - \mu_\mathcal{A}(c))\\ &= \mu^c_\mathcal{A}(a) \vee \mu^c_\mathcal{A}(c)\end{aligned}$$

It follows that $\mu^c_\mathcal{A}$ is an anti-fuzzy subgroup of G. Therefore, $\mathcal{A} = (\mu_\mathcal{A}, \mu^c_\mathcal{A})$ is an intuitionistic fuzzy subgroup of G. □

Theorem 3.4. *Consider $\mathcal{A} = (\nu^c_\mathcal{A}, \nu_\mathcal{A})$ as an intuitionistic fuzzy subset of the group G. If $\nu_\mathcal{A}$ is an anti-fuzzy subgroup of G, it follows that $\mathcal{A}$ is an intuitionistic fuzzy subgroup of G.*

Proof. Let $\nu_\mathcal{A}$ be an anti-fuzzy subgroup of G. Consequently, for all $a, c \in G$:

$$\begin{aligned}\nu^c_\mathcal{A}(ac^{-1}) &= 1 - \nu_\mathcal{A}(ac^{-1})\\ &\ge 1 - (\nu_\mathcal{A}(a) \vee \nu_\mathcal{A}(c))\\ &= (1 - \nu_\mathcal{A}(a)) \wedge (1 - \nu_\mathcal{A}(c))\\ &= \nu^c_\mathcal{A}(a) \wedge \nu^c_\mathcal{A}(c).\end{aligned}$$

It can be concluded that $\nu^c_\mathcal{A}$ is a fuzzy subgroup of G. Consequently, $\mathcal{A} = (\nu^c_\mathcal{A}, \nu_\mathcal{A})$ is an intuitionistic fuzzy subgroup of G. □

After establishing the conditions for an intuitionistic fuzzy group, we now turn our attention to the Cartesian product of type $\times_4$, as defined by Atanassov [6–8]. This type of Cartesian product has also been studied by several researchers, including Deschrijver and Kerre [12], as well as Palanivelrajan *et al.* [16]. Hereafter, we denote this Cartesian product by the symbol $\otimes$.

Definition 3.5. *Let $\mathcal{A} = (\mu_\mathcal{A}, \nu_\mathcal{A})$ and $\mathcal{B} = (\mu_\mathcal{B}, \nu_\mathcal{B})$ be an intuitionistic fuzzy subset of the group G. The Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \otimes \mathcal{B}$, is defined as:*

$$\mathcal{A} \otimes \mathcal{B} := \{((a, c), \mu_\mathcal{A} \times \mu_\mathcal{B}(a, c), \nu_\mathcal{A} \times \nu_\mathcal{B}(a, c)) \mid (a, c) \in G \times G\}$$

where $\mu_\mathcal{A} \times \mu_\mathcal{B}(a, c) := \mu_\mathcal{A}(a) \wedge \mu_\mathcal{B}(c)$ and $\nu_\mathcal{A} \times \nu_\mathcal{B}(a, c) := \nu_\mathcal{A}(a) \vee \nu_\mathcal{B}(c)$. For simplicity, we denote this as $\mathcal{A} \otimes \mathcal{B} = (\mu_\mathcal{A} \times \mu_\mathcal{B}, \nu_\mathcal{A} \times \nu_\mathcal{B})$.

Theorem 3.6. *The Cartesian product $\mathcal{A} \otimes \mathcal{B} = (\mu_\mathcal{A} \times \mu_\mathcal{B}, \nu_\mathcal{A} \times \nu_\mathcal{B})$ of two intuitionistic fuzzy subgroups $\mathcal{A} = (\mu_\mathcal{A}, \nu_\mathcal{A})$ and $\mathcal{B} = (\mu_\mathcal{B}, \nu_\mathcal{B})$ of a group G is an intuitionistic fuzzy subgroup of the group $G \times G$.*

Proof. Let $\mathcal{A} = (\mu_\mathcal{A}, \nu_\mathcal{A})$ and $\mathcal{B} = (\mu_\mathcal{B}, \nu_\mathcal{B})$ be intuitionistic fuzzy subgroups of G. Then, by Theorem 3.2, we have that $\mu_\mathcal{A}$ and $\mu_\mathcal{B}$ are fuzzy subgroups of G, while $\nu_\mathcal{A}$ and $\nu_\mathcal{B}$ are anti-fuzzy subgroups of G. Since $G \times G$ is a group, for every $(x, z) \in G \times G$, there exists $(x, z)^{-1} \in G \times G$ such that

$$(x, z)(x, z)^{-1} = (e, e) = (xx^{-1}, zz^{-1}) = (x, z)(x^{-1}, z^{-1}).$$

Consequently, based on the cancellation property in the group $G \times G$, it follows that

$$(x, z)^{-1} = (x^{-1}, z^{-1}).$$

Therefore, for all $(a, c) \in G \times G$:

$$\begin{aligned}
\mu_{\mathcal{A}} \times \mu_{\mathcal{B}} \left(((a, c)(x, z)^{-1})\right) &= \mu_{\mathcal{A}} \times \mu_{\mathcal{B}} \left(((a, c)(x^{-1}, z^{-1}))\right) \\
&= \mu_{\mathcal{A}} \times \mu_{\mathcal{B}} \left((ax^{-1}, cz^{-1})\right) \\
&= \mu_{\mathcal{A}}(ax^{-1}) \wedge \mu_{\mathcal{B}}(cz^{-1}) \\
&\geq (\mu_{\mathcal{A}}(a) \wedge \mu_{\mathcal{A}}(x)) \wedge (\mu_{\mathcal{B}}(c) \wedge \mu_{\mathcal{B}}(z)) \\
&= (\mu_{\mathcal{A}}(a) \wedge \mu_{\mathcal{B}}(c)) \wedge (\mu_{\mathcal{A}}(x) \wedge \mu_{\mathcal{B}}(z)) \\
&= \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a, c) \wedge \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(x, z)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{\mathcal{A}} \times \nu_{\mathcal{B}} \left(((a, c)(x, z)^{-1})\right) &= \nu_{\mathcal{A}} \times \nu_{\mathcal{B}} \left((ax^{-1}, cz^{-1})\right) \\
&= \nu_{\mathcal{A}}(ax^{-1}) \vee \nu_{\mathcal{B}}(cz^{-1}) \\
&\leq (\nu_{\mathcal{A}}(a) \vee \nu_{\mathcal{A}}(x)) \vee (\nu_{\mathcal{B}}(c) \vee \nu_{\mathcal{B}}(z)) \\
&= (\nu_{\mathcal{A}}(a) \vee \nu_{\mathcal{B}}(c)) \vee (\nu_{\mathcal{A}}(x) \vee \nu_{\mathcal{B}}(z)) \\
&= \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a, c) \vee \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(x, z).
\end{aligned}$$

Thus, $\mathcal{A} \otimes \mathcal{B} = (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \times \nu_{\mathcal{B}})$ is an intuitionistic fuzzy subgroup of the group $G \times G$. $\square$

A consequence of the conditions of Theorem 3.6 is the conclusion that the Cartesian product $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n$ is an intuitionistic fuzzy subgroup of the group $\underbrace{G \times G \times \cdots \times G}_{n \text{ times}}$, provided that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are intuitionistic fuzzy subgroups of a group G.

Corollary 3.7. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be intuitionistic fuzzy subgroups of a group G, where each $\mathcal{A}_i$ is defined as $\mathcal{A}_i = (\mu_{\mathcal{A}_i}, \nu_{\mathcal{A}_i})$. Then, the Cartesian product $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n$ is an intuitionistic fuzzy subgroup of the group $\underbrace{G \times G \times \cdots \times G}_{n \text{ times}}$, where the operation is defined as*

$$(\mu_{\mathcal{A}_1} \times \mu_{\mathcal{A}_2} \times \cdots \times \mu_{\mathcal{A}_n}, \nu_{\mathcal{A}_1} \times \nu_{\mathcal{A}_2} \times \cdots \times \nu_{\mathcal{A}_n}).$$

Theorems 3.3, 3.4, and 3.6 imply the following corollaries regarding intuitionistic fuzzy subgroups $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ of group G:

Corollary 3.8. *$\mathcal{C} = (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}, (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}})^c)$ is an intuitionistic fuzzy subgroup of $G \times G$.*

Corollary 3.9. *$\mathcal{D} = ((\nu_{\mathcal{A}} \times \nu_{\mathcal{B}})^c, \nu_{\mathcal{A}} \times \nu_{\mathcal{B}})$ is an intuitionistic fuzzy subgroup of $G \times G$.*

Similarly, as a result of Theorem 3.3, Theorem 3.4, and Corollary 3.7, we have the following consequence under the condition that $\mathcal{A}_i = (\mu_{\mathcal{A}_i}, \nu_{\mathcal{A}_i})$ Are intuitionistic fuzzy subgroups of G where $i = 1, 2, \ldots, n$.

Corollary 3.10. *Let* $K = (\mu_{\mathcal{A}_1} \times \mu_{\mathcal{A}_2} \times \cdots \times \mu_{\mathcal{A}_n}, (\mu_{\mathcal{A}_1} \times \mu_{\mathcal{A}_2} \times \cdots \times \mu_{\mathcal{A}_n})^c)$. *Then* K *is an intuitionistic fuzzy subgroup of* $\underbrace{G \times G \times \cdots \times G}_{n \text{ times}}$.

Corollary 3.11. *Let* $D = ((\nu_{\mathcal{B}_1} \times \nu_{\mathcal{B}_2} \times \cdots \times \nu_{\mathcal{B}_n})^c, \nu_{\mathcal{B}_1} \times \nu_{\mathcal{B}_2} \times \cdots \times \nu_{\mathcal{B}_n})$. *Then* D *is an intuitionistic fuzzy subgroup of* $\underbrace{G \times G \times \cdots \times G}_{n \text{ times}}$.

Theorem 3.12. *Let* $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ *and* $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ *be intuitionistic fuzzy subgroups of a group* G*, where* e *is the identity element of* G*. Then the following conditions hold for any* $(a, c) \in G \times G$*:*

(1) $\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(e, e) \geq \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a, c)$,
(2) $\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a, c)^{-1} = \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a, c)$,
(3) $\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(e, e) \leq \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a, c)$,
(4) $\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a, c)^{-1} = \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a, c)$.

Proof. Let $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ be intuitionistic fuzzy subgroups of G. By Theorem 3.6, $\mathcal{A} \otimes \mathcal{B} = (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \times \nu_{\mathcal{B}})$ is an intuitionistic fuzzy subgroup of $G \times G$. Thus, $\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}$ is a fuzzy subgroup of $G \times G$, and $\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}$ is an anti-fuzzy subgroup of $G \times G$. Since $G \times G$ is a group, for every $(a, c) \in G \times G$, there exists $(a, c)^{-1} = (a^{-1}, c^{-1}) \in G \times G$ such that $(a, c)(a, c)^{-1} = (e, e)$. Therefore, using Theorem 2.6 we have:

(1) $\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(e, e) \geq \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a, c)$,
(2) $\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a, c)^{-1} = \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a, c)$,

because $\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}$ is a fuzzy subgroup.

Similarly, using Theorem 2.8 we have:

(3) $\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(e, e) \leq \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a, c)$,
(4) $\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a, c)^{-1} = \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a, c)$,

because $\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}$ is an anti-fuzzy subgroup. □

Following our discussion of Theorem 3.6, which states that the Cartesian product of two intuitionistic fuzzy subgroups $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ is an intuitionistic fuzzy subgroup of the group $G \times G$, we will consider its converse. This investigation will be presented in the following theorem.

Theorem 3.13. *Let* $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ *and* $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ *be intuitionistic fuzzy subsets of a group* G*. If* $\mathcal{A} \otimes \mathcal{B} = (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \times \nu_{\mathcal{B}})$ *is an intuitionistic fuzzy subgroup of the group* $G \times G$*, then* $\mathcal{A}$ *and* $\mathcal{B}$ *are intuitionistic fuzzy subgroups of* G.

Proof. Assume that $\mathcal{A} \otimes \mathcal{B} = (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \times \nu_{\mathcal{B}})$ is an intuitionistic fuzzy subgroup of $G \times G$, and let e be the identity element of the group G. Since $\mathcal{A} \otimes \mathcal{B}$ is an intuitionistic fuzzy subgroup of $G \times G$, it follows that $\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}$ is a fuzzy subgroup of $G \times G$, and $\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}$ is an anti-fuzzy subgroup of $G \times G$.

Take arbitrary elements $a, c \in G$. Then $(a,e),(c,e),(e,a),(e,c) \in G \times G$. By Theorem 2.9 and Theorem 2.10, we have:

$$\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a,e)(c,e)^{-1} \geq \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a,e) \wedge \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(c,e),$$
$$\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(e,a)(e,c)^{-1} \geq \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(e,a) \wedge \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(e,c),$$
$$\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a,e)(c,e)^{-1} \leq \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(a,e) \vee \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(c,e),$$
$$\nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(e,a)(e,c)^{-1} \leq \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(e,a) \vee \nu_{\mathcal{A}} \times \nu_{\mathcal{B}}(e,c).$$

Note that:

$$(a,e)(c,e)^{-1} = (ac^{-1}, e), \quad (e,a)(e,c)^{-1} = (e, ac^{-1}).$$

Substituting, we obtain:

$$\mu_{\mathcal{A}}(ac^{-1}) \wedge \mu_{\mathcal{B}}(e) \geq \mu_{\mathcal{A}}(a) \wedge \mu_{\mathcal{A}}(c) \wedge \mu_{\mathcal{B}}(e),$$
$$\mu_{\mathcal{A}}(e) \wedge \mu_{\mathcal{B}}(ac^{-1}) \geq \mu_{\mathcal{A}}(e) \wedge \mu_{\mathcal{B}}(a) \wedge \mu_{\mathcal{B}}(c),$$
$$\nu_{\mathcal{A}}(ac^{-1}) \vee \nu_{\mathcal{B}}(e) \leq \nu_{\mathcal{A}}(a) \vee \nu_{\mathcal{A}}(c) \vee \nu_{\mathcal{B}}(e),$$
$$\nu_{\mathcal{A}}(e) \vee \nu_{\mathcal{B}}(ac^{-1}) \leq \nu_{\mathcal{A}}(e) \vee \nu_{\mathcal{B}}(a) \vee \nu_{\mathcal{B}}(c).$$

Since $\mu_{\mathcal{B}}(e), \mu_{\mathcal{A}}(e), \nu_{\mathcal{B}}(e), \nu_{\mathcal{A}}(e)$ are constant values, they can be simplified. Thus, we obtain:

$$\mu_{\mathcal{A}}(ac^{-1}) \geq \mu_{\mathcal{A}}(a) \wedge \mu_{\mathcal{A}}(c), \quad \nu_{\mathcal{A}}(ac^{-1}) \leq \nu_{\mathcal{A}}(a) \vee \nu_{\mathcal{A}}(c),$$
$$\mu_{\mathcal{B}}(ac^{-1}) \geq \mu_{\mathcal{B}}(a) \wedge \mu_{\mathcal{B}}(c), \quad \nu_{\mathcal{B}}(ac^{-1}) \leq \nu_{\mathcal{B}}(a) \vee \nu_{\mathcal{B}}(c).$$

Therefore, $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ are intuitionistic fuzzy subgroups of G. □

To support the bidirectional relationship established in Theorems 3.6 and 3.13, and further validated by the technical conditions in Theorem 3.12, we present the following example.

Let $G = \{a, c\}$ be a group with the binary operation defined by:

$$a \cdot a = c \cdot c = a, \quad a \cdot c = c \cdot a = c.$$

Consider two intuitionistic fuzzy subsets $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ of G, defined as follows:

$$\mu_{\mathcal{A}}(a) = 1, \quad \mu_{\mathcal{A}}(c) = 0.6, \quad \nu_{\mathcal{A}}(a) = 0, \quad \nu_{\mathcal{A}}(c) = 0.3,$$
$$\mu_{\mathcal{B}}(a) = 0.4, \quad \mu_{\mathcal{B}}(c) = 0.8, \quad \nu_{\mathcal{B}}(a) = 0.5, \quad \nu_{\mathcal{B}}(c) = 0.1.$$

It can be verified that $\mathcal{A}$ satisfies the conditions of an intuitionistic fuzzy subgroup of G. However, $\mathcal{B}$ does not. Specifically, we observe:

$$\mu_{\mathcal{B}}(cc^{-1}) = \mu_{\mathcal{B}}(a) = 0.4 \ngeq \mu_{\mathcal{B}}(c) \wedge \mu_{\mathcal{B}}(c) = 0.8,$$

which violates the fuzzy subgroup condition for $\mu_{\mathcal{B}}$.

Consequently, the Cartesian product $\mathcal{A} \otimes \mathcal{B} = (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \times \nu_{\mathcal{B}})$ also fails to be an intuitionistic fuzzy subgroup of $G \times G$. This is evident from the following computation:

$$\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}((c,c)(c,c)^{-1}) = \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(a,a) = \mu_{\mathcal{A}}(a) \wedge \mu_{\mathcal{B}}(a) = 0.4,$$
$$\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(c,c) \wedge \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(c,c) = \mu_{\mathcal{A}} \times \mu_{\mathcal{B}}(c,c) = 0.6,$$

and since $0.4 \ngeq 0.6$, the fuzzy subgroup condition is not satisfied.

Based on the example presented above, it can be concluded that the Cartesian product of two intuitionistic fuzzy subgroups forms an intuitionistic fuzzy subgroup if and only if both constituent subsets individually satisfy the conditions of an intuitionistic fuzzy subgroup. Conversely, if either of the two subsets fails to meet these conditions, then the resulting Cartesian product cannot be classified as an intuitionistic fuzzy subgroup.

Accordingly, in light of the conditions stated in Theorems 3.6 and 3.13, the following implication is obtained:

Corollary 3.14. *Let $\mathcal{A} = (\mu_{\mathcal{A}}, \nu_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \nu_{\mathcal{B}})$ be intuitionistic fuzzy subsets of a group G. Then, the Cartesian product $\mathcal{A} \otimes \mathcal{B} = (\mu_{\mathcal{A}} \times \mu_{\mathcal{B}}, \nu_{\mathcal{A}} \times \nu_{\mathcal{B}})$ is an intuitionistic fuzzy subgroup of $G \times G$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are intuitionistic fuzzy subgroups of G.*

Corollary 3.15. *Let $\mathcal{A}_1 = (\mu_{\mathcal{A}_1}, \nu_{\mathcal{B}_1})$, $\mathcal{A}_2 = (\mu_{\mathcal{A}_2}, \nu_{\mathcal{B}_2}), \ldots,$ and $\mathcal{A}_n = (\mu_{\mathcal{A}_n}, \nu_{\mathcal{B}_n})$ be intuitionistic fuzzy subsets of the group G. Then, the Cartesian product*

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n = (\mu_{\mathcal{A}_1} \times \mu_{\mathcal{A}_2} \times \cdots \times \mu_{\mathcal{A}_n}, \nu_{\mathcal{B}_1} \times \nu_{\mathcal{B}_2} \times \cdots \times \nu_{\mathcal{B}_n})$$

is an intuitionistic fuzzy subgroup of $G \times G \times \cdots \times G$ if and only if each $\mathcal{A}_i$ is an intuitionistic fuzzy subgroup of G, for every $i = 1, 2, \ldots, n$.

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