Notes on Intuitionistic Fuzzy Sets
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Direct product of finite intuitionistic anti fuzzy normed normal subrings

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Abstract: In this paper, we generalize direct product of finite intuitionistic anti fuzzy normal subrings over normed rings. In particular, we discuss the relation between intuitionistic anti characteristic function and direct product of finite intuitionistic anti fuzzy normed normal subrings. Finally, we give characterizations of direct product of finite intuitionistic anti fuzzy normed normal subrings and some relevant properties are presented.
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1 Introduction

Intuitionistic fuzzy set was introduced by Atanassov [5], as a generalization of fuzzy sets presented by Zadeh in 1965 [17]. Later, many researchers applied the notion of intuitionistic fuzzy sets to different branches of algebra such as groups, subgroups, rings, subrings and ideals. In this research we are concerned with intuitionistic anti fuzzy normed subrings and intuitionistic anti fuzzy normed normal subrings. In [10], intuitionistic anti fuzzy subgroup and intuitionistic anti fuzzy normal subgroup were defined and some properties were given. Sharma and Bansal [16], introduced the concept of intuitionistic anti fuzzy subring and ideal in a ring. Later, Sharma in [15], defined the concept of intuitionistic anti fuzzy submodule of a module and some of their properties were presented. Anitha [4], discussed some properties of intuitionistic anti fuzzy normal subrings and defined direct product of intuitionistic anti fuzzy normal subrings. In [3], Abed Alhaleem and Ahmad introduced the notion of intuitionistic anti fuzzy normed normal subrings and defined the algebraic nature of the direct product of intuitionistic anti fuzzy normed normal subrings. Followed by a study to define intuitionistic anti fuzzy normed ideals in [2]. Recently, shah et al. [14] initiated the idea of intuitionistic fuzzy normal subrings over a non-associative ring and characterized some related properties. Recently, Kausar [9] explored the direct product of finite intuitionistic anti fuzzy normal subrings over non-associative rings.

In this paper, we define direct product of finite intuitionistic anti fuzzy normed normal subrings and we identify the relationship between intuitionistic anti characteristic function and direct product of finite intuitionistic anti fuzzy normed normal subring. Finally, we specify various results related to direct product of finite intuitionistic anti fuzzy normed normal subrings and some fundamental properties will be discussed.

2 Preliminaries

In this section, we outline the most significant definitions and results needed for the following section.

Definition 2.1 (see [5]). *An intuitionistic fuzzy set (briefly,* IFS*) A in a nonempty set in fixed universe X is an object having the form* IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$*, where the functions $\mu_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, where $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$. An intuitionistic fuzzy set A is written symbolically in the form $A = (\mu_A, \nu_A)$.*

Definition 2.2 (see [5]). *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$ be two intuitionistic fuzzy sets of X. Hence:*

i. $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all* $x \in X$.

ii. $A = B$ *if and only if* $\mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$ *for all* $x \in X$.

iii. $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$ *such that* $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$ *and* $\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\} = \nu_A(x) \vee \nu_B(x)$.

iv. $A \cup B = (\mu_{A\cup B}, \nu_{A\cup B})$ *such that* $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$ *and* $\nu_{A\cup B}(x) = \min\{\nu_A(x), \nu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$.

v. $\mu^c_A(x) = 1 - \mu_A(x)$.

vi. $\nu^c_A(x) = 1 - \nu_A(x)$.

vii. $\triangle A = (\mu_A, \mu^c_A)$.

viii. $\Diamond A = (\nu^c_A, \nu_A)$.

Definition 2.3 (see [6]). *Let A and B be two intuitionistic fuzzy sets of universes X_1 and X_2, respectively. The direct product of A and B, is denoted by $A \times B$, and defined as*

$$A \times B = \{\langle (x,y), \mu_{A\times B}(x,y), \nu_{A\times B}(x,y)\rangle \ : \textit{for all } x \in X_1 \textit{ and } y \in X_2\},$$

where $\mu_{A\times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$ and $\nu_{A\times B}(x,y) = \max\{\nu_A(x), \nu_B(y)\}$.

In 1941, Genlfand [7] gave an introduction to the basic theory of Banach algebras and defined normed rings as follows:

Definition 2.4. *A ring R is said to be a normed ring (NR) if it possesses a norm $\| \ \|$, that is, a non-negative real-valued function $\| \ \| : R \to \mathbb{R}$ so that for any $x, y \in R$,*

1. $\|x\| = 0 \Leftrightarrow x = 0$,
2. $\|x+y\| \leq \|x\| + \|y\|$,
3. $\|x\| = \|-x\|$, *(and hence $\|1_R\| = \|-1_R\| = 1$ if identity exists), and*
4. $\|xy\| \leq \|x\|\|y\|$.

Example 2.1 (see [12]). *The field of real numbers $\mathbb{R}$ is a normed ring with respect to the absolute value and the field of complex numbers $\mathbb{C}$ is a normed ring with respect to the modulus. More general examples are the ring of real square matrices with the matrix norm and the ring of real polynomials with a polynomial norm.*

The definitions of T-operators have been given by many researchers, originated from the studies of probabilistic metric spaces by Menger [11] and Schweizer and Sklar [13]. Gupta and Qi in [8] gave a complete set of definitions to T-operators, which are given below:

Definition 2.5. *Let $* : [0,1] \times [0,1] \to [0,1]$ be a binary operation. Then $*$ is a t-norm if $*$ satisfies the conditions of commutativity, associativity, monotonicity and neutral element 1.*
We shortly use t-norm and write $x * y$ instead of $*(x,y)$.

Definition 2.6. *Let $\diamond : [0,1] \times [0,1] \to [0,1]$ be a binary operation. Then $\diamond$ is a s-norm if $\diamond$ satisfies the conditions of commutativity, associativity, monotonicity and neutral element 0.*
We shortly use s-norm and write $x \diamond y$ instead of $\diamond(x,y)$.

Definition 2.7 (see [3]). *Let $*$ be a continuous t-norm and $\diamond$ be a continuous s-norm. An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in NR\}$ is said to be an intuitionistic anti fuzzy normed subring (*IAFNSR*) of the normed ring $(NR, +, .)$ if it satisfies the following for all $x, y \in NR$:*

- *i.* $\mu_A(x-y) \leq \mu_A(x) \diamond \mu_A(y)$,
- *ii.* $\mu_A(xy) \leq \mu_A(x) \diamond \mu_A(y)$,
- *iii.* $\nu_A(x-y) \geq \nu_A(x) * \nu_A(y)$,
- *iv.* $\nu_A(xy) \geq \nu_A(x) * \nu_A(y)$.

Definition 2.8 (see [3]). *Let NR be a normed ring. An intuitionistic anti fuzzy normed subring A of NR is said to be an intuitionistic anti fuzzy normed normal subring (*IAFNNSR*) of NR if:*

i. $\mu_A(xy) = \mu_A(yx)$,

ii. $\nu_A(xy) = \nu_A(yx)$

Definition 2.9 (see [3])**.** *Let A and B be two intuitionistic anti fuzzy normed subrings of NR_1 and NR_2, respectively. The direct product of A and B, is denoted by $A \times B$, and defined as*

$$A \times B = \{\langle (x, y), \mu_{A\times B}(x, y), \nu_{A\times B}(x, y)\rangle \text{: for all } x \in NR_1 \text{ and } y \in NR_2\}$$

where $\mu_{A\times B}(x, y) = \max\{\mu_A(x), \mu_B(y)\}$ and $\nu_{A\times B}(x, y) = \min\{\nu_A(x), \nu_B(y)\}$.

3 Direct product of finite intuitionistic anti fuzzy normed normal subrings

In this section, we define the direct product of intuitionistic anti fuzzy subrings $A_1, A_2, \ldots, A_n$ of normed rings $NR_1, NR_2, \ldots, NR_n$, respectively, and examine some fundamental properties of direct product of finite intuitionistic anti fuzzy normed normal subrings over a normed ring $NR = NR_1 \times NR_2 \times \cdots \times NR_n$.

Definition 3.1 ([1])**.** *The direct product $NR = \times_{i\,\in\,\Omega} NR_i$ of a family of normed rings $\{NR_i : i \in \Omega\}$ has the structure of a normed ring with the operations of addition and multiplication defined for all $x, y \in NR$ as:*

$$x + y = (x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots)$$

and

$$x \circ y = (x_1, x_2, x_3, \ldots) \circ (y_1, y_2, y_3, \ldots) = (x_1y_1, x_2y_2, x_3y_3, \ldots)$$

Definition 3.2. *Let $A_1, A_2, \ldots, A_n$ be intuitionistic anti fuzzy sets of normed rings $NR_1, NR_2, \ldots,$ NR_n, respectively. The direct product of intuitionistic anti fuzzy subrings $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \cdots \times A_n$ and defined by*

$$\begin{aligned} A_1 \times A_2 \times \cdots \times A_n = \{\langle x, \mu_{A_1\times A_2\times\cdots\times A_n}(x), \nu_{A_1\times A_2\times\cdots\times A_n}(x)\rangle \\ : \forall x = (x_1, x_2, \ldots, x_n) \in NR_1 \times NR_2 \times \cdots \times NR_n\}, \end{aligned}$$

where

$$\mu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n) = \max\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\}$$

and

$$\nu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n) = \min\{\nu_{A_1}(x_1), \nu_{A_2}(x_2), \ldots, \nu_{A_n}(x_n)\}$$

Definition 3.3. *An intuitionistic fuzzy set (IFS)*

$$A = A_1 \times A_2 \times \cdots \times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n}, \nu_{A_1\times A_2\times\cdots\times A_n})$$

of a normed ring $NR_1 \times NR_2 \times \cdots \times NR_n$ is an intuitionistic anti fuzzy normrd subring (IAFNSR) *of $NR_1 \times NR_2 \times \cdots \times NR_n$ if:*

i. $\mu_{A_1 \times A_2 \times \cdots \times A_n}(x - y) \leq \mu_{A_1 \times A_2 \times \cdots \times A_n}(x) \diamond \mu_{A_1 \times A_2 \times \cdots \times A_n}(y)$,

ii. $\mu_{A_1 \times A_2 \times \cdots \times A_n}(xy) \leq \mu_{A_1 \times A_2 \times \cdots \times A_n}(x) \diamond \mu_{A_1 \times A_2 \times \cdots \times A_n}(y)$,

iii. $\nu_{A_1 \times A_2 \times \cdots \times A_n}(x - y) \geq \nu_{A_1 \times A_2 \times \cdots \times A_n}(x) * \nu_{A_1 \times A_2 \times \cdots \times A_n}(y)$,

iv. $\nu_{A_1 \times A_2 \times \cdots \times A_n}(xy) \geq \nu_{A_1 \times A_2 \times \cdots \times A_n}(x) * \nu_{A_1 \times A_2 \times \cdots \times A_n}(y)$.

Definition 3.4. *An intuitionistic anti fuzzy normed subring*

$$A_1 \times A_2 \times \cdots \times A_n = (\mu_{A_1 \times A_2 \times \cdots \times A_n}, \nu_{A_1 \times A_2 \times \cdots \times A_n})$$

of a normed ring $NR_1 \times NR_2 \times \cdots \times NR_n$ is an intuitionistic anti fuzzy normrd normal subring (IAFNNSR) of $NR_1 \times NR_2 \times \cdots \times NR_n$ if:

i. $\mu_{A_1 \times A_2 \times \cdots \times A_n}(xy) = \mu_{A_1 \times A_2 \times \cdots \times A_n}(yx)$,

ii. $\nu_{A_1 \times A_2 \times \cdots \times A_n}(xy) = \nu_{A_1 \times A_2 \times \cdots \times A_n}(yx)$.

Lemma 3.5. *If $A_1, A_2, \ldots, A_n$ are normed subrings of the normed rings $NR_1, NR_2 \ldots, NR_n$, respectively, then $A_1 \times A_2 \times \cdots \times A_n$ is a normed subring of $NR_1 \times NR_2 \times \cdots \times NR_n$ under the same operations defined on $NR_1 \times NR_2 \times \cdots \times NR_n$.*

Proposition 3.6. *Let $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$ be intuitionistic fuzzy subsets of the normed rings $NR = NR_1 \times NR_2 \times \cdots \times NR_n$ and $NR' = NR'_1 \times NR'_2 \times \cdots \times NR'_n$ with identities $e = (e_1, e_2, \ldots, e_n)$ and $e' = (e'_1, e'_2, \ldots, e'_n)$, respectively. If $A \times B$ is an intuitionistic anti fuzzy normed subring of $NR \times NR'$, then at least one of the following must hold:*

i. $\mu_A(x) \geq \mu_B(e')$ *and* $\nu_A(x) \leq \nu_B(e')$*, for all* $z \in NR$,

ii. $\mu_B(y) \geq \mu_A(e)$ *and* $\nu_B(y) \leq \nu_A(e)$*, for all* $y \in NR'$.

Proof. Let $A \times B$ be an intuitionistic anti fuzzy normed subring of $NR \times NR'$, and let the statements *(i)* and *(ii)* not hold we can find $x \in NR$ and $y \in NR'$ such that $\mu_A(x) < \mu_B(e')$, $\nu_A(x) > \nu_B(e')$ and $\mu_B(y) < \mu_A(e)$, $\nu_B(y) > \nu_A(e)$. Thus we have

$$\begin{aligned} \mu_{A \times B}(xy) &= \max\{\mu_A(x), \mu_B(y)\} \\ &< \max\{\mu_A(e), \mu_B(e')\} \\ &= \mu_{A \times B}(e, e') \end{aligned}$$

and

$$\begin{aligned} \nu_{A \times B}(xy) &= \min\{\nu_A(x), \nu_B(y)\} \\ &> \min\{\nu_A(e), \nu_B(e')\} \\ &= \nu_{A \times B}(e, e'). \end{aligned}$$

This implies that $A \times B$ is not an intuitionistic anti fuzzy normed subring of $NR \times NR'$, which is a contradiction. Therefore, at least one of the statements must hold. □

Lemma 3.7. *Let $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$ be intuitionistic fuzzy subsets of the normed rings $NR = NR_1 \times NR_2 \times \cdots \times NR_n$ and $NR^{'} = NR_1^{'} \times NR_2^{'} \times \cdots \times NR_n^{'}$ with identities $e = (e_1, e_2, \ldots, e_n)$ and $e^{'} = (e_1^{'}, e_2^{'}, \ldots, e_n^{'})$, respectively. If $A \times B$ is an intuitionistic anti fuzzy normed normal subring of $NR \times NR^{'}$, then the following are true:*

i. if $\mu_A(x) \geq \mu_B(e^{'})$ and $\nu_A(x) \leq \nu_B(e^{'})$, then A is an intuitionistic anti fuzzy normed normal subring of NR.

ii. if $\mu_B(y) \geq \mu_A(e)$ and $\nu_B(y) \leq \nu_A(e)$, then B is an intuitionistic anti fuzzy normed normal subring of $NR^{'}$.

Proof. i. Let $\mu_A(x) \geq \mu_B(e^{'})$ and $\nu_A(x) \leq \nu_B(e^{'})$ for all $x, q \in NR$ and $e^{'} \in NR^{'}$. We have to show that A is an intuitionistic anti fuzzy normed subring of NR, then:

$$\begin{aligned}
\mu_A(x-q) &= \mu_A(x+(-q)) \\
&= \max\{\mu_A(x+(-q)), \mu_B(e^{'}+(-e^{'}))\} \\
&= \mu_{A\times B}((x+(-q)),(e^{'}+(-e^{'})) \\
&= \mu_{A\times B}((x,e^{'})+(-q,-e^{'})) \\
&= \mu_{A\times B}((x,e^{'})-(q,e^{'})) \\
&\leq \mu_{A\times B}(x,e^{'}) \diamond \mu_{A\times B}(q,e^{'}) \\
&= \max\{\mu_A(x),\mu_B(e^{'})\} \diamond \max\{\mu_A(q),\mu_B(e^{'})\} \\
&= \mu_A(x) \diamond \mu_A(q).
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_A(xq) &= \max\{\mu_A(xq), \mu_B(e^{'}e^{'})\} \\
&= \mu_{A\times B}(xq, e^{'}e^{'}) \\
&= \mu_{A\times B}((x,e^{'}) \circ (q,e^{'})) \\
&\leq \mu_{A\times B}(x,e^{'}) \diamond \mu_{A\times B}(q,e^{'}) \\
&= \max\{\mu_A(x),\mu_B(e^{'})\} \diamond \max\{\mu_A(q),\mu_B(e^{'})\} \\
&= \mu_A(x) \diamond \mu_A(q)
\end{aligned}$$

and with,

$$\begin{aligned}
\mu_A(xq) &= \max\{\mu_A(xq), \mu_B(e^{'}e^{'})\} \\
&= \mu_{A\times B}((xq), e^{'}e^{'})) \\
&= \mu_{A\times B}((x,e^{'}) \circ (q,e^{'})) \\
&= \mu_{A\times B}((q,e^{'}) \circ (x,e^{'})) \\
&= \mu_{A\times B}((qx),(e^{'}e^{'})) \\
&= \max\{\mu_A(qx), \mu_B(e^{'}e^{'})\} \\
&= \mu_A(qx).
\end{aligned}$$

Similarly, we can prove that $\nu_A(x-q) \geq \nu_A(x) * \nu_A(q)$, $\nu_A(xq) \geq \nu_A(x) * \nu_A(q)$ and $\nu_A(xq) = \nu_A(qx)$ for all $x, q \in NR$. Hence, A is an intuitionistic anti fuzzy normed normal subring of NR.

ii. The proof is similar to the above. □

Definition 3.8. *Let $A = A_1 \times A_2 \times \cdots \times A_n$ be a non-empty subset of the normed ring $NR = NR_1 \times NR_2 \times \cdots \times NR_n$. The intuitionistic anti characteristic function of A is defined as $\lambda_{A_1 \times A_2 \times \cdots \times A_n} = (\mu_{\lambda_{A_1 \times A_2 \times \cdots \times A_n}}, \nu_{\lambda_{A_1 \times A_2 \times \cdots \times A_n}})$, where:*

$$\mu_{\lambda_A}(x) = \begin{cases} 0, & \text{if} \quad x \in A \\ 1, & \text{if} \quad x \notin A \end{cases} \quad \text{and} \quad \nu_{\lambda_A}(x) = \begin{cases} 1, & \text{if} \quad x \in A \\ 0, & \text{if} \quad x \notin A. \end{cases}$$

Lemma 3.9. *Let $A_1, A_2, \ldots, A_n$ be normed subrings of the normed rings $NR_1, NR_2, \ldots, NR_n$, respectively. Then $A = A_1 \times A_2 \times \cdots \times A_n$ is a normed subring of a normed ring $NR_1 \times NR_2 \times \cdots \times NR_n$ if and only if the intuitionistic anti characteristic function $\lambda_A = (\mu_{\lambda_A}, \nu_{\lambda_A})$ of A is an intuitionistic anti fuzzy normed normal subring of a normed ring $NR_1 \times NR_2 \times \cdots \times NR_n$.*

Proof. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in NR_1 \times NR_2 \times \cdots \times NR_n$. If $x, y \in A$ then by definition of the intuitionistic anti characteristic function $\mu_{\lambda_A}(x) = 0 = \mu_{\lambda_A}(y)$ and $\nu_{\lambda_A}(x) = 1 = \nu_{\lambda_A}(y)$. Since $x - y$, xy in A, then, $\mu_{\lambda_A}(x-y) = 0 = 0 \diamond 0 = \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y)$ and $\mu_{\lambda_A}(xy) = 0 = 0 \diamond 0 = \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y)$. Thus, $\mu_{\lambda_A}(x - y) \leq \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y)$ and $\mu_{\lambda_A}(xy) \leq \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y)$. Now $\nu_{\lambda_A}(x-y) = 1 = 1 * 1 = \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y)$ and $\nu_{\lambda_A}(xy) = 1 = 1 * 1 = \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y)$. Thus, $\nu_{\lambda_A}(x - y) \geq \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y)$ and $\nu_{\lambda_A}(xy) \geq \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y)$. As xy and $yx \in A$, so $\mu_{\lambda_A}(xy) = 0 = \mu_{\lambda_A}(yx)$ and $\nu_{\lambda_A}(xy) = 1 = \nu_{\lambda_A}(yx)$. Accordingly, $\mu_{\lambda_A}(xy) = \mu_{\lambda_A}(yx)$ and $\nu_{\lambda_A}(xy) = \nu_{\lambda_A}(yx)$. Similarly we have when $x, y \notin A$:

$$\begin{array}{ccc} \mu_{\lambda_A}(x-y) \leq \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y) & \text{and} & \mu_{\lambda_A}(xy) \leq \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y), \\ \nu_{\lambda_A}(x-y) \geq \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y) & \text{and} & \nu_{\lambda_C}(xy) \geq \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y), \\ \mu_{\lambda_A}(xy) = \mu_{\lambda_A}(yx) & \text{and} & \nu_{\lambda_A}(xy) = \nu_{\lambda_A}(yx). \end{array}$$

Hence, the intuitionistic anti characteristic function $\lambda_A = (\mu_{\lambda_A}, \nu_{\lambda_A})$ is an intuitionistic anti fuzzy normed normal subring of NR.

On the other hand, assume that the intuitionistic anti characteristic function $\lambda_A = (\mu_{\lambda_A}, \nu_{\lambda_A})$ of A is an intuitionistic anti fuzzy normal normed subring of NR. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in NR_1 \times NR_2 \times \cdots \times NR_n$, this implies that $\mu_{\lambda_A}(x) = 0 = \mu_{\lambda_A}(y)$ and $\nu_{\lambda_A}(x) = 1 = \nu_{\lambda_A}(y)$, then:

$$\begin{aligned} \mu_{\lambda_A}(x-y) &\leq \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y) &&= 0 \diamond 0 = 0, \\ \mu_{\lambda_A}(xy) &\leq \mu_{\lambda_A}(x) \diamond \mu_{\lambda_A}(y) &&= 0 \diamond 0 = 0, \\ \nu_{\lambda_A}(x-y) &\geq \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y) &&= 1 * 1 = 1, \\ \nu_{\lambda_A}(xy) &\geq \nu_{\lambda_A}(x) * \nu_{\lambda_A}(y) &&= 1 * 1 = 1. \end{aligned}$$

This implies that $\mu_{\lambda_A}(x - y) = 0$, $\mu_{\lambda_A}(xy) = 0$ and $\nu_{\lambda_A}(x - y) = 1$, $\nu_{\lambda_A}(xy) = 1$. Thus, $x - y$ and $xy \in A$. Hence $A = A_1 \times A_2 \times \cdots \times A_n$ is a normed subring of the normed ring $NR_1 \times NR_2 \times \cdots \times NR_n$ □

Lemma 3.10. *Let $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$ be two normed subrings of the normed ring $NR_1 \times NR_2 \times \cdots \times NR_n$. Then there intersection $A \cap B$ is also a normed subring of $NR_1 \times NR_2 \times \cdots \times NR_n$.*

Theorem 3.11. *Let $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$ be two subrings of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$. Then $A \cap B$ is a normed subring of $NR_1 \times NR_2 \times \cdots \times NR_n$ if and only if the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \nu_{\lambda_C})$ of $C = A \cap B$ is an intuitionistic anti fuzzy normed normal subring of a normed ring $NR_1 \times NR_2 \times \cdots \times NR_n$.*

Proof. Let $C = A \cap B$ be a normed subring of $NR_1 \times NR_2 \times \cdots \times NR_n$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in NR_1 \times NR_2 \times \cdots \times NR_n$. If $x, y \in C$ then by definition of the intuitionistic anti characteristic function $\mu_{\lambda_C}(x) = 0 = \mu_{\lambda_C}(y)$ and $\nu_{\lambda_C}(x) = 1 = \nu_{\lambda_C}(y)$. Since $x - y$ and $xy \in C$ and C is a subring. It follows that $\mu_{\lambda_C}(x - y) = 0 = 0 \diamond 0 = \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y)$ and $\mu_{\lambda_C}(xy) = 0 = 0 \diamond 0 = \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y)$. Thus $\mu_{\lambda_C}(x - y) \leq \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y)$ and $\mu_{\lambda_C}(xy) \leq \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y)$. Now $\nu_{\lambda_C}(x - y) = 1 = 1 * 1 = \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y)$ and $\nu_{\lambda_C}(x - y) = 0 = 0 * 0 = \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y)$. Thus $\nu_{\lambda_C}(x - y) \geq \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y)$ and $\nu_{\lambda_C}(xy) \geq \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y)$. As xy and $yx \in C$, so $\mu_{\lambda_C}(xy) = 0 = \mu_{\lambda_C}(yx)$ and $\nu_{\lambda_C}(xy) = 0 = \nu_{\lambda_C}(yx)$. This implies that $\mu_{\lambda_C}(xy) = \mu_{\lambda_C}(yx)$ and $\nu_{\lambda_C}(xy) = \nu_{\lambda_C}(yx)$. Similarly we have if $x, y \notin C$,

$$\begin{aligned}
\mu_{\lambda_C}(x - y) \leq \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y) \quad &\text{and} \quad \mu_{\lambda_C}(xy) \leq \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y), \\
\nu_{\lambda_C}(x - y) \geq \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y) \quad &\text{and} \quad \nu_{\lambda_C}(xy) \geq \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y), \\
\mu_{\lambda_C}(xy) = \mu_{\lambda_C}(yx) \quad &\text{and} \quad \nu_{\lambda_C}(xy) = \nu_{\lambda_C}(yx).
\end{aligned}$$

Hence, the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \nu_{\lambda_C})$ of $C = A \cap B$ is an intuitionistic anti fuzzy normed normal subring of $NR_1 \times NR_2 \times \cdots \times NR_n$.

Conversely, assume that the intuitionistic anti characteristic function $\lambda_C = (\mu_{\lambda_C}, \nu_{\lambda_C})$ of C is an intuitionistic anti fuzzy normal normed subring of NR. Now we have to show that $C = A \cap B$ is a subring of $NR_1 \times NR_2 \times \cdots \times NR_n$. Let $x, y \in C$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. By definition $\mu_{\lambda_C}(x) = 0 = \mu_{\lambda_C}(y)$ and $\nu_{\lambda_C}(x) = 1 = \nu_{\lambda_C}(y)$, then:

$$\begin{aligned}
\mu_{\lambda_C}(x - y) &\leq \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y) = 0 \diamond 0 = 0, \\
\mu_{\lambda_C}(xy) &\leq \mu_{\lambda_C}(x) \diamond \mu_{\lambda_C}(y) = 0 \diamond 0 = 0, \\
\nu_{\lambda_C}(x - y) &\geq \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y) = 1 * 1 = 1, \\
\nu_{\lambda_C}(xy) &\geq \nu_{\lambda_C}(x) * \nu_{\lambda_C}(y) = 1 * 1 = 1.
\end{aligned}$$

This implies that $\mu_{\lambda_C}(x - y) = 0$, $\mu_{\lambda_C}(xy) = 0$ and $\nu_{\lambda_C}(x - y) = 1$, $\nu_{\lambda_C}(xy) = 1$. Thus, $x - y$ and $xy \in C$. Hence, C is a normed subring of $NR_1 \times NR_2 \times \cdots \times NR_n$. □

Theorem 3.12. *If $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$ are two intuitionistic anti fuzzy normed normal subrings of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$, then their intersection $A \cap B$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$.*

Proof. Let

$$\begin{aligned}
A = A_1 \times A_2 \times \cdots \times A_n = \{\langle x, \mu_{A_1 \times A_2 \times \cdots \times A_n}(x), \nu_{A_1 \times A_2 \times \cdots \times A_n}(x)\rangle \\
: \forall x = (x_1, x_2, \ldots, x_n) \in NR_1 \times NR_2 \times \cdots \times NR_n\}
\end{aligned}$$

and

$$\begin{aligned}
B = B_1 \times B_2 \times \cdots \times B_n = \{\langle y, \mu_{B_1 \times B_2 \times \cdots \times B_n}(y), \nu_{B_1 \times B_2 \times \cdots \times B_n}(y)\rangle \\
: \forall y = (y_1, y_2, \ldots, y_n) \in NR_1 \times NR_2 \times \cdots \times NR_n\}
\end{aligned}$$

be two intuitionistic anti fuzzy normed normal subrings of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$. Let $W = A \cap B$ and

$$W = \{\langle w, \mu_W(w), \nu_W(w)\rangle : \forall w = (w_1, w_2, \ldots, w_n) \in NR_1 \times NR_2 \times \cdots \times NR_n\},$$

where:

$$\begin{aligned} \mu_W(w_1, w_2, \ldots, w_n) &= \mu_{A\cap B}(w_1, w_2, \ldots, w_n) \\ &= \max\{\mu_A(w_1, w_2, \ldots, w_n), \mu_B(w_1, w_2, \ldots, w_n)\} \end{aligned}$$

and

$$\begin{aligned} \nu_W(w_1, w_2, \ldots, w_n) &= \nu_{A\cap B}(w_1, w_2, \ldots, w_n) \\ &= \min\{\nu_A(w_1, w_2, \ldots, w_n), \nu_B(w_1, w_2, \ldots, w_n)\} \end{aligned}$$

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in NR_1 \times NR_2 \times \cdots \times NR_n$. Then,

$$\begin{aligned} \mu_W(x-y) &= \max\{\mu_A(x-y), \mu_B(x-y)\} \\ &= \mu_A(x-y) \diamond \mu_B(x-y) \\ &\leq \{\mu_A(x) \diamond \mu_A(y)\} \diamond \{\mu_B(x) \diamond \mu_B(y)\} \\ &= \mu_A(x) \diamond \{\mu_A(y) \diamond \mu_B(x)\} \diamond \mu_B(y) \\ &= \mu_A(x) \diamond \{\mu_B(x) \diamond \mu_A(y)\} \diamond \mu_B(y) \\ &= \{\mu_A(x) \diamond \mu_B(x)\} \diamond \{\mu_A(y) \diamond \mu_B(y)\} \\ &= \mu_{A\cap B}(x) \diamond \mu_{A\cap B}(y) \\ &= \mu_W(x) \diamond \mu_W(y) \end{aligned}$$

and

$$\begin{aligned} \mu_W(x \circ y) &= \max\{\mu_A(x \circ y), \mu_B(x \circ y)\} \\ &= \mu_A(x \circ y) \diamond \mu_B(x \circ y) \\ &\leq \{\mu_A(x) \diamond \mu_A(y)\} \diamond \{\mu_B(x) \diamond \mu_B(y)\} \\ &= \mu_A(x) \diamond \{\mu_A(y) \diamond \mu_B(x)\} \diamond \mu_B(y) \\ &= \mu_A(x) \diamond \{\mu_B(x) \diamond \mu_A(y)\} \diamond \mu_B(y) \\ &= \{\mu_A(x) \diamond \mu_B(x)\} \diamond \{\mu_A(y) \diamond \mu_B(y)\} \\ &= \mu_{A\cap B}(x) \diamond \mu_{A\cap B}(y) \\ &= \mu_W(x) \diamond \mu_W(y). \end{aligned}$$

Thus,

$$\mu_W((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)) \leq \mu_W(x_1, x_2, \ldots, x_n) \diamond \mu_W(y_1, y_2, \ldots, y_n), \text{ and}$$
$$\mu_W((x_1, x_2, \ldots, x_n) \circ (y_1, y_2, \ldots, y_n)) \leq \mu_W(x_1, x_2, \ldots, x_n) \diamond \mu_W(y_1, y_2, \ldots, y_n).$$

Similarly,

$$\nu_W((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)) \geq \nu_W(x_1, x_2, \ldots, x_n) * \nu_W(y_1, y_2, \ldots, y_n), \text{ and}$$
$$\nu_W((x_1, x_2, \ldots, x_n) \circ (y_1, y_2, \ldots, y_n)) \geq \nu_W(x_1, x_2, \ldots, x_n) * \nu_W(y_1, y_2, \ldots, y_n).$$

Therefore, $W = (\mu_W, \nu_W)$ is an intuitionistic anti fuzzy normed subring of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$. Now,

$$
\begin{aligned}
\mu_W((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) &= \mu_{A\cap B}(x_1y_1,x_2y_2,\dots,x_ny_n)\\
&= \max\{\mu_A(x_1y_1,x_2y_2,...,x_ny_n),\mu_B(x_1y_1,x_2y_2,...,x_ny_n)\}\\
&= \max\{\mu_A(y_1x_1,y_2x_2,...,y_nx_n),\mu_B(y_1x_1,y_2x_2,...,y_nz_n)\}\\
&= \mu_{A\cap B}(y_1x_1,y_2x_2,\dots,y_nx_n)\\
&= \mu_W((y_1,y_2,\dots,y_n)\circ(x_1,x_2,\dots,x_n)).
\end{aligned}
$$

Similarly, $\nu_W((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n)) = \nu_W((y_1,y_2,\dots,y_n)\circ(x_1,x_2,\dots,x_n))$.

Hence, $W = A\cap B$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. □

Theorem 3.13. *If the IFS $A = A_1\times A_2\times\cdots\times A_n$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$, then $\triangle A = \triangle A_1\times A_2\times\cdots\times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n},\mu^c_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$.*

Proof. Let $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)\in NR_1\times NR_2\times\cdots\times NR_n$. Then

$$
\begin{aligned}
&\mu^c_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,\dots,x_n)-(y_1,y_2,\dots,y_n))\\
&\quad= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,\dots,x_n)-(y_1,y_2,\dots,y_n))\\
&\quad\geq 1-(\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)\diamond\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n))\\
&\quad= 1-\max\{\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&\quad= \min\{1-\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),1-\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&\quad= \min\{\mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&\quad= \mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)*\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mu^c_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&\quad= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&\quad\geq 1-(\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)\diamond\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n))\\
&\quad= 1-\max\{\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&\quad= \min\{1-\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),1-\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&\quad= \min\{\mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&\quad= \mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)*\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n).
\end{aligned}
$$

Thus, $\triangle A_1\times A_2\times\cdots\times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n},\mu^c_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$.

$$
\begin{aligned}
\mu^c_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n)) &= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,...,x_n)\circ(y_1,y_2,...,y_n))\\
&= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n))\\
&= \mu^c_{A_1\times A_2\times\cdots\times A_n}((y_1,y_2,...,y_n)\circ(x_1,x_2,...,x_n)).
\end{aligned}
$$

Hence, $\triangle A = \triangle A_1\times A_2\times\cdots\times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n},\mu^c_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. □

Theorem 3.14. *If the* IFS $A = A_1 \times A_2 \times \cdots \times A_n$ *is an intuitionistic anti fuzzy normed normal subring of the ring* $NR_1 \times NR_2 \times \cdots \times NR_n$*. Then* $\Diamond A = \Diamond A_1 \times A_2 \times \cdots \times A_n = (\nu^c_{A_1\times A_2\times\cdots\times A_n}, \nu_{A_1\times A_2\times\cdots\times A_n})$ *is an intuitionistic anti fuzzy normed normal subring of the ring* $NR_1 \times NR_2 \times \cdots \times NR_n$*.*

Proof. Similar to the proof of Proposition 3.13. □

Corollary 3.15. *An* IFS $A = A_1 \times A_2 \times \cdots \times A_n$ *is an intuitionistic anti fuzzy normed normal subring of the ring* $NR_1 \times NR_2 \times \cdots \times NR_n$ *if and only if* $\triangle A = (\mu_A, \mu^c_A)$ *(resp.*$\Diamond A = (\nu^c_A, \nu_A)$*) is an intuitionistic anti fuzzy normed normal subring of the ring* $NR_1 \times NR_2 \times \cdots \times NR_n$*.*

Theorem 3.16. *An* IFS $A = A_1 \times A_2 \times \cdots \times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n}, \nu_{A_1\times A_2\times\cdots\times A_n})$ *is an intuitionistic anti fuzzy normed normal subring of the ring* $NR_1 \times NR_2 \times \cdots \times NR_n$ *if and only if the fuzzy subsets* $\mu_{A_1\times A_2\times\cdots\times A_n}$ *and* $\nu^c_{A_1\times A_2\times\cdots\times A_n}$ *are intuitionistic anti fuzzy normed normal subring of the ring* $NR_1 \times NR_2 \times \cdots \times NR_n$*.*

Proof. Let $A = A_1 \times A_2 \times \cdots \times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n}, \nu_{A_1\times A_2\times\cdots\times A_n})$ be an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$. This implies that $\mu_{A_1\times A_2\times\cdots\times A_n}$ is an intuitionistic anti fuzzy normed normal subring of $NR_1 \times NR_2 \times \cdots \times NR_n$. We have to show that $\nu^c_{A_1\times A_2\times\cdots\times A_n}$ is also an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$. Then

$$\begin{aligned}
&\nu^c_{A_1\times A_2\times\cdots\times A_n}((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n))\\
&\qquad = 1 - \nu_{A_1\times A_2\times\cdots\times A_n}((x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n))\\
&\qquad \leq 1 - (\nu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n) * \nu_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n))\\
&\qquad = 1 - \min\{\nu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n), \nu_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n)\}\\
&\qquad = \max\{1 - \nu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n), 1 - \nu_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n)\}\\
&\qquad = \max\{\nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n), \nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n)\}\\
&\qquad = \nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n) \diamond \nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n)
\end{aligned}$$

and

$$\begin{aligned}
&\nu^c_{A_1\times A_2\times\cdots\times A_n}((x_1, x_2, \ldots, x_n) \circ (y_1, y_2, \ldots, y_n))\\
&\qquad = 1 - \nu_{A_1\times A_2\times\cdots\times A_n}((x_1, x_2, \ldots, x_n) \circ (y_1, y_2, \ldots, y_n))\\
&\qquad \leq 1 - (\nu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n) * \nu_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n))\\
&\qquad = 1 - \min\{\nu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n), \nu_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n)\}\\
&\qquad = \max\{1 - \nu_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n), 1 - \nu_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n)\}\\
&\qquad = \max\{\nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n), \nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n)\}\\
&\qquad = \nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1, x_2, \ldots, x_n) \diamond \nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1, y_2, \ldots, y_n).
\end{aligned}$$

Thus, $\nu^c_{A_1\times A_2\times\cdots\times A_n}$ is an intuitionistic anti fuzzy normed subring of $NR_1 \times NR_2 \times \cdots \times NR_n$. Also,

$$\begin{aligned}
\nu^c_{A_1\times A_2\times\cdots\times A_n}((x_1, x_2, ..., x_n)\circ(y_1, y_2, ..., y_n)) &= 1-\nu_{A_1\times A_2\times\cdots\times A_n}((x_1, x_2, ..., x_n) \circ (y_1, y_2, ..., y_n))\\
&= 1-\nu_{A_1\times A_2\times\cdots\times A_n}((y_1, y_2, ..., y_n)\circ(x_1, x_2, ..., x_n))\\
&= \nu^c_{A_1\times A_2\times\cdots\times A_n}((y_1, y_2, ..., y_n)\circ(x_1, x_2, ..., x_n)).
\end{aligned}$$

Hence, $\nu^c_{A_1\times A_2\times\cdots\times A_n}$ is an intuitionistic anti fuzzy normed normal subring of $NR_1 \times NR_2 \times \cdots \times NR_n$.

Conversely, suppose that $\mu_{A_1\times A_2\times\cdots\times A_n}$ and $\nu^c_{A_1\times A_2\times\cdots\times A_n}$ are intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. We have to show that $A=(\mu_{A_1\times A_2\times\cdots\times A_n},$ $\nu_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots$ $\times NR_n$. Now,

$$\begin{aligned}
1-\nu_{A_1\times A_2\times\cdots\times A_n}&((x_1,x_2,\dots,x_n)-(y_1,y_2,\dots,y_n))\\
&=\nu^c_{A\times B}((x_1,x_2,\dots,x_n)-(y_1,y_2,\dots,y_n))\\
&\le \nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)\diamond\nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\\
&=\max\{\nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=\max\{1-\nu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),1-\nu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=1-\min\{\nu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\nu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=1-(\nu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)*\nu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n))
\end{aligned}$$

and

$$\begin{aligned}
1-\nu_{A_1\times A_2\times\cdots\times A_n}&((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&=\nu^c_{A\times B}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&\le \nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)\diamond\nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\\
&=\max\{\nu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\nu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=\max\{1-\nu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),1-\nu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=1-\min\{\nu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\nu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=1-(\nu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)*\nu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)).
\end{aligned}$$

Hence, $A=A_1\times A_2\times\cdots\times A_n=(\mu_{A_1\times A_2\times\cdots\times A_n},\nu_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. Also,

$$\begin{aligned}
1-\nu_{A\times B}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))&=\nu^c_{A\times B}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&=\nu^c_{A\times B}((y_1,y_2,\dots,y_n)\circ(x_1,x_2,\dots,x_n))\\
&=1-\nu_{A\times B}((y_1,y_2,\dots,y_n)\circ(x_1,x_2,\dots,x_n)).
\end{aligned}$$

Therefore, $A=A_1\times A_2\times\cdots\times A_n=(\mu_{A_1\times A_2\times\cdots\times A_n},\nu_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. $\square$

Theorem 3.17. *An* IFS $A=A_1\times A_2\times\cdots\times A_n=(\mu_{A_1\times A_2\times\cdots\times A_n},\nu_{A_1\times A_2\times\cdots\times A_n})$ *is an intuitionistic anti fuzzy normed normal subring of the ring* $NR_1\times NR_2\times\cdots\times NR_n$ *if and only if the fuzzy subsets* $\mu^c_{A_1\times A_2\times\cdots\times A_n}$ *and* $\nu_{A_1\times A_2\times\cdots\times A_n}$ *are intuitionistic anti fuzzy normed normal subrings of the ring* $NR_1\times NR_2\times\cdots\times NR_n$.

Proof. Let $A=A_1\times A_2\times\cdots\times A_n=(\mu_{A_1\times A_2\times\cdots\times A_n},\nu_{A_1\times A_2\times\cdots\times A_n})$ be an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. This implies that $\nu_{A_1\times A_2\times\cdots\times A_n}$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2\times\cdots\times NR_n$. We have to show that $\mu^c_{A_1\times A_2\times\cdots\times A_n}$ is also an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. Now,

$$
\begin{aligned}
\mu^c_{A_1\times A_2\times\cdots\times A_n}&((x_1,x_2,\ldots,x_n)-(y_1,y_2,\ldots,y_n))\\
&= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,\ldots,x_n)-(y_1,y_2,\ldots,y_n))\\
&\geq 1-(\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n)\diamond\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n))\\
&= 1-\max\{\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= \min\{1-\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),1-\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= \min\{\mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= \mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n)*\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)
\end{aligned}
$$

and

$$
\begin{aligned}
\mu^c_{A_1\times A_2\times\cdots\times A_n}&((x_1,x_2,\ldots,x_n)\circ(y_1,y_2,\ldots,y_n))\\
&= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,\ldots,x_n)\circ(y_1,y_2,\ldots,y_n))\\
&\geq 1-(\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n)\diamond\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n))\\
&= 1-\max\{\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= \min\{1-\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),1-\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= \min\{\mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= \mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n)*\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n).
\end{aligned}
$$

Thus, $\mu^c_{A_1\times A_2\times\cdots\times A_n}$ is an intuitionistic anti fuzzy normed subring of $NR_1\times NR_2\times\cdots\times NR_n$. Also,

$$
\begin{aligned}
\mu^c_{A_1\times A_2\times\cdots\times A_n}&((x_1,x_2,\ldots,x_n)\circ(y_1,y_2,\ldots,y_n))\\
&= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((x_1,x_2,\ldots,x_n)\circ(y_1,y_2,\ldots,y_n))\\
&= 1-\mu_{A_1\times A_2\times\cdots\times A_n}((y_1,y_2,\ldots,y_n)\circ(x_1,x_2,\ldots,x_n))\\
&= \mu^c_{A_1\times A_2\times\cdots\times A_n}((y_1,y_2,\ldots,y_n)\circ(x_1,x_2,\ldots,x_n)).
\end{aligned}
$$

Hence, $\mu^c_{A_1\times A_2\times\cdots\times A_n}$ is an intuitionistic anti fuzzy normed normal subring of $NR_1\times NR_2\times\cdots\times NR_n$.

Conversely, suppose that $\mu^c_{A_1\times A_2\times\cdots\times A_n}$ and $\nu_{A_1\times A_2\times\cdots\times A_n}$ are intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. We have to show that $A=(\mu_{A_1\times A_2\times\cdots\times A_n},\nu_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1\times NR_2\times\cdots\times NR_n$. Then,

$$
\begin{aligned}
1-\mu_{A_1\times A_2\times\cdots\times A_n}&((x_1,x_2,\ldots,x_n)-(y_1,y_2,\ldots,y_n))\\
&= \mu^c_{A\times B}((x_1,x_2,\ldots,x_n)-(y_1,y_2,\ldots,y_n))\\
&\geq \mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n)*\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\\
&= \min\{\mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= \min\{1-\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),1-\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= 1-\max\{\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n),\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n)\}\\
&= 1-(\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\ldots,x_n)\diamond\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\ldots,y_n))
\end{aligned}
$$

and

$$
\begin{aligned}
1-\mu_{A_1\times A_2\times\cdots\times A_n}&((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&=\mu^c_{A\times B}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&\geq \mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)*\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\\
&=\min\{\mu^c_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\mu^c_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=\min\{1-\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),1-\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=1-\max\{\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n),\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)\}\\
&=1-(\mu_{A_1\times A_2\times\cdots\times A_n}(x_1,x_2,\dots,x_n)\diamond\mu_{A_1\times A_2\times\cdots\times A_n}(y_1,y_2,\dots,y_n)).
\end{aligned}
$$

Hence, $A = A_1 \times A_2 \times \cdots \times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n}, \nu_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed subring of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$. Also,

$$
\begin{aligned}
1-\mu_{A\times B}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n)) &=\mu^c_{A\times B}((x_1,x_2,\dots,x_n)\circ(y_1,y_2,\dots,y_n))\\
&=\mu^c_{A\times B}((y_1,y_2,\dots,y_n)\circ(x_1,x_2,\dots,x_n))\\
&=1-\mu_{A\times B}((y_1,y_2,\dots,y_n)\circ(x_1,x_2,\dots,x_n)).
\end{aligned}
$$

Therefore, $A = A_1 \times A_2 \times \cdots \times A_n = (\mu_{A_1\times A_2\times\cdots\times A_n}, \nu_{A_1\times A_2\times\cdots\times A_n})$ is an intuitionistic anti fuzzy normed normal subring of the ring $NR_1 \times NR_2 \times \cdots \times NR_n$. □

4 Conclusion

In this paper, we defined the nature of the direct product of finite intuitionistic anti fuzzy normed normal subrings. We extended direct product of intuitionistic anti fuzzy normed normal subrings to finite intuitionistic anti fuzzy normed normal subrings. Also, we established the relation between intuitionistic anti characteristic function and direct product of finite intuitionistic anti fuzzy normed normal subrings and derived some related properties.

References

[1] Abed Alhaleem, N., & Ahmad, A. G. (2021). Intuitionistic Fuzzy Normal Subrings over Normed Rings. *International Journal of Analysis and Applications*, 19(3), 341–359.

[2] Abed Alhaleem, N., & Ahmad, A. G. (2021). Intuitionistic anti fuzzy normed ideals. *Notes on Intuitionistic Fuzzy Sets*, 27(1), 60–71.

[3] Abed Alhaleem, N., & Ahmad, A. G. (2022). Intuitionistic Anti Fuzzy Normal Subrings over Normed Rings. *Sains Malaysiana*, 51(2), 609–618.

[4] Anitha, B. (2019). Properties of intuitionistic anti fuzzy normal subrings. *Malaya Journal of Matematik*, 7(2), 304–308.

[5] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[6] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Springer, Berlin, Germany, 1–137.

[7] Gelfand, I. (1941). Normierte ringe. *Rec. Math. [Matematicheskii Sbornik] N.S.*, 9(1), 3–24.

[8] Gupta, M., & Qi, J. (1991). Theory of T-norms and fuzzy inference methods. *Fuzzy Sets and Systems*, 40(3), 431–450.

[9] Kausar, N. (2019). Direct Product of Finite Intuitionistic Anti Fuzzy Normal Subrings over Non-associative Rings. *European Journal of Pure and Applied Mathematics*, 12(2), 622–648.

[10] Li, D., Zhang, C., & Ma, S. (2009). The Intuitionistic Anti-fuzzy Subgroup in Group G. *Fuzzy Information and Engineering*. Advances in Soft Computing, vol 54. Springer, Berlin, 145–151.

[11] Menger, K. (1942). Statistical metrics. *Proceedings of the National Academy of Sciences of the United States of America*, 28(12), 535–537.

[12] Naimark, M. A. (1959). *Normed Rings*. Groningen, Noordhoff.

[13] Schweizer, B., & Sklar, A. (1960). Statistical metric space. *Pacific Journal of Mathematics*, 10(1), 313–334.

[14] Shah, T., Kausar, N., & Rehman, I. (2012). Intuitionistic fuzzy normal subrings over a non-associative ring. *Analele Universitatii "Ovidius" Constanta-Seria Matematica*, 20(1), 369–386.

[15] Sharma, P. K. (2012). On Intuitionistic Anti-Fuzzy Submodule of a Module. *International Journal of Fuzzy Mathematics & Systems*, 2(2), 127–132.

[16] Sharma, P. K., & Bansal, V. (2012). On Intuitionistic Anti-Fuzzy Ideal in Rings. *International Journal of Mathematical Sciences*, 11(3–4), 237–243.

[17] Zadeh, L. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.