**Notes on Intuitionistic Fuzzy Sets**
**Print ISSN 1310–4926, Online ISSN 2367–8283**
**2022, Volume 28, Number 4, 428–435**
**DOI: 10.7546/nifs.28.4.428-435**

# Intuitionistic fuzzy interpretations of formula $(A \to B) \to ((\neg A \to B) \to B)$

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**Received:** 21 June 2022 **Revised:** 8 October 2022 **Accepted:** 10 November 2022

**Abstract:** One of the essential formulas of the classical mathematical logic is

$$(A \to B) \to ((\neg A \to B) \to B).$$

In the present paper, its intuitionistic fuzzy interpretation is introduced, and lists of all defined intuitionistic fuzzy implications that satisfy it as a tautology and an intuitionistic fuzzy tautology are given.

**Keywords:** Intuitionistic fuzzy implication, Intuitionistic fuzzy pair, Intuitionistic fuzzy logic.
**2020 Mathematics Subject Classification:** 03E72.

* Paper presented at the International Workshop on Intuitionistic Fuzzy Sets, founded by Prof. Beloslav Riečan, 2 December 2022, Banská Bystrica, Slovakia.

# 1 Introduction

In a series of papers, some of the basic properties of the intuitionistic fuzzy implications are discussed (see, e.g., [2, 5–15, 17, 18]).

In the present paper, a new interesting formula from the classical logic obtains an intuitionistic fuzzy interpretation and for it, the lists of the intuitionistic fuzzy implications that satisfy it as tautologies and as intuitionistic fuzzy tautologies are given.

In the paper, we use the notation from [3]. Here, we will mention only that the ordered pair $\langle a, b \rangle$ in [4] is called an Intuituinistic Fuzzy Pair (IFP) if $a, b, a + b \in [0, 1]$. Here we are going to work with these pairs.

The IFP $\langle a, b \rangle$ is:

- a tautology if and only if (iff) $a = 1$ and $b = 0$;
- an intuitionistic fuzzy tautology (IFT) iff $a \geq b$.

Let everywhere the IFPs $A$ and $B$ have the forms: $A = \langle a, b \rangle, B = \langle c, d \rangle$, i.e., $a, b, c, d, a + b$, $c + d \in [0, 1]$.

In some definitions we shall use functions sg and $\overline{\text{sg}}$:

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \qquad \overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}.$$

# 2 Main results

In [16], the following formula of the classical mathematical logic is given:

$$(A \to B) \to ((\neg A \to B) \to B).$$

Since at the moment in the intuitionistic fuzzy logic more than 200 different intuitionistic fuzzy implications have been defined, we will rewrite the above formula to the form:

$$(A \to_i B) \to_i ((\neg_{\sigma(i)} A \to_i B) \to_i B) \tag{1}$$

and will study which intuitionistic fuzzy implications satisfy it and in which form - as tautologies or as IFTs.

**Theorem 1.** Formula (1) is an IFT for for $i = 1, 4, 5, 6, 7, 9, 12, 13, 17, 18, 20, 21, 22, 23, 25, 27, 28, 29, 30, 33, 34, 35, 36, 38, 42, 43, 44, 45, 46, 49, 50, 51, 53, 57, 61, 64, 66, 71, 72, 74, 75, 76, 77, 79, 80, 81, 82, 85, 88, 89, 90, 91, 94, 97, 100, 101, 102, 103, 104, 105, 106, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 120, 121, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 151, 153, 158, 159, 160, 161, 166, 167, 168, 169, 170, 177, 178, 179, 180, 187, 188, 189, 190, 192, 193, 194, 195, 196, 197, 198, 204, 206$ when the negation $\neg_{\sigma(i)}$ is generated by the implication $\to_i$.

*Proof.* For $i = 1$, we obtain that $\sigma(1) = 1$ and

$$\begin{aligned}
&(A \to_1 B) \to_1 ((\neg_1 A \to_1 B) \to_1 B)\\
&\quad = (\langle a, b\rangle \to_1 \langle c, d\rangle) \to_1 ((\neg_1\langle a, b\rangle \to_1 \langle c, d\rangle) \to_1 \langle c, d\rangle)\\
&\quad = \langle \max(b, \min(a, c)), \min(a, d)\rangle \to_1 ((\langle b, a\rangle \to_1 \langle c, d\rangle) \to_1 \langle c, d\rangle)\\
&\quad = \langle \max(b, \min(a, c)), \min(a, d)\rangle \to_1 (\langle \max(a, \min(b, c)), \min(b, d)\rangle \to_1 \langle c, d\rangle)\\
&\quad = \langle \max(b, \min(a, c)), \min(a, d)\rangle \to_1 \langle \max(\min(b, d), \min(\max(a, \min(b, c)), c)),\\
&\qquad\qquad \min(\max(a, \min(b, c)), d)\rangle\\
&\quad = \langle \max(\min(a, d), \min(\max(b, \min(a, c)), \max(\min(b, d), \min(\max(a, \min(b, c)), c)))),\\
&\qquad \min(\max(b, \min(a, c)), \min(\max(a, \min(b, c)), d))\rangle.
\end{aligned}$$

Let

$$\begin{aligned}
X \equiv{}& \max(\min(a, d), \min(\max(b, \min(a, c)), \max(\min(b, d), \min(\max(a, \min(b, c)), c))))\\
&- \min(\max(b, \min(a, c)), \min(\max(a, \min(b, c)), d)).
\end{aligned}$$

If $b \geq c$, then

$$\begin{aligned}
X ={}& \max(\min(a, d), \min(\max(b, \min(a, c)), \max(\min(b, d), \min(\max(a, c), c))))\\
&- \min(\max(b, \min(a, c)), \min(\max(a, c), d))\\
={}& \max(\min(a, d), \min(b, \max(\min(b, d), c))) - \min(b, \max(a, c), d).
\end{aligned}$$

If $b \leq d$, then

$$\begin{aligned}
X &= \max(\min(a, d), \min(b, \max(b, c))) - \min(b, \max(a, c), d)\\
&= \max(\min(a, d), b) - \min(b, \max(a, c), d)\\
&\geq b - b\\
&= 0.
\end{aligned}$$

If $b > d$, then

$$\begin{aligned}
X &= \max(\min(a, d), \min(b, \max(d, c))) - \min(b, \max(a, c), d)\\
&= \max(\min(a, d), \max(c, d)) - \min(\max(a, c), d)\\
&= \max(\min(a, d), c, d) - \min(\max(a, c), d)\\
&\geq d - d\\
&= 0.
\end{aligned}$$

If $b < c$, then

$$\begin{aligned}
X ={}& \max(\min(a, d), \min(\max(b, \min(a, c)), \max(\min(b, d), \min(\max(a, b), c))))\\
&- \min(\max(b, \min(a, c)), \min(\max(a, b), d)).
\end{aligned}$$

If $b \leq d$, then

$$
\begin{aligned}
X &= \max(\min(a,d), \min(\max(b,\min(a,c)), \max(b, \min(\max(a,b),c)))) \\
&\quad - \min(\max(b,\min(a,c)), \min(\max(a,b),d)) \\
&= \max(\min(a,d), \min(\max(b,\min(a,c)), \max(b, \max(\min(a,c),\min(b,c))))) \\
&\quad - \min(\max(b,\min(a,c)), \min(\max(a,b),d)) \\
&= \max(\min(a,d), \min(\max(b,\min(a,c)), \max(b, \min(a,c)))) \\
&\quad - \min(\max(b,\min(a,c)), \min(\max(a,b),d)) \\
&\geq \max(b,\min(a,c)) - \max(b,\min(a,c)) \\
&= 0.
\end{aligned}
$$

If $b > d$, then

$$
\begin{aligned}
X &= \max(\min(a,d), \min(\max(b,\min(a,c)), \max(d, \min(\max(a,b),c)))) \\
&\quad - \min(\max(b,\min(a,c)), \min(\max(a,b),d)) \\
&= \max(\min(a,d), \min(\max(b,\min(a,c)), \max(d, \max(\min(a,c),\min(b,c))))) \\
&\quad - \min(\max(b,\min(a,c)), \min(\max(a,b),d)) \\
&= \max(\min(a,d), \min(\max(b,\min(a,c)), \max(d, \max(\min(a,c),b)))) \\
&\quad - \min(\max(b,\min(a,c)), \min(\max(a,b),d)) \\
&= \max(\min(a,d), \min(\max(b,\min(a,c)), \max(d, \min(a,c),b))) \\
&\quad - \min(\max(b,\min(a,c)), \min(\max(a,b),d)) \\
&\geq \max(b,\min(a,c)) - \max(b,\min(a,c)) \\
&= 0.
\end{aligned}
$$

Therefore, for $i = 1$, (1) is an IFT. By the same manner, it is checked that the Theorem 1 is valid for the other values of $i$. □

**Theorem 2.** Formula (1) is a tautology for 20, 23, 42, 45, 57, 74, 77, 88, 90, 97, 153, 206 when the negation $\neg_{\sigma(i)}$ is generated by the implication $\rightarrow_i$.

*Proof.* Let $i = 20$. Then $\sigma(20) = 2$ and

$$
\begin{aligned}
&(A \rightarrow_{20} B) \rightarrow_{20} ((\neg_2 A \rightarrow_{20} B) \rightarrow_{20} B) \\
&\quad = (\langle a,b\rangle \rightarrow_{20} \langle c,d\rangle) \rightarrow_{20} ((\neg_2\langle a,b\rangle \rightarrow_{20} \langle c,d\rangle) \rightarrow_{20} \langle c,d\rangle) \\
&\quad = \langle \max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c)), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(c))\rangle \rightarrow_{20} ((\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle \rightarrow_{20} \langle c,d\rangle) \rightarrow_{20} \langle c,d\rangle) \\
&\quad = \langle \max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c)), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(c))\rangle \\
&\qquad \rightarrow_{20} (\langle \max(\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)), \mathrm{sg}(c)), \min(\mathrm{sg}(\overline{\mathrm{sg}}(a)), \overline{\mathrm{sg}}(c))\rangle \rightarrow_{20} \langle c,d\rangle) \\
&\qquad\qquad \textit{(because for each } x \in [0,1] : \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(x)) = \mathrm{sg}(x) \textit{ and } \mathrm{sg}(\overline{\mathrm{sg}}(x)) = 1 - \mathrm{sg}(x)\textit{)} \\
&\quad = \langle \max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c)), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(c))\rangle \\
&\qquad \rightarrow_{20} (\langle \max(\mathrm{sg}(a), \mathrm{sg}(c)), \min(1 - \mathrm{sg}(a), \overline{\mathrm{sg}}(c))\rangle \rightarrow_{20} \langle c,d\rangle)
\end{aligned}
$$

$$= \langle \max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c)), \min(\mathrm{sg}(a), \overline{\mathrm{sg}}(c))\rangle$$
$$\rightarrow_{20} \langle \max(\overline{\mathrm{sg}}(\max(\mathrm{sg}(a), \mathrm{sg}(c))), \mathrm{sg}(c)), \min(\mathrm{sg}(\max(\mathrm{sg}(a), \mathrm{sg}(c))), \overline{\mathrm{sg}}(c))\rangle$$
$$= \langle \max(\overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c))), \mathrm{sg}(\max(\overline{\mathrm{sg}}(\max(\mathrm{sg}(a), \mathrm{sg}(c))), \mathrm{sg}(c)))),$$
$$\min(\mathrm{sg}(\max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c))), \overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(\max(\mathrm{sg}(a), \mathrm{sg}(c))), \mathrm{sg}(c))))\rangle.$$

Let
$$X \equiv \max(\overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c))), \mathrm{sg}(\max(\overline{\mathrm{sg}}(\max(\mathrm{sg}(a), \mathrm{sg}(c))), \mathrm{sg}(c)))).$$

If $a = 0$, then $\mathrm{sg}(a) = 0, \overline{\mathrm{sg}}(a) = 1$ and
$$X = \max(\overline{\mathrm{sg}}(\max(1, \mathrm{sg}(c))), \mathrm{sg}(\max(\overline{\mathrm{sg}}(\max(0, \mathrm{sg}(c))), \mathrm{sg}(c))))$$
$$= \max(\overline{\mathrm{sg}}(1), \mathrm{sg}(\max(\overline{\mathrm{sg}}(\mathrm{sg}(c)), \mathrm{sg}(c))))$$
$$= \max(0, \mathrm{sg}(\max(\overline{\mathrm{sg}}(\mathrm{sg}(c)), \mathrm{sg}(c))))$$
$$= \mathrm{sg}(\max(\overline{\mathrm{sg}}(\mathrm{sg}(c)), \mathrm{sg}(c)))$$
*(because for each* $x \in [0, 1] : \overline{\mathrm{sg}}(\mathrm{sg}(x)) = \overline{\mathrm{sg}}(x)$*)*
$$= \mathrm{sg}(\max(\overline{\mathrm{sg}}(c), \mathrm{sg}(c)))$$
*(because for each* $x \in [0, 1] : \max(\overline{\mathrm{sg}}(x), \mathrm{sg}(x)) = 1$*)*
$$= \mathrm{sg}(1)$$
$$= 1.$$

If $a > 0$, then $\mathrm{sg}(a) = 1, \overline{\mathrm{sg}}(a) = 0$ and
$$X = \max(\overline{\mathrm{sg}}(\max(0, \mathrm{sg}(c))), \mathrm{sg}(\max(\overline{\mathrm{sg}}(\max(1, \mathrm{sg}(c))), \mathrm{sg}(c))))$$
$$= \max(\overline{\mathrm{sg}}(\mathrm{sg}(c)), \mathrm{sg}(\max(\overline{\mathrm{sg}}(1), \mathrm{sg}(c))))$$
$$= \max(\overline{\mathrm{sg}}(c), \mathrm{sg}(\max(0, \mathrm{sg}(c))))$$
$$= \max(\overline{\mathrm{sg}}(c), \mathrm{sg}(\mathrm{sg}(c)))$$
*(because for each* $x \in [0, 1] : \mathrm{sg}(\mathrm{sg}(x)) = \mathrm{sg}(x)$*)*
$$= \max(\overline{\mathrm{sg}}(c), \mathrm{sg}(c))$$
$$= 1.$$

Let
$$Y \equiv \min(\mathrm{sg}(\max(\overline{\mathrm{sg}}(a), \mathrm{sg}(c))), \overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(\max(\mathrm{sg}(a), \mathrm{sg}(c))), \mathrm{sg}(c)))).$$

If $a = 0$, then
$$Y = \min(\mathrm{sg}(\max(1, \mathrm{sg}(c))), \overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(\max(0, \mathrm{sg}(c))), \mathrm{sg}(c))))$$
$$= \min(\mathrm{sg}(1), \overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(\mathrm{sg}(c)), \mathrm{sg}(c))))$$
$$= \min(1, \overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(c), \mathrm{sg}(c))))$$
$$= \overline{\mathrm{sg}}(\max(\overline{\mathrm{sg}}(c), \mathrm{sg}(c)))$$
$$= \overline{\mathrm{sg}}(1)$$
$$= 0.$$

If $a > 0$, then

$$\begin{aligned}
Y &= \min(\text{sg}(\max(0, \text{sg}(c))), \overline{\text{sg}}(\max(\overline{\text{sg}}(\max(1, \text{sg}(c))), \text{sg}(c)))) \\
&= \min(\text{sg}(\text{sg}(c)), \overline{\text{sg}}(\max(\overline{\text{sg}}(1), \text{sg}(c)))) \\
&= \min(\text{sg}(c), \overline{\text{sg}}(\max(0, \text{sg}(c)))) \\
&= \min(\text{sg}(c), \overline{\text{sg}}(\text{sg}(c))) \\
&= \min(\text{sg}(c), \overline{\text{sg}}(c)) \\
&= 0.
\end{aligned}$$

Therefore, for $i = 20$, (1) is a tautology. By the same manner, it is checked that the Theorem 2 is valid for the other values of $i$. □

A modification of formula (1) is the following formula

$$(A \to_i B) \to_i ((\neg A \to_i B) \to_i B). \tag{2}$$

In it, the negation is the classical negation $\neg_1$. In this case, the following two assertions are valid and they are checked by the above manner.

**Theorem 3.** Formula (2) is an IFT for for $i = 1, 4, 5, 6, 7, 9, 12, 13, 14, 17, 18, 21, 22, 23, 25, 27, 28, 29, 46, 48, 49, 50, 51, 53, 57, 61, 64, 66, 71, 72, 75, 80, 81, 91, 94, 100, 101, 102, 108, 109, 110, 111, 112, 113, 117, 120, 121, 122, 124, 125, 126, 127, 128, 133, 134, 135, 136, 137, 151, 153, 158, 159, 160, 161, 166, 169, 170, 179, 180, 187, 189, 190, 192, 193, 197, 198, 201, 202, 204, 206.$

**Theorem 4.** Formula (2) is a tautology for $i = 23, 57, 153.$

The verification of Theorems 3 and 4 is done with the aid of the software for intuitionistic fuzzy sets IFSTOOL, [1].

# 3 Conclusion

The present formula will be used for the intuitionistic fuzzy implications classification and as a criterion for the determination which are the implications that have the best properties, i.e., that satisfy the biggest number of well-known and useful properties.

# Acknowledgement

This research was partially funded by the Bulgarian National Science Fund, grant number KP-06-N22/1/2018 “Theoretical research and applications of InterCriteria Analysis”.
The authors are grateful for the support from Mobility project BAS-SAS-21-01, by the Joint Polish - Slovak project under the agreement on scientific cooperation between the Polish Academy of Sciences and the Slovak Academy of Sciences, reg. num. 15 and from the Operational Programme Integrated Infrastructure (OPII) under project 313011BWH2: “InoCHF – Research and development in the field of innovative technologies in the management of patients with CHF”, co-financed by the European Regional Development Fund.

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