

# **Level operators over primary interval-valued intuitionistic fuzzy $M$ group**

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**Abstract:** The concept of interval-valued intuitionistic fuzzy  $M$  group is extended by introducing primary interval-valued intuitionistic fuzzy  $M$  group and primary interval-valued intuitionistic fuzzy anti  $M$  group using this concept primary interval-valued intuitionistic fuzzy  $M$  group and primary interval-valued intuitionistic fuzzy anti  $M$  group is defined and using level operators and their properties are established.

**Keywords:** Intuitionistic fuzzy set, Interval-valued intuitionistic fuzzy set, Primary interval-valued intuitionistic fuzzy  $M$  group, Primary interval-valued intuitionistic fuzzy anti  $M$  group.

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# 1 Introduction

The concept of fuzzy sets was initiated by L. A. Zadeh [8] then it has become a vigorous area of research in engineering, medical science, graph theory. Rosenfeld [7] gave the idea of fuzzy subgroup. H. J. Zimmermann [10] gave the idea of fuzzy set theory. The concept of IFS and IVIFS was introduced by K. T. Atanassov [1,2]. The author W. R. Zhang [9] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. K. Chakrabarty, R. Biwas and S. Nanda [4] investigated a note on union and intersection of intuitionistic fuzzy sets. G. Prasannavengeteswari, K. Gunasekaran and S. Nandakumar [5] introduced the definition of Primary Bipolar Intuitionistic  $M$  Fuzzy Group and anti  $M$  Fuzzy Group. A. Balasubramanian, K. L. Muruganantha Prasad, K. Arjunan [3] introduced the definition of Bipolar Interval Valued Fuzzy Subgroups of a Group. G. Prasannavengeteswari, K. Gunasekaran and S. Nandakumar [6] introduced the definition of primary interval-valued intuitionistic fuzzy  $M$  group and fuzzy anti  $M$  Group. In this study Level Operators over Primary interval-valued Intuitionistic Fuzzy  $M$  Group and Fuzzy anti  $M$  Group and some properties of the same are proved.

# 2 Preliminaries

**Definition 1.** An interval-valued intuitionistic fuzzy set (IVIFS)  $A$  over the set  $E$  is an object of the form  $A = \{(x, M_A(x), N_A(x))|x \in E\}$ , where  $M_A(x) \subset [0,1]$  and  $N_A(x) \subset [0,1]$  are intervals and  $\sup M_A(x) + \sup N_A(x) \leq 1$ , for every  $x \in E$ . Thus we can write IVIFS  $A$  as  $A = \{[x, [\inf M_A(x), \sup M_A(x)], [\inf N_A(x), \sup N_A(x)]]|x \in E\}$ . For simplicity, we write the intervals

$$[\inf M_A(x), \sup M_A(x)] = [\mu_A^-(x), \mu_A^+(x)]$$

and

$$[\inf N_A(x), \sup N_A(x)] = [\nu_A^-(x), \nu_A^+(x)],$$

where  $\mu_A^+(x), \nu_A^+(x), \mu_A^-(x), \nu_A^-(x)$  are functions from  $E$  into  $[0, 1]$  and  $(\forall x \in E)$  ( $\mu_A^-(x) \leq \mu_A^+(x), \nu_A^-(x) \leq \nu_A^+(x), \mu_A^+(x) + \nu_A^+(x) \leq 1$ ) are called the degree of positive membership, degree of negative membership, degree of positive non-membership, and the degree of negative non-membership, respectively. Note that we denote here  $\mu_A^-(x) = \inf M_A(x), \mu_A^+(x) = \sup M_A(x), \nu_A^-(x) = \inf N_A(x), \nu_A^+(x) = \sup N_A(x)$ .

**Definition 2.** Let  $G$  be an  $M$  group and  $A$  be an interval-valued intuitionistic fuzzy subgroup of  $G$ , then  $A$  is called a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ . If for all  $x, y \in G$  and  $m \in M$ , then either  $\mu_A^+(mxy) \leq \mu_A^+(x^p)$  and  $\nu_A^+(mxy) \geq \nu_A^+(x^p)$ , for some  $p \in Z_+$  or else  $\mu_A^+(mxy) \leq \mu_A^+(y^q)$  and  $\nu_A^+(mxy) \geq \nu_A^+(y^q)$ , for some  $q \in Z_+$  and either  $\mu_A^-(mxy) \geq \mu_A^-(x^p)$  and  $\nu_A^-(mxy) \leq \nu_A^-(x^p)$ , for some  $p \in Z_+$  or else  $\mu_A^-(mxy) \geq \mu_A^-(y^q)$  and  $\nu_A^-(mxy) \leq \nu_A^-(y^q)$ , for some  $q \in Z_+$ .

**Example 1.**

$$\mu_A^+(x) = \begin{cases} 0.7 & \text{if } x = 1 \\ 0.6 & \text{if } x = -1 \\ 0.4 & \text{if } x = i, -i \end{cases} \quad \nu_A^+(x) = \begin{cases} 0.2 & \text{if } x = 1 \\ 0.3 & \text{if } x = -1 \\ 0.5 & \text{if } x = i, -i \end{cases}$$

$$\mu_A^-(x) = \begin{cases} 0.6 & \text{if } x = 1 \\ 0.5 & \text{if } x = -1 \\ 0.3 & \text{if } x = i, -i \end{cases} \quad \nu_A^-(x) = \begin{cases} 0.1 & \text{if } x = 1 \\ 0.2 & \text{if } x = -1 \\ 0.5 & \text{if } x = i, -i \end{cases}$$

**Definition 3.** Let  $G$  be an  $M$  group and  $A$  be an interval-valued intuitionistic anti fuzzy subgroup of  $G$ , then  $A$  is called a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ . If for all  $x, y \in G$  and  $m \in M$ , then either  $\mu_A^+(mxy) \geq \mu_A^+(xp)$  and  $\nu_A^+(mxy) \leq \nu_A^+(xp)$ , for some  $p \in Z_+$  or else  $\mu_A^+(mxy) \geq \mu_A^+(yq)$  and  $\nu_A^+(mxy) \leq \nu_A^+(yq)$ , for some  $q \in Z_+$  and either  $\mu_A^-(mxy) \leq \mu_A^-(xp)$  and  $\nu_A^-(mxy) \geq \nu(xp)$ , for some  $p \in Z_+$  or else  $\mu_A^-(mxy) \leq \mu_A^-(yq)$  and  $\nu_A^-(mxy) \geq \nu_A^-(yq)$ , for some  $q \in Z_+$ .

**Example 2.**

$$\mu_A^+(x) = \begin{cases} 0.4 & \text{if } x = 1 \\ 0.6 & \text{if } x = -1 \\ 0.7 & \text{if } x = i, -i \end{cases} \quad \nu_A^+(x) = \begin{cases} 0.5 & \text{if } x = 1 \\ 0.3 & \text{if } x = -1 \\ 0.2 & \text{if } x = i, -i \end{cases}$$

$$\mu_A^-(x) = \begin{cases} 0.3 & \text{if } x = 1 \\ 0.5 & \text{if } x = -1 \\ 0.4 & \text{if } x = i, -i \end{cases} \quad \nu_A^-(x) = \begin{cases} 0.4 & \text{if } x = 1 \\ 0.2 & \text{if } x = -1 \\ 0.1 & \text{if } x = i, -i \end{cases}$$

**Definition 4.** Let  $A$  be an IVIFS over a set  $E$ , then the level operator  $!A$  and  $?A$  are defined as

$$!A = \left\{ \langle x, \left[ \min \left\{ \frac{1}{2}, \mu_A^+(x) \right\}, \max \left\{ \frac{1}{2}, \nu_A^+(x) \right\} \right], \left[ \max \left\{ \frac{1}{2}, \mu_A^-(x) \right\}, \min \left\{ \frac{1}{2}, \nu_A^-(x) \right\} \right] \rangle \mid x \in E \right\}$$

$$?A = \left\{ \langle x, \left[ \max \left\{ \frac{1}{2}, \mu_A^+(x) \right\}, \min \left\{ \frac{1}{2}, \nu_A^+(x) \right\} \right], \left[ \min \left\{ \frac{1}{2}, \mu_A^-(x) \right\}, \max \left\{ \frac{1}{2}, \nu_A^-(x) \right\} \right] \rangle \mid x \in E \right\}$$

### 3 Some operations on primary interval-valued intuitionistic fuzzy $M$ group and primary interval-valued intuitionistic fuzzy anti $M$ group

**Theorem 1.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ , then  $!A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\begin{aligned} \text{Consider } \mu_{!A}^+(mxy) &= \min \left( \frac{1}{2}, \mu_A^+(mxy) \right) = \min \left( \frac{1}{2}, \sup M_A(mxy) \right) \\ &\leq \min \left( \frac{1}{2}, \sup M_A(xp) \right) = \min \left( \frac{1}{2}, \mu_A^+(xp) \right) \\ &= \mu_{!A}^+(xp) \end{aligned}$$

Therefore,  $\mu_{!A}^+(mxy) \leq \mu_{!A}^+(xp)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } v_{!A}^+(mxy) &= \max\left(\frac{1}{2}, v_A^+(mxy)\right) = \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ &\geq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) = \max\left(\frac{1}{2}, v_A^+(x^p)\right) \\ &= v_{!A}^+(x^p) \end{aligned}$$

Therefore,  $v_{!A}^+(mxy) \geq v_{!A}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } \mu_{!A}^-(mxy) &= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) = \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\ &\geq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) = \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ &= \mu_{!A}^-(x^p) \end{aligned}$$

Therefore,  $\mu_{!A}^-(mxy) \geq \mu_{!A}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } v_{!A}^-(mxy) &= \min\left(\frac{1}{2}, v_A^-(mxy)\right) = \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ &\leq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) = \min\left(\frac{1}{2}, v_A^-(x^p)\right) \\ &= v_{!A}^-(x^p) \end{aligned}$$

Therefore,  $v_{!A}^-(mxy) \leq v_{!A}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 2.** If  $A$  and  $B$  are primary interval-valued intuitionistic fuzzy  $M$  groups of  $G$ , then  $!(A \cap B) = !A \cap !B$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $x, y \in B$  and  $m \in M$ .

$$\begin{aligned} \text{Consider } \mu_{!(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) = \min\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\ &= \min\left(\frac{1}{2}, \min(\sup M_A(mxy), \sup M_B(mxy))\right) \\ &\leq \min\left(\frac{1}{2}, \min(\sup M_A(x^p), \sup M_B(x^p))\right) \\ &= \min\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\ &= \min\left(\min\left(\frac{1}{2}, \mu_A^+(x^p)\right), \min\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\ &= \min(\mu_{!A}^+(x^p), \mu_{!B}^+(x^p)) \\ &= \mu_{!A \cap !B}^+(x^p) \end{aligned}$$

Therefore,  $\mu_{!(A \cap B)}^+(mxy) \leq \mu_{!A \cap !B}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } v_{!(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, v_{A \cap B}^+(mxy)\right) = \max\left(\frac{1}{2}, \max(v_A^+(mxy), v_B^+(mxy))\right) \\ &= \max\left(\frac{1}{2}, \max(\sup N_A(mxy), \sup N_B(mxy))\right) \\ &\geq \max\left(\frac{1}{2}, \max(\sup N_A(x^p), \sup N_B(x^p))\right) \\ &= \max\left(\frac{1}{2}, \max(v_A^+(x^p), v_B^+(x^p))\right) \\ &= \max\left(\max\left(\frac{1}{2}, v_A^+(x^p)\right), \max\left(\frac{1}{2}, v_B^+(x^p)\right)\right) \\ &= \max(v_{!A}^+(x^p), v_{!B}^+(x^p)) \\ &= v_{!A \cap !B}^+(x^p) \end{aligned}$$

Therefore,  $\nu_{!(A \cap B)}^+(mxy) \geq \nu_{!A \cap !B}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \mu_{!(A \cap B)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^-(mxy)\right) = \max\left(\frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy))\right) \\
&= \max\left(\frac{1}{2}, \max(\inf M_A(mxy), \inf M_B(mxy))\right) \\
&\geq \max\left(\frac{1}{2}, \max(\inf M_A(x^p), \inf M_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p))\right) \\
&= \max\left(\max\left(\frac{1}{2}, \mu_A^-(x^p)\right), \max\left(\frac{1}{2}, \mu_B^-(x^p)\right)\right) \\
&= \max(\mu_{!A}^-(x^p), \mu_{!B}^-(x^p)) \\
&= \mu_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(A \cap B)}^-(mxy) \geq \mu_{!A \cap !B}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \nu_{!(A \cap B)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{A \cap B}^-(mxy)\right) = \min\left(\frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy))\right) \\
&= \min\left(\frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy))\right) \\
&\leq \min\left(\frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \min(\nu_A^-(x^p), \nu_B^-(x^p))\right) \\
&= \min\left(\min\left(\frac{1}{2}, \nu_A^-(x^p)\right), \min\left(\frac{1}{2}, \nu_B^-(x^p)\right)\right) \\
&= \min(\nu_{!A}^-(x^p), \nu_{!B}^-(x^p)) \\
&= \nu_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(A \cap B)}^-(mxy) \leq \nu_{!A \cap !B}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!(A \cap B) = !A \cap !B$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 3.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ , then  $?A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\begin{aligned}
\text{Consider } \mu_{?A}^+(mxy) &= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) = \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \sup M_A(x^p)\right) = \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{?A}^+(mxy) \leq \mu_{?A}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \nu_{?A}^+(mxy) &= \min\left(\frac{1}{2}, \nu_A^+(mxy)\right) = \min\left(\frac{1}{2}, \sup N_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{?A}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{?A}^+(mxy) \geq \nu_{?A}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\mu_{?A}^-(mxy) &= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{?A}^-(mxy) \geq \mu_{?A}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{?A}^-(mxy) &= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{?A}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{?A}^-(mxy) \leq \nu_{?A}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $?A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 4.** If  $A$  and  $B$  are primary interval-valued intuitionistic fuzzy  $M$  groups of  $G$ , then  $?(A \cap B) = ?A \cap ?B$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $x, y \in B$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{?(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\
&= \max\left(\frac{1}{2}, \min(sup M_A(mxy), sup M_B(mxy))\right) \\
&\leq \max\left(\frac{1}{2}, \min(sup M_A(x^p), sup M_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\
&= \min\left(\max\left(\frac{1}{2}, \mu_A^+(x^p)\right), \max\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\
&= \min(\mu_{?A}^+(x^p), \mu_{?B}^+(x^p)) \\
&= \mu_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(A \cap B)}^+(mxy) \leq \mu_{?A \cap ?B}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{?(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{A \cap B}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max(\nu_A^+(mxy), \nu_B^+(mxy))\right) \\
&= \min\left(\frac{1}{2}, \max(sup N_A(mxy), sup N_B(mxy))\right) \\
&\geq \min\left(\frac{1}{2}, \max(sup N_A(x^p), sup N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \max(\nu_A^+(x^p), \nu_B^+(x^p))\right)
\end{aligned}$$

$$\begin{aligned}
&= \max \left( \min \left( \frac{1}{2}, \nu_A^+(x^p) \right), \min \left( \frac{1}{2}, \nu_B^+(x^p) \right) \right) \\
&= \max(\nu_{?A}^+(x^p), \nu_{?B}^+(x^p)) \\
&= \nu_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{?(A \cap B)}^+(mxy) \geq \nu_{?A \cap ?B}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \mu_{?(A \cap B)}^-(mxy) &= \min \left( \frac{1}{2}, \mu_{A \cap B}^-(mxy) \right) \\
&= \min \left( \frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy)) \right) \\
&= \min \left( \frac{1}{2}, \max(\inf M_A(mxy), \inf M_B(mxy)) \right) \\
&\geq \min \left( \frac{1}{2}, \max(\inf M_A(x^p), \inf M_B(x^p)) \right) \\
&= \min \left( \frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p)) \right) \\
&= \max \left( \min \left( \frac{1}{2}, \mu_A^-(x^p) \right), \min \left( \frac{1}{2}, \mu_B^-(x^p) \right) \right) \\
&= \max(\mu_{?A}^-(x^p), \mu_{?B}^-(x^p)) \\
&= \mu_{?A \cap ?B}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(A \cap B)}^-(mxy) \geq \mu_{?A \cap ?B}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \nu_{?(A \cap B)}^-(mxy) &= \max \left( \frac{1}{2}, \nu_{A \cap B}^-(mxy) \right) \\
&= \max \left( \frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy)) \right) \\
&= \max \left( \frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy)) \right) \\
&\leq \max \left( \frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p)) \right) \\
&= \max \left( \frac{1}{2}, \min(\nu_A^-(x^p), \nu_B^-(x^p)) \right) \\
&= \min \left( \max \left( \frac{1}{2}, \nu_A^-(x^p) \right), \max \left( \frac{1}{2}, \nu_B^-(x^p) \right) \right) \\
&= \min(\nu_{?A}^-(x^p), \nu_{?B}^-(x^p)) \\
&= \nu_{?A \cap ?B}^-(x^p).
\end{aligned}$$

Therefore,  $\nu_{?(A \cap B)}^-(mxy) \leq \nu_{?A \cap ?B}^-(x^p)$ , for some  $p \in Z_+$ . Therefore,  $?A \cap B = ?A \cap ?B$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 5.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ , then  $\overline{?A} = !A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\begin{aligned}
\text{Consider } \mu_{?\overline{A}}^+(mxy) &= \nu_{?A}^+(mxy) \\
&= \min \left( \frac{1}{2}, \nu_A^+(mxy) \right) \\
&= \min \left( \frac{1}{2}, \mu_A^+(mxy) \right) \\
&= \min \left( \frac{1}{2}, \sup M_A(mxy) \right) \\
&\leq \min \left( \frac{1}{2}, \sup M_A(x^p) \right)
\end{aligned}$$

$$= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\ = \mu_{!A}^+(x^p)$$

Therefore,  $\mu_{?A}^+(mxy) \leq \mu_{!A}^+(x^p)$ , for some  $p \in Z_+$ .

Consider  $\nu_{?A}^+(mxy) = \mu_{?A}^+(mxy)$

$$= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ \geq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ = \nu_{!A}^+(x^p)$$

Therefore,  $\nu_{?A}^+(mxy) \geq \nu_{!A}^+(x^p)$ , for some  $p \in Z_+$ .

Consider  $\mu_{?A}^-(mxy) = \nu_{?A}^-(mxy)$

$$= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ = \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\ = \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\ \geq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\ = \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ = \mu_{!A}^-(x^p)$$

Therefore,  $\mu_{?A}^-(mxy) \geq \mu_{!A}^-(x^p)$ , for some  $p \in Z_+$

Consider  $\nu_{?A}^-(mxy) = \mu_{?A}^-(mxy)$

$$= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\ = \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ = \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ \leq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\ = \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\ = \nu_{!A}^-(x^p)$$

Therefore,  $\nu_{?A}^-(mxy) \leq \nu_{!A}^-(x^p)$ , for some  $p \in Z_+$ . Therefore,  $?A = !A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 6.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ , then  $!(?A) = ?(!A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider  $\mu_{!(?A)}^+(mxy) = \min\left(\frac{1}{2}, \mu_{?A}^+(mxy)\right)$

$$= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(mxy)\right)\right)$$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(mxy)\right)\right) \\
&\leq \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \mu_{!A}^+(x^p)\right) \\
&= \mu_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(!A)}^+(mxy) \leq \mu_{?(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider  $\nu_{!(!A)}^+(mxy) = \max\left(\frac{1}{2}, \nu_{?A}^+(mxy)\right)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \nu_A^+(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(mxy)\right)\right) \\
&\geq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \nu_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \nu_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \nu_{!A}^+(x^p)\right) \\
&= \nu_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(!A)}^+(mxy) \geq \nu_{?(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider  $\mu_{!(!A)}^-(mxy) = \max\left(\frac{1}{2}, \mu_{?A}^-(mxy)\right)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(mxy)\right)\right) \\
&\geq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \mu_{!A}^-(x^p)\right) \\
&= \mu_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(!A)}^-(mxy) \geq \mu_{?(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
v_{! (?A)}^-(mxy) &= \min \left( \frac{1}{2}, v_{?A}^-(mxy) \right) \\
&= \min \left( \frac{1}{2}, \max \left( \frac{1}{2}, v_A^-(mxy) \right) \right) \\
&= \min \left( \frac{1}{2}, \max \left( \frac{1}{2}, \inf N_A(mxy) \right) \right) \\
&\leq \min \left( \frac{1}{2}, \max \left( \frac{1}{2}, \inf N_A(x^p) \right) \right) \\
&= \min \left( \frac{1}{2}, \max \left( \frac{1}{2}, v_A^-(x^p) \right) \right) \\
&= \max \left( \frac{1}{2}, \min \left( \frac{1}{2}, v_A^-(x^p) \right) \right) \\
&= \max \left( \frac{1}{2}, v_{!A}^-(x^p) \right) \\
&= v_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $v_{! (?A)}^-(mxy) \leq v_{?(!A)}^-(x^p)$ , for some  $p \in Z_+$ . Therefore,  $! (?A) = ? (!A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 7.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ , then  $! (\square A) = \square (!A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{! (\square A)}^+(mxy) &= \min \left( \frac{1}{2}, \mu_{\square A}^+(mxy) \right) \\
&= \min \left( \frac{1}{2}, \mu_A^+(mxy) \right) \\
&= \min \left( \frac{1}{2}, \sup M_A(mxy) \right) \\
&\leq \min \left( \frac{1}{2}, \sup M_A(x^p) \right) \\
&= \min \left( \frac{1}{2}, \mu_A^+(x^p) \right) \\
&= \mu_{!A}^+(x^p) \\
&= \mu_{\square (!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{! (\square A)}^+(mxy) \leq \mu_{\square (!A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
v_{! (\square A)}^+(mxy) &= \max \left( \frac{1}{2}, v_{\square A}^+(mxy) \right) \\
&= \max \left( \frac{1}{2}, 1 - \mu_A^+(mxy) \right) \\
&= \max \left( \frac{1}{2}, 1 - \sup M_A(mxy) \right) \\
&\geq \max \left( \frac{1}{2}, 1 - \sup M_A(x^p) \right) \\
&= \max \left( \frac{1}{2}, 1 - \mu_A^+(x^p) \right) \\
&= \max \left( \frac{1}{2}, v_A^+(x^p) \right) \\
&= v_{!A}^+(x^p)
\end{aligned}$$

$$\begin{aligned}
&= 1 - \mu_{!A}^+(x^p) \\
&= 1 - \mu_{\square(!A)}^+(x^p) \\
&= \nu_{\square(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(\square A)}^+(mxy) \geq \nu_{\square(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \mu_{!(\square A)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{\square A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p) \\
&= \mu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(\square A)}^-(mxy) \geq \mu_{\square(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \nu_{!(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= 1 - \mu_{!A}^-(x^p) \\
&= 1 - \mu_{\square(!A)}^-(x^p) \\
&= \nu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(\square A)}^-(mxy) \leq \nu_{\square(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!(\square A) = \square(!A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 8.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group, then  $?(\square A) = \square(?A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\begin{aligned}
\text{Consider } \mu_{?(\square A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\square A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \sup M_A(x^p)\right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= \mu_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{\square(A)}^+(mxy) \leq \mu_{\square(?A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{?(\square A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \sup M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, 1 - \sup M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{?A}^+(x^p) \\
&= 1 - \mu_{?A}^+(x^p) \\
&= 1 - \mu_{\square(?A)}^+(x^p) \\
&= \nu_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{?(\square A)}^+(mxy) \geq \nu_{\square(?A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\mu_{?(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p) \\
&= \mu_{\square(?A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(\square A)}^-(mxy) \geq \mu_{\square(?A)}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{?(\square A)}^-(mxy) &= \max\left(\frac{1}{2}, \nu_{\square A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{?A}^-(x^p) \\
&= 1 - \mu_{?A}^-(x^p)
\end{aligned}$$

$$= 1 - \mu_{\square(?A)}^-(x^p) \\ = \nu_{\square(?A)}^-(x^p)$$

Therefore,  $\nu_{\square(?A)}^-(mxy) \leq \nu_{\square(?A)}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $?(\square A) = \square(?A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 9.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ , then  $!(\Diamond A) = \Diamond(!A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\text{Consider } \mu_{!(\Diamond A)}^+(mxy) = \min\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\ = \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\ = \min\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\ = \min\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\ \leq \min\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\ = \min\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\ = \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\ = \mu_{!A}^+(x^p) \\ = 1 - \nu_{!A}^+(x^p) \\ = 1 - \nu_{\Diamond(!A)}^+(x^p) \\ = \mu_{\Diamond(!A)}^+(x^p)$$

Therefore,  $\mu_{!(\Diamond A)}^+(mxy) \leq \mu_{\Diamond(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\text{Consider } \nu_{!(\Diamond A)}^+(mxy) = \max\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ = \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ \geq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ = \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ = \nu_{!A}^+(x^p) \\ = \nu_{\Diamond(!A)}^+(x^p),$$

Therefore,  $\nu_{!(\Diamond A)}^+(mxy) \geq \nu_{\Diamond(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\text{Consider } \mu_{!(\Diamond A)}^-(mxy) = \max\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\ = \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\ = \max\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right)$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p) \\
&= 1 - \nu_{!A}^-(x^p) \\
&= 1 - \nu_{\Diamond(!A)}^-(x^p) \\
&= \mu_{\Diamond(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{\Diamond(!A)}^-(mxy) \geq \mu_{\Diamond(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{\Diamond(!A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&\leq \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= \nu_{\Diamond(!A)}^-(x^p).
\end{aligned}$$

Therefore,  $\nu_{\Diamond(!A)}^-(mxy) \leq \nu_{\Diamond(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!(\Diamond A) = \Diamond(!A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 10.** If  $A$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ , then  $?(\Diamond A) = \Diamond(?A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{?(\Diamond A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= 1 - \nu_{?A}^+(x^p) \\
&= 1 - \nu_{\Diamond(?A)}^+(x^p)
\end{aligned}$$

$$= \mu_{\Diamond(\Diamond A)}^+(x^p)$$

Therefore,  $\mu_{\Diamond(\Diamond A)}^+(mxy) \leq \mu_{\Diamond(\Diamond A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } \nu_{\Diamond(\Diamond A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\ &= \min\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ &= \min\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ &= \nu_{\Diamond A}^+(x^p) \\ &= \nu_{\Diamond(\Diamond A)}^+(x^p) \end{aligned}$$

Therefore,  $\nu_{\Diamond(\Diamond A)}^+(mxy) \geq \nu_{\Diamond(\Diamond A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } \mu_{\Diamond(\Diamond A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\ &= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ &= \mu_{\Diamond A}^-(x^p) \\ &= 1 - \nu_{\Diamond A}^-(x^p) \\ &= 1 - \nu_{\Diamond(\Diamond A)}^-(x^p) \\ &= \mu_{\Diamond(\Diamond A)}^-(x^p) \end{aligned}$$

Therefore,  $\mu_{\Diamond(\Diamond A)}^-(mxy) \geq \mu_{\Diamond(\Diamond A)}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } \nu_{\Diamond(\Diamond A)}^-(mxy) &= \max\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\ &= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ &= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ &\leq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\ &= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\ &= \nu_{\Diamond A}^-(x^p) \\ &= \nu_{\Diamond(\Diamond A)}^-(x^p) \end{aligned}$$

Therefore,  $\nu_{\Diamond(\Diamond A)}^-(mxy) \leq \nu_{\Diamond(\Diamond A)}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $\Diamond(\Diamond A) = \Diamond(\Diamond A)$  is a primary interval-valued intuitionistic fuzzy  $M$  group of  $G$ .  $\square$

**Theorem 11.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $!A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\begin{aligned} \text{Consider } \mu_{!A}^+(mxy) &= \min\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\ &= \min\left(\frac{1}{2}, \sup M_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, \sup M_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\ &= \mu_{!A}^+(x^p) \end{aligned}$$

Therefore,  $\mu_{!A}^+(mxy) \geq \mu_{!A}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } \nu_{!A}^+(mxy) &= \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\ &= \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\ &\leq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\ &= \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\ &= \nu_{!A}^+(x^p) \end{aligned}$$

Therefore,  $\nu_{!A}^+(mxy) \leq \nu_{!A}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } \mu_{!A}^-(mxy) &= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\ &= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\ &\leq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\ &= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\ &= \mu_{!A}^-(x^p) \end{aligned}$$

Therefore,  $\mu_{!A}^-(mxy) \leq \mu_{!A}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned} \text{Consider } \nu_{!A}^-(mxy) &= \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\ &= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\ &\geq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\ &= \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\ &= \nu_{!A}^-(x^p) \end{aligned}$$

Therefore,  $\nu_{!A}^-(mxy) \geq \nu_{!A}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 12.** If  $A$  and  $B$  are primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $!(A \cap B) = !A \cap !B$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $x, y \in B$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{!(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\
&= \min\left(\frac{1}{2}, \min(supM_A(mxy), supM_B(mxy))\right) \\
&\geq \min\left(\frac{1}{2}, \min(supM_A(x^p), supM_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\
&= \min\left(\min\left(\frac{1}{2}, \mu_A^+(x^p)\right), \min\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\
&= \min(\mu_{!A}^+(x^p), \mu_{!B}^+(x^p)) \\
&= \mu_{!A \cap !B}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(A \cap B)}^+(mxy) \geq \mu_{!A \cap !B}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{!(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, \nu_{A \cap B}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \max(\nu_A^+(mxy), \nu_B^+(mxy))\right) \\
&= \max\left(\frac{1}{2}, \max(supN_A(mxy), supN_B(mxy))\right) \\
&\leq \max\left(\frac{1}{2}, \max(supN_A(x^p), supN_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \max(\nu_A^+(x^p), \nu_B^+(x^p))\right) \\
&= \max\left(\max\left(\frac{1}{2}, \nu_A^+(x^p)\right), \max\left(\frac{1}{2}, \nu_B^+(x^p)\right)\right) \\
&= \max(\nu_{!A}^+(x^p), \nu_{!B}^+(x^p)) \\
&= \nu_{!A \cap !B}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(A \cap B)}^+(mxy) \leq \nu_{!A \cap !B}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\mu_{!(A \cap B)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy))\right) \\
&= \max\left(\frac{1}{2}, \max(infM_A(mxy), infM_B(mxy))\right) \\
&\leq \max\left(\frac{1}{2}, \max(infM_A(x^p), infM_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p))\right) \\
&= \max\left(\max\left(\frac{1}{2}, \mu_A^-(x^p)\right), \max\left(\frac{1}{2}, \mu_B^-(x^p)\right)\right) \\
&= \max(\mu_{!A}^-(x^p), \mu_{!B}^-(x^p)) \\
&= \mu_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(A \cap B)}^-(mxy) \leq \mu_{!A \cap !B}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{!(A \cap B)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{A \cap B}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy))\right)
\end{aligned}$$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy))\right) \\
&\geq \min\left(\frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \min(v_A^-(x^p), v_B^-(x^p))\right) \\
&= \min\left(\min\left(\frac{1}{2}, v_A^-(x^p)\right), \min\left(\frac{1}{2}, v_B^-(x^p)\right)\right) \\
&= \min(v_{!A}^-(x^p), v_{!B}^-(x^p)) \\
&= v_{!A \cap !B}^-(x^p)
\end{aligned}$$

Therefore,  $v_{!A \cap !B}^-(mxy) \geq v_{!A \cap !B}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!(A \cap B) = !A \cap !B$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 13.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $?A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{?A}^+(mxy) &= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \sup M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{?A}^+(mxy) \geq \mu_{?A}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
v_{?A}^+(mxy) &= \min\left(\frac{1}{2}, v_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \sup N_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, v_A^+(x^p)\right) \\
&= v_{?A}^+(x^p)
\end{aligned}$$

Therefore,  $v_{?A}^+(mxy) \leq v_{?A}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\mu_{?A}^-(mxy) &= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{?A}^-(mxy) \leq \mu_{?A}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
v_{?A}^-(mxy) &= \max\left(\frac{1}{2}, v_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, v_A^-(x^p)\right) \\
&= v_{?A}^-(x^p)
\end{aligned}$$

Therefore,  $v_{?A}^-(mxy) \geq v_{?A}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $?A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 14.** If  $A$  and  $B$  are primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $?A \cap B = ?A \cap ?B$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $x, y \in B$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{?(A \cap B)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{A \cap B}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(mxy), \mu_B^+(mxy))\right) \\
&= \max\left(\frac{1}{2}, \min(\sup M_A(mxy), \sup M_B(mxy))\right) \\
&\geq \max\left(\frac{1}{2}, \min(\sup M_A(x^p), \sup M_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \min(\mu_A^+(x^p), \mu_B^+(x^p))\right) \\
&= \min\left(\max\left(\frac{1}{2}, \mu_A^+(x^p)\right), \max\left(\frac{1}{2}, \mu_B^+(x^p)\right)\right) \\
&= \min(\mu_{?A}^+(x^p), \mu_{?B}^+(x^p)) \\
&= \mu_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(A \cap B)}^+(mxy) \geq \mu_{?A \cap ?B}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
v_{?(A \cap B)}^+(mxy) &= \min\left(\frac{1}{2}, v_{A \cap B}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max(v_A^+(mxy), v_B^+(mxy))\right) \\
&= \min\left(\frac{1}{2}, \max(\sup N_A(mxy), \sup N_B(mxy))\right) \\
&\leq \min\left(\frac{1}{2}, \max(\sup N_A(x^p), \sup N_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \max(v_A^+(x^p), v_B^+(x^p))\right) \\
&= \max\left(\min\left(\frac{1}{2}, v_A^+(x^p)\right), \min\left(\frac{1}{2}, v_B^+(x^p)\right)\right) \\
&= \max(v_{?A}^+(x^p), v_{?B}^+(x^p)) \\
&= v_{?A \cap ?B}^+(x^p)
\end{aligned}$$

Therefore,  $v_{?(A \cap B)}^+(mxy) \leq v_{?A \cap ?B}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\mu_{?(A \cap B)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{A \cap B}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max(\mu_A^-(mxy), \mu_B^-(mxy))\right) \\
&= \min\left(\frac{1}{2}, \max(\inf M_A(mxy), \inf M_B(mxy))\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \min\left(\frac{1}{2}, \max(\inf M_A(x^p), \inf M_B(x^p))\right) \\
&= \min\left(\frac{1}{2}, \max(\mu_A^-(x^p), \mu_B^-(x^p))\right) \\
&= \max\left(\min\left(\frac{1}{2}, \mu_A^-(x^p)\right), \min\left(\frac{1}{2}, \mu_B^-(x^p)\right)\right) \\
&= \max(\mu_{?A}^-(x^p), \mu_{?B}^-(x^p)) \\
&= \mu_{?A \cap ?B}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(A \cap B)}^-(mxy) \leq \mu_{?A \cap ?B}^-(x^p)$ , for some  $p \in Z_+$ .

Consider  $\nu_{?(A \cap B)}^-(mxy) = \max\left(\frac{1}{2}, \nu_{A \cap B}^-(mxy)\right)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \min(\nu_A^-(mxy), \nu_B^-(mxy))\right) \\
&= \max\left(\frac{1}{2}, \min(\inf N_A(mxy), \inf N_B(mxy))\right) \\
&\geq \max\left(\frac{1}{2}, \min(\inf N_A(x^p), \inf N_B(x^p))\right) \\
&= \max\left(\frac{1}{2}, \min(\nu_A^-(x^p), \nu_B^-(x^p))\right) \\
&= \min\left(\max\left(\frac{1}{2}, \nu_A^-(x^p)\right), \max\left(\frac{1}{2}, \nu_B^-(x^p)\right)\right) \\
&= \min(\nu_{?A}^-(x^p), \nu_{?B}^-(x^p)) \\
&= \nu_{?A \cap ?B}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{?(A \cap B)}^-(mxy) \geq \nu_{?A \cap ?B}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $?A \cap B = ?A \cap ?B$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 15.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $\overline{?A} = !A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider  $\mu_{?\overline{A}}^+(mxy) = \nu_{?\overline{A}}^+(mxy)$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \nu_{\overline{A}}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \sup M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{!A}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{?\overline{A}}^+(mxy) \geq \mu_{!A}^+(x^p)$ , for some  $p \in Z_+$ .

Consider  $\nu_{?\overline{A}}^+(mxy) = \mu_{?\overline{A}}^+(mxy)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \mu_{\overline{A}}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup N_A(mxy)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, v_A^+(x^p)\right) \\
&= v_{!A}^+(x^p)
\end{aligned}$$

Therefore,  $v_{?A}^+(mxy) \leq v_{!A}^+(x^p)$ , for some  $p \in Z_+$ .

Consider  $\mu_{?A}^-(mxy) = v_{?A}^-(mxy)$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, v_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{?A}^-(mxy) \leq \mu_{!A}^-(x^p)$ , for some  $p \in Z_+$ .

Consider  $v_{?A}^-(mxy) = \mu_{?A}^-(mxy)$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, v_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, v_A^-(x^p)\right) \\
&= v_{!A}^-(x^p)
\end{aligned}$$

Therefore,  $v_{?A}^-(mxy) \geq v_{!A}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $?A = !A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 16.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $!(?A) = ?(!A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\begin{aligned}
\text{Consider } \mu_{!(?A)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{?A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(mxy)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(mxy)\right)\right) \\
&\geq \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \sup M_A(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^+(x^p)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \mu_{!A}^+(x^p)\right) \\
&= \mu_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(!A)}^+(mxy) \geq \mu_{?(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } v_{!(!A)}^+(mxy) &= \max\left(\frac{1}{2}, v_{?A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, v_A^+(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(mxy)\right)\right) \\
&\leq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \sup N_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, v_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, v_A^+(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, v_{!A}^+(x^p)\right) \\
&= v_{?(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $v_{!(!A)}^+(mxy) \leq v_{?(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \mu_{!(!A)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{?A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(mxy)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(mxy)\right)\right) \\
&\leq \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \inf M_A(x^p)\right)\right) \\
&= \max\left(\frac{1}{2}, \min\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \mu_A^-(x^p)\right)\right) \\
&= \min\left(\frac{1}{2}, \mu_{!A}^-(x^p)\right) \\
&= \mu_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(!A)}^-(mxy) \leq \mu_{?(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } v_{!(!A)}^-(mxy) &= \min\left(\frac{1}{2}, v_{?A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, v_A^-(mxy)\right)\right) \\
&= \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \inf N_A(mxy)\right)\right) \\
&\geq \min\left(\frac{1}{2}, \max\left(\frac{1}{2}, \inf N_A(x^p)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \min \left( \frac{1}{2}, \max \left( \frac{1}{2}, \nu_A^-(x^p) \right) \right) \\
&= \max \left( \frac{1}{2}, \min \left( \frac{1}{2}, \nu_A^-(x^p) \right) \right) \\
&= \max \left( \frac{1}{2}, \nu_{!A}^-(x^p) \right) \\
&= \nu_{?(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(!A)}^-(mxy) \geq \nu_{?(!A)}^-(x^p)$ , for some  $p \in Z_+$ . Therefore,  $!(!A) = ?(!A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 17.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $!(\square A) = \square(!A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{!(\square A)}^+(mxy) &= \min \left( \frac{1}{2}, \mu_{\square A}^+(mxy) \right) \\
&= \min \left( \frac{1}{2}, \mu_A^+(mxy) \right) \\
&= \min \left( \frac{1}{2}, \sup M_A(mxy) \right) \\
&\geq \min \left( \frac{1}{2}, \sup M_A(x^p) \right) \\
&= \min \left( \frac{1}{2}, \mu_A^+(x^p) \right) \\
&= \mu_{!A}^+(x^p) \\
&= \mu_{\square(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(\square A)}^+(mxy) \geq \mu_{\square(!A)}^+(x^p)$ , for some  $p \in Z_+$

Consider

$$\begin{aligned}
\nu_{!(\square A)}^+(mxy) &= \max \left( \frac{1}{2}, \nu_{\square A}^+(mxy) \right) \\
&= \max \left( \frac{1}{2}, 1 - \mu_A^+(mxy) \right) \\
&= \max \left( \frac{1}{2}, 1 - \sup M_A(mxy) \right) \\
&\leq \max \left( \frac{1}{2}, 1 - \sup M_A(x^p) \right) \\
&= \max \left( \frac{1}{2}, 1 - \mu_A^+(x^p) \right) \\
&= \max \left( \frac{1}{2}, \nu_A^+(x^p) \right) \\
&= \nu_{!A}^+(x^p) \\
&= 1 - \mu_{!A}^+(x^p) \\
&= 1 - \mu_{\square(!A)}^+(x^p) \\
&= \nu_{\square(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(\square A)}^+(mxy) \leq \nu_{\square(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\mu_{!(\square A)}^-(mxy) &= \max \left( \frac{1}{2}, \mu_{\square A}^-(mxy) \right) \\
&= \max \left( \frac{1}{2}, \mu_A^-(mxy) \right)
\end{aligned}$$

$$\begin{aligned}
&= \max\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p) \\
&= \mu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(\square A)}^-(mxy) \leq \mu_{\square(!A)}^-(x^p)$ , for some  $p \in Z_+$

Consider

$$\begin{aligned}
\nu_{!(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= 1 - \mu_{!A}^-(x^p) \\
&= 1 - \mu_{\square(!A)}^-(x^p) \\
&= \nu_{\square(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(\square A)}^-(mxy) \geq \nu_{\square(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!(\square A) = \square(!A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 18.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $?(\square A) = \square(?A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{?(\square A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\square A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \sup M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= \mu_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(\square A)}^+(mxy) \geq \mu_{\square(?A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{?(\square A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\square A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(mxy)\right)
\end{aligned}$$

$$\begin{aligned}
&= \min\left(\frac{1}{2}, 1 - \sup M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, 1 - \sup M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \mu_A^+(x^p)\right) \\
&= \min\left(\frac{1}{2}, v_A^+(x^p)\right) \\
&= v_{?A}^+(x^p) \\
&= 1 - \mu_{?A}^+(x^p) \\
&= 1 - \mu_{\square(?A)}^+(x^p) \\
&= v_{\square(?A)}^+(x^p)
\end{aligned}$$

Therefore,  $v_{?(\square A)}^+(mxy) \leq v_{\square(?A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\mu_{?(\square A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\square A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf M_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, \inf M_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p) \\
&= \mu_{\square(?A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(\square A)}^-(mxy) \leq \mu_{\square(?A)}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
v_{?(\square A)}^-(mxy) &= \max\left(\frac{1}{2}, v_{\square A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \inf M_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, 1 - \inf M_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \mu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, v_A^-(x^p)\right) \\
&= v_{?A}^-(x^p) \\
&= 1 - \mu_{?A}^-(x^p) \\
&= 1 - \mu_{\square(?A)}^-(x^p) \\
&= v_{\square(?A)}^-(x^p).
\end{aligned}$$

Therefore,  $v_{?(\square A)}^-(mxy) \geq v_{\square(?A)}^-(x^p)$ , for some  $p \in Z_+$ . Therefore,  $?(\square A) = \square(?A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 19.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $!(\diamond A) = \diamond !(A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

$$\begin{aligned}
\text{Consider } \mu_{!(\Diamond A)}^+(mxy) &= \min\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\
&\geq \min\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{!A}^+(x^p) \\
&= 1 - \nu_{!A}^+(x^p) \\
&= 1 - \nu_{\Diamond(!A)}^+(x^p) \\
&= \mu_{\Diamond(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{!(\Diamond A)}^+(mxy) \geq \mu_{\Diamond(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \nu_{!(\Diamond A)}^+(mxy) &= \max\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, \sup N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{!A}^+(x^p) \\
&= \nu_{\Diamond(!A)}^+(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(\Diamond A)}^+(mxy) \leq \nu_{\Diamond(!A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \mu_{!(\Diamond A)}^-(mxy) &= \max\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\
&\leq \max\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{!A}^-(x^p)
\end{aligned}$$

$$\begin{aligned}
&= 1 - \nu_{!A}^-(x^p) \\
&= 1 - \nu_{\Diamond(!A)}^-(x^p) \\
&= \mu_{\Diamond(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{\Diamond(!A)}^-(mxy) \leq \mu_{\Diamond(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{!(\Diamond A)}^-(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&= \min\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&\geq \min\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{!A}^-(x^p) \\
&= \nu_{\Diamond(!A)}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{!(\Diamond A)}^-(mxy) \geq \nu_{\Diamond(!A)}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $!(\Diamond A) = \Diamond(!A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

**Theorem 20.** If  $A$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ , then  $?(\Diamond A) = \Diamond(?A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .

*Proof.* Consider  $x, y \in A$  and  $m \in M$ .

Consider

$$\begin{aligned}
\mu_{?(\Diamond A)}^+(mxy) &= \max\left(\frac{1}{2}, \mu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(mxy)\right) \\
&= \max\left(\frac{1}{2}, 1 - \sup N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, 1 - \sup N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, 1 - \nu_A^+(x^p)\right) \\
&= \max\left(\frac{1}{2}, \mu_A^+(x^p)\right) \\
&= \mu_{?A}^+(x^p) \\
&= 1 - \nu_{?A}^+(x^p) \\
&= 1 - \nu_{\Diamond(?A)}^+(x^p) \\
&= \mu_{\Diamond(?A)}^+(x^p)
\end{aligned}$$

Therefore,  $\mu_{?(\Diamond A)}^+(mxy) \geq \mu_{\Diamond(?A)}^+(x^p)$ , for some  $p \in Z_+$ .

Consider

$$\begin{aligned}
\nu_{?(\Diamond A)}^+(mxy) &= \min\left(\frac{1}{2}, \nu_{\Diamond A}^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(mxy)\right) \\
&= \min\left(\frac{1}{2}, \sup N_A(mxy)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \min\left(\frac{1}{2}, \sup N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, \nu_A^+(x^p)\right) \\
&= \nu_{?A}^+(x^p) \\
&= \nu_{\Diamond(?A)}^+(x^p).
\end{aligned}$$

Therefore,  $\nu_{\Diamond(?A)}^+(mxy) \leq \nu_{\Diamond(?A)}^+(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \mu_{\Diamond(?A)}^-(mxy) &= \min\left(\frac{1}{2}, \mu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_{\Diamond A}^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^-(mxy)\right) \\
&= \min\left(\frac{1}{2}, 1 - \inf N_A(mxy)\right) \\
&\leq \min\left(\frac{1}{2}, 1 - \inf N_A(x^p)\right) \\
&= \min\left(\frac{1}{2}, 1 - \nu_A^-(x^p)\right) \\
&= \min\left(\frac{1}{2}, \mu_A^-(x^p)\right) \\
&= \mu_{?A}^-(x^p) \\
&= 1 - \nu_{?A}^-(x^p) \\
&= 1 - \nu_{\Diamond(?A)}^-(x^p) \\
&= \mu_{\Diamond(?A)}^-(x^p)
\end{aligned}$$

Therefore,  $\mu_{\Diamond(?A)}^-(mxy) \leq \mu_{\Diamond(?A)}^-(x^p)$ , for some  $p \in Z_+$ .

$$\begin{aligned}
\text{Consider } \nu_{\Diamond(?A)}^-(mxy) &= \max\left(\frac{1}{2}, \nu_{\Diamond A}^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(mxy)\right) \\
&= \max\left(\frac{1}{2}, \inf N_A(mxy)\right) \\
&\geq \max\left(\frac{1}{2}, \inf N_A(x^p)\right) \\
&= \max\left(\frac{1}{2}, \nu_A^-(x^p)\right) \\
&= \nu_{?A}^-(x^p) \\
&= \nu_{\Diamond(?A)}^-(x^p)
\end{aligned}$$

Therefore,  $\nu_{\Diamond(?A)}^-(mxy) \geq \nu_{\Diamond(?A)}^-(x^p)$ , for some  $p \in Z_+$ .

Therefore,  $?(\Diamond A) = \Diamond(?A)$  is a primary interval-valued intuitionistic fuzzy anti  $M$  group of  $G$ .  $\square$

## 4 Conclusion

In this paper the main idea of primary interval-valued intuitionistic fuzzy  $M$  group and primary interval-valued intuitionistic fuzzy anti  $M$  group are a new algebraic structures of fuzzy algebra and it is used through the level operators. We believe that our ideas can also applied for other algebraic system.

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