**Notes on Intuitionistic Fuzzy Sets**
**ISSN 1310–4926**
**Vol. 21, 2015, No. 1, 1–5**

# A property of the intuitionistic fuzzy modal logic operator $X_{a,b,c,d,e,f}$

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**Abstract:** It is proved that for every two intuitionistic fuzzy pairs $\langle \mu, \nu, \rangle$ and $\langle \rho, \sigma \rangle$, there are real numbers $a, b, c, d, e, f \in [0, 1]$ satisfying the conditions for existing of operator $X_{a,b,c,d,e,f}$, such that

$$X_{a,b,c,d,e,f}(\langle \mu, \nu, \rangle) = \langle \rho, \sigma \rangle.$$

**Keywords:** Intuitionistic fuzzy pair, Extended modal operator.
**AMS Classification:** 03E72.

## 1 Introduction

Operator $X_{a,b,c,d,e,f}$ is the most general extension of the modal operators, defined over Intuitionistic Fuzzy Sets (IFSs, see [3, 4]). It was introduced in [2] in 1993, but in its definition there was an omission, that was discussed firstly in [5]. Here, we use this corrected form of operator $X_{a,b,c,d,e,f}$, but we will use it for the case of Intuitionistic Fuzzy Modal Logic (IFML, see [4]).

First, following [6], we mention that the Intuitionistic Fuzzy Pair (IFP) is an object with the form $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process and whose conponents ($a$ and $b$) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let everywhere $\pi = 1 - \mu - \nu$.

In [1, 3, 4], the clasical logic modal operators (see, e.g., [7]) are extended to the following forms.

Let $\alpha, \beta \in [0, 1]$ and let:

$$\begin{aligned}
&F_{\alpha,\beta}(x) = \langle \mu + \alpha.\pi, \nu_A + \beta.\pi\rangle, \text{ where } \alpha + \beta \leq 1\\
&G_{\alpha,\beta}(x) = \langle \alpha\mu, \beta\nu\rangle,\\
&H_{\alpha,\beta}(A) = \langle \alpha\mu, \nu_A + \beta\pi\rangle,\\
&H^*_{\alpha,\beta}(x) = \langle \alpha\mu, \nu + \beta(1 - \alpha\mu - \nu)\rangle,\\
&J_{\alpha,\beta}(x) = \langle \mu + \alpha\pi, \beta\nu\rangle,\\
&J^*_{\alpha,\beta}(x) = \langle \mu + \alpha(1 - \mu - \beta\nu), \beta\nu\rangle.
\end{aligned}$$

The operators from modal type, defined in IFS theory or inn IFML, were extended in some directions as described in [3, 4]. The most general of them is the operator with the form

$$X_{a,b,c,d,e,f}(x) = \langle a.\mu + b.(1 - \mu - c.\nu), d.\nu + e.(1 - f.\mu - \nu)\rangle,$$

where $a, b, c, d, e, f \in [0, 1]$ and

$$a + e - e.f \leq 1, \tag{1}$$

$$b + d - b.c \leq 1, \tag{2}$$

$$b + e \leq 1. \tag{3}$$

The third condition was added to the definition in [5], because without it for the IFS

$$U^* = \{\langle x, 0, 0\rangle | x \in E\}$$

we obtain

$$X_{0,1,0,0,1,0}(U^*) = \{\langle x, 1, 1\rangle | x \in E\},$$

that is impossible. The same is valid in IFML for $1^* = \langle 1, 0\rangle$: we obtain

$$X_{0,1,0,0,1,0}(1^*) = \langle 1, 1\rangle.$$

On the other hand, as it is mentioned in [5], this condition is valid in all cases when operator $X_{a,b,c,d,e,f}$ represents some of the above described modal type of operators and, probably, this is the reason why the author had not seen it from the beginning.

# 2 Main results

Here, we prove the following

**Theorem.** For every two IFPs $\langle \mu, \nu, \rangle$ and $\langle \rho, \sigma\rangle$, there are real numbers $a, b, c, d, e, f \in [0, 1]$ satisfying (1)–(3), such that

$$X_{a,b,c,d,e,f}(\langle \mu, \nu, \rangle) = \langle \rho, \sigma\rangle.$$

*Proof.* Let $\mu, \nu, \rho, \sigma \in [0,1]$, so that $\mu + \nu \leq 1, \rho + \sigma \leq 1$. We search $a, b, c, d, e, f \in [0,1]$ that satisfy (1)–(3) and for which

$$\langle \rho, \sigma \rangle = X_{a,b,c,d,e,f}(\langle \mu, \nu, \rangle) = \langle a.\mu + b.(1 - \mu - c.\nu), d.\nu + e.(1 - f.\mu - \nu) \rangle, \tag{4}$$

i.e.

$$\rho = a.\mu + b.(1 - \mu - c.\nu), \tag{5}$$

$$\sigma = d.\nu + e.(1 - f.\mu - \nu). \tag{6}$$

We discuss nine cases.

**Case 1.** $\pi = \mu = 0$. Then $\nu = 1$. We put

$$a = c = e = f = 0,\ b = \rho,\ d = \sigma.$$

Then conditions (1)–(3) are valid and

$$X_{0,\rho,0,\sigma,0,0}(\langle \mu, \nu, \rangle) = \langle 0 + \rho.(1 - 0 - 0 \times 1), \sigma \times 1 + 0.(1 - 0 \times \mu - 1) \rangle = \langle \rho, \sigma \rangle.$$

**Case 2.** $\pi = \nu = 0$. Then $\mu = 1$. We put

$$a = \rho,\ b = c = d = f = 0,\ e = \sigma.$$

Then conditions (1)–(3) are valid and

$$X_{\rho,0,0,0,\sigma,0}(\langle \mu, \nu, \rangle) = \langle \rho + 0 \times (1 - 0 \times 1 - 0), 0 + \sigma.(1 - 0 \times 1 - 1) \rangle = \langle \rho, \sigma \rangle.$$

When $\pi = 0$ and $\mu, \nu > 0$, there are three (sub)cases. It is important to mention that now $\mu, \nu < 1$.

**Case 3.** $\rho > \mu$. Then from $\pi = 0$ it follows that $\mu = 1 - \nu$ and hence $\sigma \leq 1 - \rho < 1 - \mu = \nu$. So, we put

$$a = 1,\ b = \frac{\rho - \mu}{1 - \mu}, c = e = f = 0,\ d = \frac{\sigma}{\nu}.$$

Then conditions (1)–(3) are valid and

$$X_{1,\frac{\rho-\mu}{1-\mu},0,\frac{\sigma}{\nu},0,0}(\langle \mu, \nu, \rangle) = \langle \mu + \frac{\rho - \mu}{1 - \mu}.(1 - \mu), \frac{\sigma}{\nu}.\nu \rangle = \langle \rho, \sigma \rangle.$$

**Case 4.** $\sigma > \nu$. Then from $\pi = 0$ again it follows that $\mu = 1 - \nu$ and hence $\rho \leq 1 - \sigma < 1 - \nu = \mu$. So, we put

$$a = \frac{\rho}{\mu},\ b = c = f = 0,\ d = 1, e = \frac{\sigma - \nu}{1 - \nu}.$$

Then conditions (1)–(3) are valid and

$$X_{\frac{\rho}{\mu},0,0,1,\frac{\sigma-\nu}{1-\nu},0,0}(\langle \mu, \nu, \rangle) = \langle \frac{\rho}{\mu}.\mu + 0, \nu + \frac{\sigma - \nu}{1 - \nu}.(1 - \nu) \rangle = \langle \rho, \sigma \rangle.$$

**Case 5.** $\rho \leq \mu$ and $\sigma \leq \nu$. Then we put

$$a = \frac{\rho}{\mu},\ b = c = e = f = 0,\ d = \frac{\sigma}{\nu}.$$

Then conditions (1)–(3) are valid and

$$X_{\frac{\rho}{\mu},0,0,\frac{\sigma}{\nu},0,0}(\langle \mu, \nu, \rangle) = \langle \frac{\rho}{\mu}.\mu + 0, \frac{\sigma}{\nu}.\nu + 0\rangle = \langle \rho, \sigma\rangle.$$

When $\pi > 0$, then $\mu, \nu < 1$.

**Case 6.** $\rho > \mu$ and $\sigma > \nu$. Then we put

$$a = c = d = f = 1,\ b = \frac{\rho - \mu}{\pi},\ e = \frac{\sigma - \nu}{\pi}.$$

Then conditions (1)–(3) are valid, because:

$$a + e - e.f = 1 + e - e = 1 \leq 1,$$

$$b + d - b.c = d = \frac{\sigma - \nu}{\pi} \leq 1,$$

$$b + e = \frac{\rho - \mu}{\pi} + \frac{\sigma - \nu}{\pi} = \frac{\rho + \sigma - \mu - \nu}{\pi} \leq 1.$$

All other similar checks are doing by similar way. Now,

$$X_{1,\frac{\rho-\mu}{\pi},1,1,\frac{\sigma-\nu}{\pi},1}(\langle \mu, \nu, \rangle) = \langle \mu + \frac{\rho - \mu}{\pi}.\pi, \nu + \frac{\sigma - \nu}{\pi}.\pi\rangle = \langle \rho, \sigma\rangle.$$

**Case 7.** $\rho > \mu$ and $\sigma \leq \nu$. Then, as in Case 3, we put

$$a = 1,\ b = \frac{\rho - \mu}{1 - \mu}, c = e = f = 0,\ d = \frac{\sigma}{\nu}.$$

Then conditions (1)–(3) are valid and

$$X_{1,\frac{\rho-\mu}{1-\mu},0,\frac{\sigma}{\nu},0,0}(\langle \mu, \nu, \rangle) = \langle \mu + \frac{\rho - \mu}{1 - \mu}.(1 - \mu), \frac{\sigma}{\nu}.\nu\rangle = \langle \rho, \sigma\rangle.$$

**Case 8.** $\rho \leq \mu$ and $\sigma > \nu$. Then, as in Case 4, we put

$$a = \frac{\rho}{\mu},\ b = c = f = 0,\ d = 1, e = \frac{\sigma - \nu}{1 - \nu}.$$

Then conditions (1)–(3) are valid and

$$X_{\frac{\rho}{\mu},0,0,1,\frac{\sigma-\nu}{1-\nu},0,0}(\langle \mu, \nu, \rangle) = \langle \frac{\rho}{\mu}.\mu + 0, \nu + \frac{\sigma - \nu}{1 - \nu}.(1 - \nu)\rangle = \langle \rho, \sigma\rangle.$$

**Case 9.** $\rho \leq \mu$ and $\sigma \leq \nu$. Then, as in Case 5, we put

$$a = \frac{\rho}{\mu},\ b = c = e = f = 0,\ d = \frac{\sigma}{\nu}.$$

Then conditions (1)–(3) are valid and

$$X_{\frac{\rho}{\mu},0,0,\frac{\sigma}{\nu},0,0}(\langle \mu, \nu, \rangle) = \langle \frac{\rho}{\mu}.\mu + 0, \frac{\sigma}{\nu}.\nu + 0\rangle = \langle \rho, \sigma\rangle.$$

This completes the proof. $\square$

# 3 Conclusion

We finish with the following **Open problems**:

1. Which other values of the arguments of operator $X$ are possible values, leading to solution of the above formulated problem?

2. To represent operator $X$ as composition of some of the operators $F, G, H, H^*, J, J^*$.


## Acknowledgements

The author is thankful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. DFNI-I-02-5 "InterCriteria Analysis: A New Approach to Decision Making".


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