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Four extended level operators of membership/non-membership over Ituitionistic Fuzzy Sets

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Abstract. We present 4 new operators over IFSs which are extension of the already defined level operators of membership/non-membership in [1]. After that we define the new terms: cell, semicell and perfect n-net.

In [1] are defined the following 3 operators over IFS A:

$$N_{\alpha,\beta}(A)=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in E,\mu_A(x)\geq\alpha,\nu_A(x)\leq\beta,\alpha+\beta\leq 1,0\leq\alpha\leq 1,0\leq\beta\leq 1\}$$

$$N_{\alpha}(A)=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in E,\mu_A(x)\geq\alpha,0\leq\alpha\leq 1\}$$

$$N^{\beta}(A)=\{\langle x,\mu_A(x),\nu_A(x)\rangle \mid x\in E,\nu_A(x)\leq\beta,0\leq\beta\leq 1\}$$

$N_{\alpha,\beta}(A)$ is called a set of (α,β)-level, generated by an IFS A. We have to note that it is obeyed $\alpha+\beta\leq 1$ in this definition. $N_{\alpha}(A)$ is called a set of level of membership α, generated by A. $N^{\beta}(A)$ is called a set of level of non-membership β, generated by A. Lets see the geometric interpretation of sets $N_{\alpha,\beta}(A)$, $N_{\alpha}(A)$, $N^{\beta}(A)$ (Fig.1,2,3,4):

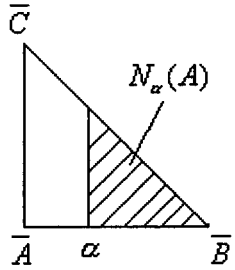

Fig.1

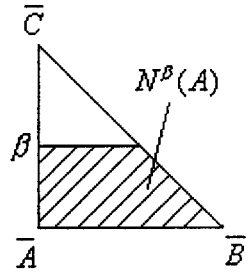

Fig.2

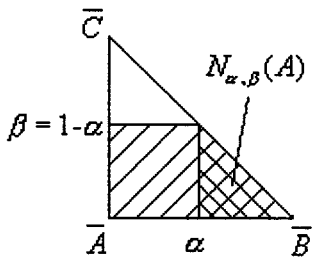

Fig.3

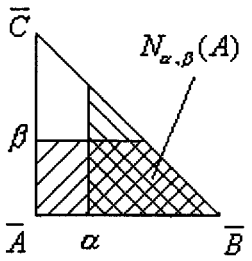

Fig.4

Definition.1 We define the sets:

$V_\alpha(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E, \mu_A(x) = \alpha, 0 \le \alpha \le 1\}$

$H^\beta(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E, \nu_A(x) = \beta, 0 \le \beta \le 1\}$

(Fig.5,6)

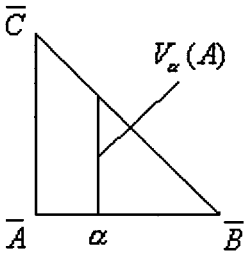

Fig.5

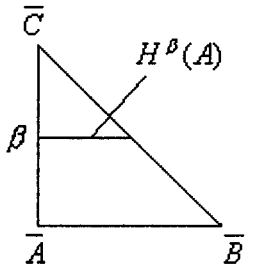

Fig.6

Obviously $V_\alpha(A) \parallel \overline{AC}$ and $H^\beta(A) \parallel \overline{AB}$. Every couple of sets $(V_\alpha(A), H^\beta(A))$ where $\alpha, \beta \in [0,1]$ corresponds with exactly one ordered triad $\langle x, \alpha, \beta\rangle$, i.e. exactly one point in the square $\overline{ABDC}$. This point(ordered triad $\langle x, \alpha, \beta\rangle$) we will note with P_α^β for our convenience. When $\alpha, \beta \in [0,1]$ for the point P_α^β are possible the following three cases according to the sum $\alpha + \beta$:

1) If $\alpha + \beta < 1$ then P_α^β is an inside point for $\triangle\overline{ABC}$, i.e. $P_\alpha^\beta \in \triangle\overline{ABC}, P_\alpha^\beta \notin \overline{BC}$
2) If $\alpha + \beta = 1$ then $P_\alpha^\beta \in \overline{BC}$. In this case we say that P_α^β is a boundary point for $\triangle\overline{ABC}$
3) If $1 < \alpha + \beta < 2$ then P_α^β is an outside point for $\triangle\overline{ABC}$

(Fig.7,8,9)

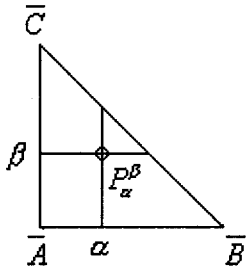

Fig.7

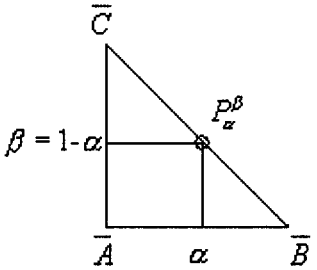

Fig.8

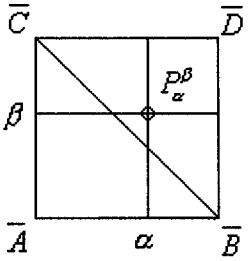

Fig.9

If we put the restriction $\alpha+\beta\le 1$ and $\alpha,\beta\in[0,1]$ then every couple of sets $(V_\alpha(A),H^\beta(A))$ corresponds with only one point $P_\alpha^\beta\in\triangle\overline{ABC}$. In this case P_α^β can be an inside or a boundary point for $\triangle\overline{ABC}$.

If we put the restriction $\alpha+\beta\le 2$ and $\alpha,\beta\in[0,1]$ then every couple of sets $(V_\alpha(A),H^\beta(A))$ corresponds with only one point $P_\alpha^\beta\in\overline{ABDC}$. In this case P_α^β can be an inside, ouside or boundary point for $\triangle\overline{ABC}$.

We have to note that every set $V_\alpha(A)$ corresponds with the set $N_\alpha(A)$ and vice versa. Furthermore, the set $V_\alpha(A)$ is included in $N_\alpha(A)$. Therefore when we have $N_\alpha(A)$ we can define $V_\alpha(A)$. The opposite is not true, i.e. when we have $V_\alpha(A)$ we can not define $N_\alpha(A)$. The same dependences are observed between the sets $H^\beta(A)$ and $N^\beta(A)$.

Every set $N_{\alpha,\beta}(A)$ corresponds with one couple of sets $(V_\alpha(A),H^\beta(A))$ (here we have $\alpha+\beta\le 1$ according to the definition of $N_{\alpha,\beta}(A)$). Therefore every set $N_{\alpha,\beta}(A)$ corresponds with one point P_α^β from the triangle $\triangle\overline{ABC}$. There are 2 cases about P_α^β: to be an inside or a boundary point for $\triangle\overline{ABC}$.

We are ready to define the following 4 new operators which we will name "extended level operators of membership/non-membership, generated by an IFS A ".

Definition.2 The set

$${}_{\alpha_1}N_{\alpha_2}(A)=\{<x,\mu_A(x),\nu_A(x)>|\,x\in E,\alpha_1\le\mu_A(x)\le\alpha_2,0\le\alpha_1\le\alpha_2\le 1\}$$

is called a set of level of membership between α_1 and α_2, generated by an IFS A. (Fig.10,11)

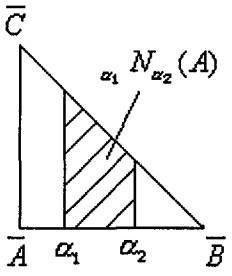

Fig.10

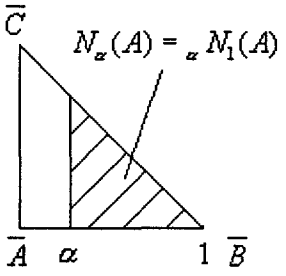

Fig.11

Note 2.1 For every $\alpha\in[0,1]$ we have that $N_\alpha(A)={}_\alpha N_1(A)$.

Note 2.2 If ${}_{\alpha_1}N_{\alpha_2}(A)$ and $\alpha_1=\alpha=\alpha_2$ then we get the set $V_\alpha(A)$.

Definition.3 The set

$${}^{\beta_1}N^{\beta_2}(A)=\{<x,\mu_A(x),\nu_A(x)>|\,x\in E,\beta_1\le\nu_A(x)\le\beta_2,0\le\beta_1\le\beta_2\le 1\}$$

is called a set of level of non-membership between β_1 and β_2, generated by an IFS A. (Fig.12,13)

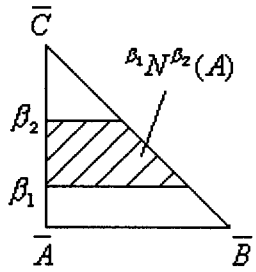

Fig.12

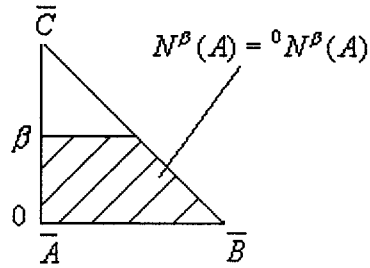

Fig.13

Note 3.1 For every $\beta \in [0,1]$ we have that $N^{\beta}(A) = {}^{0}N^{\beta}(A)$

Note 3.2 If ${}^{\beta_1}N^{\beta_2}(A)$ and $\beta_1 = \beta = \beta_2$ then we get the set $H^{\beta}(A)$.

Definition.4 A set of (α, β)-level, generated by an IFS A, is defined as

$$N_{\alpha}^{\beta}(A) = \{< x, \mu_A(x), \nu_A(x) > | \, x \in E, \mu_A(x) + \nu_A(x) \le 1,$$
$$\mu_A(x) \ge \alpha, \nu_A(x) \le \beta, \alpha \in [0,1], \beta \in [0,1] \ \}$$

Here, the special feature is $\alpha + \beta \le 2$. Therefore every set $N_{\alpha}^{\beta}(A)$ corresponds with one point P_{α}^{β} from the square $\overline{ABDC}$ (whereas in the definition of $N_{\alpha,\beta}(A)$ we have $\alpha + \beta \le 1$. So in $N_{\alpha,\beta}(A)$ the point P_{α}^{β} is inside or boundary for $\triangle\overline{ABC}$). But here, in $N_{\alpha}^{\beta}(A)$, there are 3 cases about the point P_{α}^{β}: to be inside, boundary or outside for $\triangle\overline{ABC}$.

Note 4.1 In the definition of $N_{\alpha}^{\beta}(A)$ the restriction $\mu_A(x) + \nu_A(x) \le 1$ is observed. Therefore $N_{\alpha}^{\beta}(A)$ is really an intuitionistic fuzzy set.

There is the following dependence in the geometric interpretation:

1) If $\alpha + \beta \le 1$ then $N_{\alpha}^{\beta}(A) = N_{\alpha,\beta}(A)$

2) If $1 < \alpha + \beta < 2$ then $N_{\alpha}^{\beta}(A) = N_{\alpha,1-\alpha}(A)$ (Fig.14)

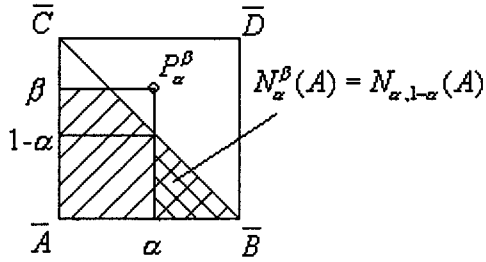

Fig.14

Definition.5 A set of $((\alpha_1, \alpha_2), (\beta_1, \beta_2))$-level, generated by an IFS A, is defined as:

$${}_{\alpha_1}^{\beta_1}N_{\alpha_2}^{\beta_2}(A) = \{ \ < x, \mu_A(x), \nu_A(x) > | \ x \in E, \mu_A(x) + \nu_A(x) \le 1,$$

$$\alpha_1 \le \mu_A(x) \le \alpha_2, \beta_1 \le \nu_A(x) \le \beta_2,$$
$$0 \le \alpha_1 \le \alpha_2 \le 1,\ 0 \le \beta_1 \le \beta_2 \le 1 \ \}$$

Here we have again that $\alpha_1 + \beta_1 \le 2,\ \alpha_2 + \beta_2 \le 2$ like in the definition of $N_\alpha^\beta(A)$.

We will say that ${}_{\alpha_1}^{\beta_1}N_{\alpha_2}^{\beta_2}(A)$ is an trivial set of $((\alpha_1,\alpha_2),(\beta_1,\beta_2))$ -level if one of the following 3 cases is executed:

1) $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ 2) $\alpha_1 \ne \alpha_2$, $\beta_1 = \beta_2$ 3) $\alpha_1 = \alpha_2$, $\beta_1 \ne \beta_2$

Such trivial sets we will note with ${}_{\alpha_1}^{\beta_1}\widetilde{N}_{\alpha_2}^{\beta_2}(A)$ to make difference between trivials sets and the others.

The set ${}_{\alpha_1}^{\beta_1}N_{\alpha_2}^{\beta_2}(A)$ where $\alpha_1 \ne \alpha_2$ and $\beta_1 \ne \beta_2$ we will name non-trivial. We will work only with non-trivial sets.

From the two previous definitions follows that every set ${}_{\alpha_1}^{\beta_1}N_{\alpha_2}^{\beta_2}(A)$ corresponds with exactly 4 sets $N_{\alpha_1}^{\beta_1}(A),\ N_{\alpha_2}^{\beta_2}(A),\ N_{\alpha_1}^{\beta_2}(A),\ N_{\alpha_2}^{\beta_1}(A)$. On the other hand we know that every set $N_\alpha^\beta(A)$ corresponds with a point P_α^β from the square $\overline{ABDC}$. Therefore ${}_{\alpha_1}^{\beta_1}N_{\alpha_2}^{\beta_2}(A)$ corresponds with exactly 4 points $P_{\alpha_1}^{\beta_1},\ P_{\alpha_2}^{\beta_2},\ P_{\alpha_1}^{\beta_2},\ P_{\alpha_2}^{\beta_1}$ (we will note them with $\mathrm{P}_1^1,\ \mathrm{P}_2^2,\mathrm{P}_1^2,\mathrm{P}_2^1$ for our facilitation).

(Fig. 15.1, 15.2)

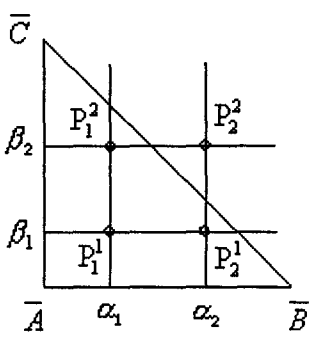

Fig.15.1

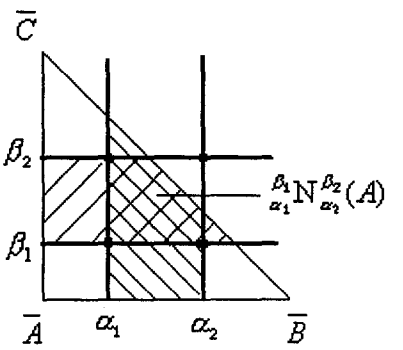

Fig.15.2

Since we have 4 points and have 3 cases for every point(P_α^β is inside, boundary or outside for $\triangle\overline{ABC}$), then we have the most $3\times3\times3\times3 = 81$ possibiliteies about the mutual position of $\mathrm{P}_1^1,\ \mathrm{P}_2^2,\mathrm{P}_1^2,\mathrm{P}_2^1$, thence there are the most 81 cases about the view of ${}_{\alpha_1}^{\beta_1}N_{\alpha_2}^{\beta_2}(A)$. But we have the restrictions $\alpha_1 < \alpha_2,\ \beta_1 < \beta_2\ (\Rightarrow\ \alpha_1 + \beta_1 < \alpha_2 + \beta_2)$ and therefore we have to look at only these 3 cases:

1) $\alpha_2 + \beta_2 \le 1$

In this case $\mathrm{P}_2^2 \in \triangle\overline{ABC}$ (inside or boundary).

2) $\alpha_1 + \beta_1 = 1$

We get ${}_{\alpha_1}^{\beta_1}N_{\alpha_2}^{\beta_2}(A) = \{< x,\ \alpha_1,\ 1-\alpha_1 > | \ x \in E\ \}$.

3) $\alpha_1+\beta_1<1,\ \alpha_2+\beta_2>1$

In this case P_1^1 is inside and P_2^2 is outside. Therefore the only thing that we have to do is to define if P_1^2 and P_2^1 are inside,boundary or outside. Therefore we have exactly $3\times 3=9$ possibilities.

If $\alpha_1+\beta_1>1$, then $\mu_A(x)+\nu_A(x)>1$ and therefore ${}^{\beta_1}_{\alpha_1}N^{\beta_2}_{\alpha_2}(A)\equiv\varnothing$.

From the three cases we get together 12 possibilities about the view of the set ${}^{\beta_1}_{\alpha_1}N^{\beta_2}_{\alpha_2}(A)$.

Definition.6 A cell, generated by an IFS A, is defined as:

$$
\begin{aligned}
{}^{\beta_1}_{\alpha_1}Cell^{\beta_2}_{\alpha_2}(A)=\{<x,\mu_A(x),\nu_A(x)>|\, x\in E,\ \mu_A(x)+\nu_A(x)\le 1,\\
\alpha_1\le\mu_A(x)\le\alpha_2,\ \beta_1\le\nu_A(x)\le\beta_2,\\
\alpha_1+\beta_2<1,\ \alpha_2+\beta_1<1,\ \alpha_2+\beta_2\le 1,\\
0\le\alpha_1<\alpha_2\le 1,\ 0\le\beta_1<\beta_2\le 1\quad\}
\end{aligned}
$$

One cell has the following geometric interpretation (Fig.16,17):

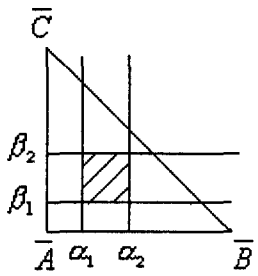

Fig.16

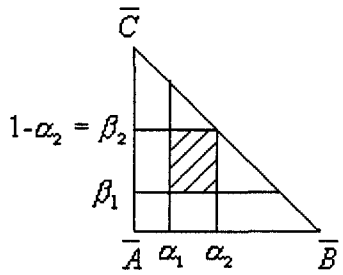

Fig.17

Definition.7 A semicell, generated by an IFS A, is defined as:

$$
\begin{aligned}
{}^{\beta_1}_{\alpha_1}SCell^{\beta_2}_{\alpha_2}(A)=\{<x,\mu_A(x),\nu_A(x)>|\, x\in E,\ \mu_A(x)+\nu_A(x)\le 1,\\
\alpha_1\le\mu_A(x)\le\alpha_2,\ \beta_1\le\nu_A(x)\le\beta_2,\\
\alpha_1+\beta_2=1,\ \alpha_2+\beta_1=1,\\
0\le\alpha_1<\alpha_2\le 1,\ 0\le\beta_1<\beta_2\le 1\quad\}
\end{aligned}
$$

Here is the geometric interpretation of a semicell(Fig.18):

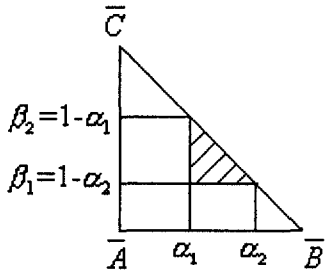

Fig.18

We have to note that a cell and a semicell ${}^{\beta_1}_{\alpha_1}Cell^{\beta_2}_{\alpha_2}(A)$ are also IFSs because the restriction $\mu_A(x)+\nu_A(x)\le 1$ is observed in the two previous definitions.

Definition.8 Lets $0=\alpha_0<\alpha_1<\alpha_2<\ldots<\alpha_{n-1}<\alpha_n=1$.

We put $\beta_0=1-\alpha_n=0$

$\ldots$

$\beta_i=1-\alpha_{n-i} \qquad i=0,1,2,\ldots,n-1,n$

$\ldots$

$\beta_n=1-\alpha_0=1$

Obviously $0=\beta_0<\beta_1<\beta_2<\ldots<\beta_{n-1}<\beta_n=1$.

Perfect n-net, generated by an IFS A, is defined as:

$Net_n(A)=\{\ <x,\mu_A(x),\nu_A(x)>|\ x\in E,\ \mu_A(x)\in\{\alpha_0,\alpha_1,\ldots,\alpha_n\},$

$\nu_A(x)\in\{\beta_0,\beta_1,\ldots,\beta_n\},\ \mu_A(x)+\nu_A(x)\le 1\ \}$ □

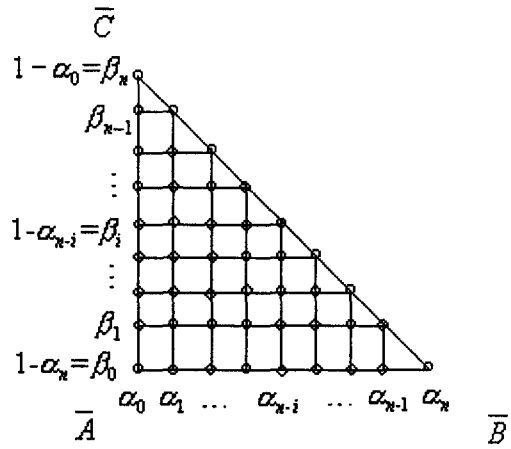

Fig.19

Note 8.1 From the definition of perfect n-net we get that $\triangle\overline{ABC}$ can be splitted in exactly $\frac{n.(n-1)}{2}$ cells and n semicells.

Note 8.2 One finit set from n real numbers between 0 and 1(where the first element is 0 and the last element is 1) defines a perfect net. The opposite statement is also true.

Similary a perfect n-net can be defined from every finit set of points which lie on the hypotenuse $\overline{BC}$ of $\triangle\overline{ABC}$.

Note 8.3 From the definition of perfect n-net follows that:

1) $\alpha_i+\beta_j=1 \leftrightarrow i+j=n$

2) $\alpha_i+\beta_j<1 \leftrightarrow i+j<n$,

where $<x,\alpha_i,\beta_j>\in Grid_n(A)$.

We can check easy that from the manner of defining $\{\alpha_i\}_{i=0}^n$ и $\{\beta_j\}_{j=0}^n$ from the Definition.8 follows the next equivalence : $\alpha_i + \beta_j > 1 \leftrightarrow i + j > n$. Then we have thah $< x, \alpha_i, \beta_j > \notin \triangle \overline{ABC}$ for these ordered triads $< x, \alpha_i, \beta_j >$ for which $\alpha_i + \beta_j > 1$.

Definition.9 Uniform perfect n-net with step h, $h \in (0,1]$, generated by an IFS A, is defined as:

$$Net_{h,n}(A) = \{ < x, \mu_A(x), \nu_A(x) > | \ x \in E, \ \mu_A(x) \in \{\alpha_0, \alpha_1, ..., \alpha_n\},$$
$$\nu_A(x) \in \{\beta_0, \beta_1, ..., \beta_n\}, \ \mu_A(x) + \nu_A(x) \le 1 \ \},$$

where $\{\alpha_i\}_{i=0}^n$ и $\{\beta_j\}_{j=0}^n$ are defined in the following inductive way:

$$\alpha_0 = 0, \ 0 \le \alpha_k \le 1, \ \alpha_k = \alpha_{k-1} + h \qquad \text{for } k = 1,2,3,...,n$$
$$0 \le b_l \le 1, \ b_l = 1 - \alpha_{n-l} \qquad \text{for } l = 0,1,2,...,n$$

□

Reference

[1]Atanassov, K., Ituitionistic Fuzzy Sets, Springer Physica-Verlag, Heidelberg, 1999