Grey Systems and Interval Valued Fuzzy Sets

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Abstract

Grey systems and interval valued fuzzy sets appear to be similar concepts. This abstract provides a brief introduction to grey systems and compares them to interval valued fuzzy sets. We argue that grey system is similar to an interval fuzzy set, but has different properties. Grey systems focus on the interval range of the system property itself, but interval fuzzy sets consider the fuzzy memberships as interval values. We also introduce the notion of grey sets.

Keywords: Grey Systems, Interval Valued Fuzzy Sets.

1 Introduction

As a new model for systems with incomplete information, grey systems are proving to be a useful approach for prediction and control applications [1]. However, because of their similarity with fuzzy sets (especially interval valued fuzzy sets) it is important that we understand the relationship between the two paradigms. Fuzzy logic has had a profound influence in the application of soft computing in different areas, and it has well established theoretical basis. In particular, possibility theory [2,3,4] provides one approach to modeling uncertainty. In this paper we are only interested in contrast interval valued fuzzy sets and grey systems. Grey systems are not well established and, indeed, there is still a lot of work to do in terms of defining them properly and explaining their position with regard to uncertainty. In this paper we do this, perhaps for the first time. Dubois et al [5] and later Deschrijver and Kerre [6] comment that grey systems are interval valued fuzzy sets. Y. Huang and C. Huang [7] applied grey systems in complementary with fuzzy logic. We believe that they are different (although there are similarities) and provide in this paper a comparison between grey systems and interval valued fuzzy set theory so as to benefit the further application of grey systems. We also introduce the notion of a grey set.

2 Grey Systems

The Grey System was proposed by Professor Julong Deng in 1982 [8]. In this system, the information is classified into three categories: white with completely certain information, grey with insufficient information, and black with totally unknown information. Grey systems are concerned with, in particular, the information belonging to the grey category. Because of insufficient information, most of the statistical characteristics of the system may not be clearly identified. However, the data available may reveal the range of information. We now provide a number of definitions.

Definition 1 - Grey number. A grey number is a number with clear upper and lower boundaries but which has an unknown position within the boundaries. A grey number for the system is expressed mathematically as [9]

$$a^{\pm} = [a^{-}, a^{+}] = \{t \in a^{\pm} \mid a^{-} \le t \le a^{+}\}$$

where $a^{\pm}$ is a grey number, t is information, a^{+} and a^{-} are the upper and lower limits of the information.

For instance, the time table for a bus is a typical grey number. The bus time given really only implies a time interval. If the arrival time is 10:00AM, then it may come 10 minutes early or later. However, it is clear that the bus time will not have difference over one hour from reality. Then, the grey time would be [9:00AM,11:00AM].

Clearly, a grey number represents the range of the possible variance of the underlying number, hence it implies that there is uncertainty surrounding the number itself. To represent this uncertainty, a degree of greyness g° is associated with a grey number $a^{\pm}$.

Definition 2 – Degree of greyness. The significance of the unknown interval to the white number represented by a grey number is called the degree of greyness. It is a function of the interval and the underlying white number. Because the underlying white number is unknown, the degree of greyness is expressed as a function of two boundaries of a grey number.

$$g^{\circ}(a^{\pm}) = f(a^{+}, a^{-})$$

For example, we could have

$$g^{\circ}(a^{\pm}) = |2\times(a^{+} - a^{-})/(a^{+} + a^{-})|$$

Obviously, the interval between a^{+} and a^{-} will directly effect the degree of greyness g°.

Take the previous example, the degree of greyness for that bus timetable would be

$$g^{\circ}(a^{\pm}) = |2\times(a^{+} - a^{-})/(a^{+} + a^{-})|$$

$$g^{\circ}(a^{\pm}) = |2\times(11-9)/(11+9)| = 0.2$$

We can carry out operations on grey numbers. The operation on a grey number is defined as

$$a^{\pm} \otimes b^{\pm} = [f^{-}(a^{+}, b^{+}, a^{-}, b^{-}), f^{+}(a^{+}, b^{+}, a^{-}, b^{-})]$$

where $\otimes$ is an operator, $f^{-}(a^{+}, b^{+}, a^{-}, b^{-})$ and $f^{+}(a^{+}, b^{+}, a^{-}, b^{-})$ are the functions for minimum and maximum values. The operation of the grey function is defined as

$$\begin{aligned}&[f(a_1^{\pm}, a_2^{\pm}, \ldots, a_n^{\pm})]^{\pm}\\ &= \{[f(a_1^{\pm}, a_2^{\pm}, \ldots, a_n^{\pm})]^{-}, [f(a_1^{\pm}, a_2^{\pm}, \ldots, a_n^{\pm})]^{+}\}\end{aligned}$$

where $[f(a_1^{\pm}, a_2^{\pm}, \ldots a_n^{\pm})]^{\pm}$ is the value of the grey function, $[f(a_1^{\pm}, a_2^{\pm}, \ldots a_n^{\pm})]^{-}$ and $[f(a_1^{\pm}, a_2^{\pm}, \ldots a_n^{\pm})]^{+}$ are the minimum and maximum values of the function.

For grey systems, the operation is not limited with the operation of interval values, it considers also the whitenisation of its output interval values. Regardless of the interval of a grey number, its real value could only be one value within this scope. The technique to transfer a grey number into a white number is called whitenisation. There is a weight function W($a^{\pm}$) for whitenisation:

$$a = \mathrm{W}(a^{\pm})$$

where a is the white number after the whitenisation, which is the number with the highest possibility to be the real value.

Grey systems' modelling is carried out mainly through the generation of grey numbers or functions of series operators to find hidden patterns in data. A grey system considers the specific data series as a white track of the underlined grey behavior of a system, hence the values of the data could be transferred using special operators such as accumulating generation operator and reverse accumulating generation operator [10]. Based on filtering, refinement, extension and analogy of a small amount of grey information, a data series appearing as chaos may reveal some consistent relationships. Compared with probability theory and fuzzy set theory, the main advantage of a grey system is its low requirement on sample data and flexible capability in pattern identification. Grey system theory has been widely applied to various scientific and engineering fields as a powerful tool for modelling and prediction of complex time series data, such as image processing [11], computer graphics [12], automatic control [13] and reservoir operation [9]. As a modelling system for uncertain and nonlinear systems, grey systems have proved a powerful capability in dealing with uncertain and incomplete data.

3 Interval Valued Fuzzy Sets

This Section discusses interval valued fuzzy sets. Interval analysis and computation [14] is a vast research area that offers much in the way of dealing with various forms of imprecision. The interval valued fuzzy sets community is a growing community as a subset of the wider interval research field.

The term interval valued fuzzy sets means different things to different people. They can be considered as a form of type-2 fuzzy sets [15,16]. However, most people consider them to be as defined below. Interval-valued (i-v) fuzzy sets, relax the requirement for precise membership functions.

There are a number of ways to define i-v sets [14]. We use the definition provided by Sambuc [17].

Definition 3 – Interval valued fuzzy set. An interval fuzzy set in X is given by expression A

$$A = \{< x, M_A(x) >| x \in X\}$$

where the function $M_A : X \to D[0,1]$ defines degree of membership of an element x to A.

For interval valued fuzzy sets A and B, if the lower bounds of their fuzzy membership (for a given x) are M_{AL} and M_{BL}, and upper bounds are M_{AU} and M_{BU}, then we have

1. $A \le B$ if and only if $M_{AL}(x) \le M_{BL}(x)$ and $M_{AU}(x) \le M_{BU}(x)$ for all $x \in X$
2. $A \prec B$ if and only if $M_{AL}(x) \le M_{BL}(x)$ and $M_{AU}(x) \ge M_{BU}(x)$ for all $x \in X$
3. $A = B$ if and only if $M_{AL}(x) = M_{BL}(x)$ and $M_{AU}(x) = M_{BU}(x)$ for all $x \in X$

4 Grey Sets

As previously mentioned, the grey number is a number with unknown position within a clear boundary. In this sense, there are a set of candidate numbers within that boundary, and we call this a grey set.

Definition 4 – Grey set. For a given lower and upper boundary B_l and B_u ($B_l < B_u$), there is always a grey number $G_i = [B_{il}, B_{iu}]$ which satisfies $B_l \le B_{il}$ and $B_{iu} \le B_u$. The set of all the grey numbers G_i (i=1,2,3,…) satisfying this condition, S_G, is called a grey set.

Similar to grey number, a grey degree is associated with grey set:

Definition 5 – Degree of greyness. For a given grey set S_G, the degree of greyness for this grey set satisfies:

$$g^\circ(S_G) = Max\{g^\circ(S_{G_1}), g^\circ(S_{G_2}), \ldots\}$$

Obviously, the degree of greyness of a grey set is the same as of the grey number with the same boundary as grey set.

According to this definition of a grey set, any pair of boundaries within the set boundary could constitute another grey set which locates within the original set. Therefore, we have the definition for a grey sub set.

Definition 6 – Grey subset. For two given grey sets G_a with boundary $B_{la} < B_{ua}$ and G_b with boundary $B_{lb} < B_{ub}$, G_a is called as a grey subset of G_b if their boundaries satisfy the following condition:

$$B_{lb} \le B_{la}$$

$$B_{ua} \le B_{ub}$$

Therefore, a grey set includes all its grey subsets.

With these definitions, the union, intersection and difference between two grey sets could be considered.

Definition 7 – Union of grey sets. Given two grey sets G_a with boundary $B_{la} < B_{ua}$ and G_b with boundary $B_{lb} < B_{ub}$, if $B_{la} \le B_{lb} \le B_{ua}$ or $B_{lb} \le B_{la} \le B_{ub}$, the union operation produces a new grey set G_c with boundary $B_{lc} < B_{uc}$:

$$B_{lc} = \mathrm{Min}(B_{la}, B_{lb})$$

$$B_{uc} = \mathrm{Max}(B_{ua}, B_{ub})$$

Any grey number in G_c is either a member of G_a or G_b.

Definition 8 – Intersection of grey sets. Given two grey sets G_a with boundary $B_{la} < B_{ua}$ and G_b with boundary $B_{lb} < B_{ub}$, if $B_{la} \le B_{lb} \le B_{ua}$ or $B_{lb} \le B_{la} \le B_{ub}$, the intersect operation results in a new grey set G_c with boundary $B_{lc} < B_{uc}$:

$$B_{lc} = \mathrm{Max}(B_{la}, B_{lb})$$

$$B_{uc} = \mathrm{Min}(B_{ua}, B_{ub})$$

Any grey number in G_c must be a member of both G_a and G_b.

For a grey set S_G, the following properties exist

1. If grey set S_s is a subset of S_G, then $g^\circ(S_G) \ge g^\circ(S_s)$
2. For grey set $S_A(a_1, a_2)$ and $S_B(b_1, b_2)$, if $a_2 \le b_1$, then $S_A \le S_B$

3. For grey set $S_A(a_1, a_2)$ and $S_B(b_1, b_2)$, if $b_2 \geq a_2 \geq b_1$ and $a_1 \leq b_1$, then the following holds: $S_A \cup S_B = S_U(a_1, b_2)$, $S_A \cap S_B = S_I(b_1, a_2)$, $g^\circ(S_U) \geq g^\circ(S_A)$, $g^\circ(S_U) \geq g^\circ(S_B)$, $g^\circ(S_I) \leq g^\circ(S_A)$ and $g^\circ(S_I) \leq g^\circ(S_B)$

4. $S_G = \bigcup_{i=1}^{n} S^i{}_S$, $S^i{}_s$ is an element of S_G for all i in {1,2,…,n}

5. For grey set $S_A(a_1, a_2)$ and $S_B(b_1, b_2)$, if $a_1 = b_1$, $a_2 = b_2$ and the whitenisation $W_A = W_B$ for all grey numbers in the two set with identical boundary, then $S_A = S_B$.

6. For grey set $S_A(a_1, a_2)$ and $S_B(b_1, b_2)$, if $a_1 \leq b_1$ and $a_2 \geq b_2$, and every grey number in S_B has the same whitenisation function with its counterpart in S_A, then $S_A \supset S_B$

5 A Comparison

We have given a brief introduction to both grey systems and interval valued fuzzy sets. To enable a comparison, we focus on the set comparison between grey sets and interval fuzzy sets. As previously mentioned, the grey number is a number with unknown position within a clear boundary. In this sense, there are a set of candidate numbers within that boundary, and we call this a grey set.

1. The degree of greyness is defined for the whole set, but fuzziness is defined for each individual member of that set.
2. The interval of a grey number is the domain for the value of an underlying white number, hence it is for the object itself. The interval of interval fuzzy set is about the scope of its membership, it is not directly related to the object itself.
3. $A \leq B$, $A \prec B$ and $A = B$ in interval fuzzy set mean the membership relationships between two fuzzy sets with identical members, but $S_A \leq S_B, S_A \supset S_B$ and $S_A = S_B$ in grey set show the relationship between the components of two grey set with different members.
4. Greyness in grey sets represents lack of knowledge about data. Membership in fuzzy sets represents a measure of belief in some concepts.
5. As we acquire knowledge about the grey number it becomes white - precise. Fuzzy logic represents uncertainty and so extra information allows us to be surer about the membership value. In the case of interval valued fuzzy set, this will narrow the interval perhaps eventually become a zero interval and, therefore, a precise membership value. But the object itself is still fuzzy.

There is still further work to be done on grey and fuzzy and their relationships.

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