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ON A GENERALIZATION OF INTUITIONISTIC FUZZY GRAPHS

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1 Introduction

The concept of an Intuitionistic Fuzzy Graph (IFG) was introduced in 1994 in [9]. It was subsequently an object of some extensions (see [4, 10, 6]), representations (see [2, 3, 5]) and applications (see [5]).

Here we discuss a new generalization of the IFGs, using as a basis the concepts of Intuitionistic Fuzzy Sets (IFSs), Intuitionistic Fuzzy Relations (IFRs) and Index Matrices (IMs). All necessary definitions are collected in [5]. We shall introduce only some of the basic concepts.

2 Some necessary definitions

Let a set E be fixed. An IFS A in E is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ determine the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let I be a fixed set of indices and $\mathcal{R}$ be the set of all real numbers. By an IM with index sets K and L ($K, L \subset I$) we will mean the object (see [1]):

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \dots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \dots & a_{k_2,l_n} \\ \vdots & & & & \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \dots & a_{k_m,l_n} \end{array}$$

where $K = \{k_1, k_2, ..., k_m\}, L = \{l_1, l_2, ..., l_n\}$, for $1 \leq i \leq m$, and for $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{R}$. For the IMs $A = [K, L, \{a_{k_i,l_j}\}]$ and $B = [P, Q, \{b_{p_r,q_s}\}]$ the usual matrix operations "addition" and "multiplication" are defined, and also the following operations

$$A + B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i,l_j} + b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ 0, & \text{otherwise} \end{cases}$$

$$A \times B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j}.b_{p_r,q_s}, \text{ for } t_u = k_i = p_r \in K \cap P \text{ and} v_w = l_j = q_s \in L \cap Q;$$

$$A.B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ \sum\limits_{l_j = A_r \in L \cap P} a_{k_i,l_j}.b_{p_r,q_s}, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}$$

The above mathematical apparatus may be applied to the IMs with elements from the sets $\{0,1\}$, [0,1], or from the class of all predicates, etc. In the first two cases, the operations "+" and "." in $\mathcal{R}$ will be substituted by "max" and "min", respectively, and in the third case - by the operations "$\vee$" and "$\wedge$".

Let E_1 and E_2 be two universes and let

$$\begin{aligned} A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E_1\}, \\ B &= \{\langle x, \mu_B(y), \nu_B(y)\rangle | y \in E_2\}. \end{aligned}$$

be two IFSs; A - over E_1 and B - over E_2. We shall define (see [5]):

$$\begin{aligned} A \times_1 B &= \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle | \langle x, y\rangle \in E_1 \times E_2\}, \\ A \times_2 B &= \{\langle\langle x, y\rangle, \mu_A(x) + \mu_B(y) - \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle | \langle x, y\rangle \in E_1 \times E_2\}, \\ A \times_3 B &= \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x) + \nu_B(y) - \nu_A(x).\nu_B(y)\rangle | \langle x, y\rangle \in E_1 \times E_2\}, \\ A \times_4 B &= \{\langle\langle x, y\rangle, min(\mu_A(x), \mu_B(y)), max(\nu_A(x).\nu_B(y))\rangle | \langle x, y\rangle \in E_1 \times E_2\}, \\ A \times_5 B &= \{\langle\langle x, y\rangle, max(\mu_A(x), \mu_B(y)), min(\nu_A(x).\nu_B(y))\rangle | \langle x, y\rangle \in E_1 \times E_2\}. \end{aligned}$$

We must note that $A \times_i B$ is an IFS, but it is an IFS over the universe $E_1 \times E_2$, where "$\times_i$" is one of the five Cartesian products above and "$\times$" is the classical Cartesian product on ordinary sets (E_1 and E_2).

Let X and Y are arbitrary finite non-empty sets. IFR is an IFS $R \subset X \times Y$ of the form:

$$R = \{\langle\langle x, y\rangle, \mu_R(x, y), \nu_R(x, y)\rangle | x \in X \ \& \ y \in Y\},$$

where $\mu_R : X \times Y \to [0, 1]$, $\nu_R : X \times Y \to [0, 1]$ are the degrees of membership and non-membership as the ordinary IFSs or degrees of validity and non-validity of the relation R; and for every $\langle x, y\rangle \in X \times Y$:

$$0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1.$$

3 Main results

Let $IFR_o(X,Y)$ be the set of all IFRs over $X \times Y$, where $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are fixed finite sets (universes), $\times$ is the ordinary Cartesian product and $o \in \{\times_1, \times_2, ..., \times_5\}$. Therefore, the set $R \in IFR_o(X,Y)$ can be represented in the form:

$$\begin{array}{c|ccc} & y_1 & \ldots & y_n \\ \hline x_1 & \langle \mu(x_1,y_1), \nu(x_1,y_1) \rangle & \ldots & \langle \mu(x_1,y_1), \nu(x_1,y_1) \rangle \\ x_2 & \langle \mu(x_1,y_1), \nu(x_1,y_1) \rangle & \ldots & \langle \mu(x_1,y_1), \nu(x_1,y_1) \rangle \\ \vdots & \ldots & & \\ x_m & \langle \mu(x_1,y_1), \nu(x_1,y_1) \rangle & \ldots & \langle \mu(x_1,y_1), \nu(x_1,y_1) \rangle \end{array}$$

This IM-representation allows the graphical representation of the elements of R and their degrees of membership and non-membership. Moreover, different universes can be used when different operations over IFRs have to be defined. Let $R \in IFR_o(X_1,Y_1)$ and $S \in IFR_o(X_2,Y_2)$, where X_1, Y_1, X_2 and Y_2 are fixed finite sets. Then operations "$\cup$" and "$\cap$" can be defined (cf. [5]), as follows

$$R \cup S \in IFR_o(X_1 \cup X_2, Y_1 \cup Y_2)$$

and $R \cup S$ has the form

$$\begin{array}{c|ccc} & y_1 & \ldots & y_N \\ \hline x_1 & \langle \mu(x_1,y_1), \nu(x_1,y_1) \rangle & \ldots & \langle \mu(x_1,y_N), \nu(x_1,y_N) \rangle \\ x_2 & \langle \mu(x_2,y_1), \nu(x_2,y_1) \rangle & \ldots & \langle \mu(x_2,y_N), \nu(x_2,y_N) \rangle \\ \vdots & & & \\ x_M & \langle \mu(x_M,y_1), \nu(x_M,y_1) \rangle & \ldots & \langle \mu(x_M,y_N), \nu(x_M,y_N) \rangle \end{array}$$

where

$$X_1 \cup X_2 = \{x_1, x_2, \ldots, x_M\} \text{ and } Y_1 \cup Y_2 = \{y_1, y_2, \ldots, y_N\}, \text{ and}$$

$$\langle \mu(x_i,y_j), \nu(x_i,y_j) \rangle = \begin{cases} \langle \mu(x'_a,y'_b), \nu(x'_a,y'_b) \rangle, \\ \quad \text{if } x_i = x'_a \in X_1 \text{ and } y_j = y'_b \in Y_1 - Y_2 \\ \quad \text{or } x_i = x'_a \in X_1 - X_2 \text{ and } y_j = y'_b \in Y_1 \\ \\ \langle \mu(x''_c,y''_d), \nu(x''_c,y''_d) \rangle, \\ \quad \text{if } x_i = x''_c \in X_2 \text{ and } y_j = y''_d \in Y_2 - Y_1 \\ \quad \text{or } x_i = x''_c \in X_2 - X_1 \text{ and } y_j = y''_d \in Y_2 \\ \\ \langle max(\mu(x',y'), \mu(x'',y'')), min(\nu(x',y'), \nu(x'',y'')) \rangle, \\ \quad \text{if } x_i = x'_a = x''_c \in X_1 \cap X_2 \text{ and} \\ \quad y_j = y'_b = y''_d \in Y_1 \cap Y_2 \\ \\ 0, \text{ otherwise} \end{cases}$$

$$R \cap S \in IFR_o(X_1 \cap X_2, Y_1 \cap Y_2)$$

and for an arbitrary matrix-element of the new IM is valid:

$$\langle \mu(x_i, y_j), \nu(x_i, y_j)\rangle = \begin{cases} \langle \mu(x'_a, y'_b), \nu(x'_a, y'_b)\rangle, \\ \quad \text{if } x_i = x'_a \in X_1 \text{ and } y_j = y'_b \in Y_1 - Y_2 \\ \quad \text{or } x_i = x'_a \in X_1 - X_2 \text{ and } y_j = y'_b \in Y_1 \\ \\ \langle \mu(x''_c, y''_d), \nu(x''_c, y''_d)\rangle, \\ \quad \text{if } x_i = x''_c \in X_2 \text{ and } y_j = y''_d \in Y_2 - Y_1 \\ \quad \text{or } x_i = x''_c \in X_2 - X_1 \text{ and } y_j = y''_d \in Y_2 \\ \\ \langle min(\mu(x', y'), \mu(x'', y'')), max(\nu(x', y'), \nu(x'', y''))\rangle, \\ \quad \text{if } x_i = x'_a = x''_c \in X_1 \cap X_2 \text{ and} \\ \quad y_j = y'_b = y''_d \in Y_1 \cap Y_2 \\ \\ 0, \text{ otherwise} \end{cases}$$

Let the oriented graph $G = (V, A)$ be given, where V is a set of vertices and A is a set of arcs. Every graph arc connects two graph vertices (see, e.g., [8]).

In [5, 9] an approach for introducing of an IFG is given. Here we will modify it in two directions on the basis of some ideas generated from IFS-theoretical and from IFS-decision making (see, e.g, [7]) points of view. We shall start with the oldest version of the concept.

Let operation $\times$ denote the standard Cartesian product operation, while operation $o \in \{\times_1, \times_2, \ldots, \times_5\}$.

Following [9] we shall note that the set

$$G^* = \{\langle\langle x, y\rangle, \mu_G(x, y), \nu_G(x, y)\rangle \mid \langle x, y\rangle \in V \times V\}$$

is called an o-IFG (or briefly, an IFG) if the functions $\mu_G : V \times V \to [0, 1]$ and $\nu_G : V \times V \to [0, 1]$ define the respective degrees of membership and non-membership of the element $\langle x, y\rangle \in V \times V$. These functions have the forms of the corresponding components of the o-Cartesian product over IFSs, and for all $\langle x, y\rangle \in V \times V$:

$$0 \le \mu_G(x, y) + \nu_G(x, y) \le 1.$$

This approach supposes that the given set V and the operation o are choised and fixed previously and they will be used without changes.

On the other hand, following the IFS-interpretations in decision making, we can construct set V and values of functions μ_G and ν_G in the current time, e.g., on the basis of expert knowledge and we can change their forms on the next steps of the process of IFG's use.

Now, we shall introduce a definition of a new type of an IFG.

Let E be an universe, containing fixed graph-vertices and let $V \subset E$ be an fixed set. For it, we construct the IFS

$$V = \{\langle x, \mu_V(x), \nu_V(x)\rangle | x \in E\},$$

where functions $\mu_V : E \to [0, 1]$ and $\nu_V : E \to [0, 1]$ determine the degree of membership and the degree of non-membership to set V of the element (vertex) $x \in E$, respectively, and for every $x \in E$:

$$0 \le \mu_V(x) + \nu_V(x) \le 1.$$

Now, we shall use the idea for an IFS over universe that is an IFS over another universe (see [5]) and will define: the set

$$G^* = \{\langle\langle x, y\rangle, \mu_G(x, y), \nu_G(x, y)\rangle \mid \langle x, y\rangle \in E \times E\}$$

is called an o-Generalized IFG (or briefly, an GIFG) if the functions $\mu_G : V \times V \to [0, 1]$ and $\nu_G : V \times V \to [0, 1]$ define the respective degrees of membership and non-membership of the element (the graph arc) $\langle x, y\rangle \in V \times V$. As above, these functions have the forms of the corresponding components of the o-Cartesian product over IFSs, and for all $\langle x, y\rangle \in V \times V$:

$$0 \leq \mu_G(x, y) + \nu_G(x, y) \leq 1.$$

Let us note by $card(X)$ the cardinality of set X.

We must note that in the present case (for instance of the two above cases) $card(V) = card(E)$, i.e., the cardinality of the graph vertices set can be very large, that is not suitable for real applications. By this reason, by analogy with [10], we can use (α, β)-level operator N_α^β, defined by

$$N_\alpha^\beta(V) = \{\langle x, \mu_V(x), \nu_V(x)\rangle | \mu_V(x) \geq \alpha, \nu_V(x) \leq \beta, x \in E\}.$$

Therefore,

$$card(V) \geq card(N_\alpha^\beta(V)).$$

4 Conclusion

In later research the authors plan to study in more detail the properties of these new graphs and essentially, their IM-properties, thus generalizing standard graphs.

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