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Intuitionistic fuzzy-valued possibility and necessity measures

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Abstract We introduce intuitionistic fuzzy-valued possibility and necessity measures and we study their relations with the intuitionistic fuzzy-valued fuzzy measures proposed in a recent paper. Properties with respect to continuity and some examples are also given.

1 Preliminaries

A fuzzy measure (see e. g. [5]) is a set function $\mu : \mathcal{A} \to [0,1]$ which satisfies $\mu(\emptyset) = 0, \mu(X) = 1$ and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$. A fuzzy measure (or generally a set function) μ defined on a σ-algebra $\mathcal{A}$ is called continuous from below if for every sequence $(A_n)_{n\in\mathsf{Q}} \subset \mathcal{A}$ such that $A_n \subseteq A_{n+1}, \forall n \in \mathsf{Q}$ we have

$$\mu\left(\underset{n\in\mathsf{Q}}{\cup} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

and continuous from above if for every sequence $(A_n)_{n\in\mathsf{Q}} \subset \mathcal{A}$ such that $A_n \supseteq A_{n+1}, \forall n \in \mathsf{Q}$ we have

$$\mu\left(\underset{n\in\mathsf{Q}}{\cap} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

The important set in intuitionistic fuzzy set theory (see [1], [2])

$$\mathcal{L} = \{(x_1, x_2) \in [0,1] \times [0,1]; x_1 + x_2 \leq 1\}$$

is a complete lattice (see [4]) if we consider

$$(x_1, x_2) \leq_{\mathcal{L}} (y_1, y_2) \text{ iff } x_1 \leq y_1 \text{ and } x_2 \geq y_2,$$

$$\begin{aligned} \vee_{\mathcal{L}} A &= (\vee \{x \in [0,1] \,|\exists y \in [0,1] : (x,y) \in A\}, \\ &\quad \wedge \{y \in [0,1] \,|\exists x \in [0,1] : (x,y) \in A\}) \\ \wedge_{\mathcal{L}} A &= (\wedge \{x \in [0,1] \,|\exists y \in [0,1] : (x,y) \in A\}, \\ &\quad \vee \{y \in [0,1] \,|\exists x \in [0,1] : (x,y) \in A\}), \end{aligned}$$

for each $A \subseteq \mathcal{L}$.

The following concepts are introduced in the paper [3].

Definition 1.1 *An intuitionistic fuzzy-valued fuzzy measure over a measurable space* $(X, \mathcal{A})$ *is a map* $v : \mathcal{A} \to \mathcal{L}$ *with the following properties:*
(i) $v(\emptyset) = (0, 1)$;
(ii) $v(X) = (1, 0)$;
(iii) $A \subseteq B$ *implies* $v(A) \leq_{\mathcal{L}} v(B)$.

Definition 1.2 *The intuitionistic fuzzy-valued fuzzy measure* $v : \mathcal{A} \to \mathcal{L}, v(A) = (v_1(A), v_2(A)$ *is called:*
(i) *continuous from below if* v_1 *and* v_2 *are continuous from below;*
(ii) *continuous from above if* v_1 *and* v_2 *are continuous from above;*
(iii) $*$*-decomposable if there exists the composition law* $* : \mathcal{L} \to \mathcal{L}$ *such that* $v(A \cup B) = v(A) * v(B)$, *for every* $A, B \in \mathcal{A}, A \cap B = \emptyset$.

Let us point out that the above definition can be given even if v is any intuitionistic fuzzy-valued set function on a σ-algebra.

The dual $\overline{v}$ of an intuitionistic fuzzy-valued fuzzy measure $v = (v_1, v_2)$, that is $\overline{v}(A) = (v_2(A^c), v_1(A^c))$, where A^c denotes the complement set of A, is also an intuitionistic fuzzy-valued fuzzy measure and the following property is proved in [3].

Theorem 1.3 *If* v *is continuous from below (continuous from above) then* $\overline{v}$ *is continuous from above (continuous from below).*

In this paper, by convention $\underset{t \in \emptyset}{\vee_{\mathcal{L}}} A_t = (0, 1)$ and, only for simplicity, we consider $\mathcal{A} = \mathcal{P}(X)$.

In the book [6], pp. 63-66 the following definitions are given and the below results are proved.

Definition 1.4 *The set function* $\pi : \mathcal{P}(X) \to [0, 1]$ *is called a possibility measure if* $\pi(X) = 1$ *and*

$$\pi\left(\underset{t \in T}{\cup} A_t\right) = \underset{t \in T}{\vee} \pi(A_t),$$

for any subclass $(A_t)_{t \in T} \subset \mathcal{P}(X)$, *where* T *is an arbitrary index set.*

Definition 1.5 *The dual set function* ω *of a possibility measure* π, *that is*

$$\omega(A) = 1 - \pi(A^c),$$

for any $A \in \mathcal{P}(X)$, *is called a necessity measure (or consonant belief function) on* $\mathcal{P}(X)$.

Theorem 1.6 *Any possibility measure is a continuous from below fuzzy measure. Any necessity measure is a continuous from above fuzzy measure.*

Theorem 1.7 *A set function* $\omega : \mathcal{P}(X) \to [0, 1]$ *is a necessity measure if and only if it satisfies*

$$\omega\left(\underset{t \in T}{\cap} A_t\right) = \underset{t \in T}{\wedge} \omega(A_t),$$

for any subclass $(A_t)_{t \in T} \subset \mathcal{P}(X)$, *where* T *is an arbitrary index set.*

2 Intuitionistic fuzzy-valued possibility measures

The concept of possibility measure on classical sets with intuitionistic fuzzy values is introduced and studied in this section.

Definition 2.1 *The set function $\widetilde{\pi}:\mathcal{P}(X)\rightarrow\mathcal{L}$ is called an intuitionistic fuzzy-valued possibility measure if $\widetilde{\pi}(X)=(1,0)$ and*

$$\widetilde{\pi}\left(\underset{t\in T}{\cup}A_t\right)=\underset{t\in T}{\vee_{\mathcal{L}}}\widetilde{\pi}(A_t),$$

for any subfamily $(A_t)_{t\in T}\subset\mathcal{P}(X)$, where T is an arbitrary index set.

Theorem 2.2 *Any intuitionistic fuzzy-valued possibility measure is a $\vee_{\mathcal{L}}$-decomposable intuitionistic fuzzy-valued fuzzy measure.*

Proof. According to the convention, when $T=\emptyset$, we have $\underset{t\in T}{\cup}A_t=\emptyset$ and $\underset{t\in T}{\vee_{\mathcal{L}}}\widetilde{\pi}(A_t)=(0,1)$. So $\widetilde{\pi}(\emptyset)=(0,1)$. Furthermore, if $A_1,A_2\subseteq X$ and $T=\{1,2\}$ then we have

$$\widetilde{\pi}(A_1\cup A_2)=\widetilde{\pi}(A_1)\vee_{\mathcal{L}}\widetilde{\pi}(A_2)$$

even if $A_1\cap A_2\neq\emptyset$. If $A_1\subseteq A_2$ then $A_1\cup A_2=A_2$, therefore

$$\widetilde{\pi}(A_1)\leq_{\mathcal{L}}\widetilde{\pi}(A_1)\vee_{\mathcal{L}}\widetilde{\pi}(A_2)=\widetilde{\pi}(A_2).$$

■

Theorem 2.3 *If $\widetilde{\pi}:\mathcal{P}(X)\rightarrow\mathcal{L},\widetilde{\pi}=(\pi_1,\pi_2)$ is an intuitionistic fuzzy-valued possibility measure then $\pi_1,\pi_2^{\sim}:\mathcal{P}(X)\rightarrow[0,1]$, where $\pi_2^{\sim}(A)=1-\pi_2(A)$, are possibility measures.*

Proof. Because $\widetilde{\pi}(X)=(1,0)$ the equalities $\pi_1(X)=1$ and $\pi_2^{\sim}(X)=1$ are immediate.

Let T be an arbitrary index set and $(A_t)_{t\in T}\subseteq\mathcal{P}(X)$. Because

$$\begin{aligned}\left(\pi_1\left(\underset{t\in T}{\cup}A_t\right),\pi_2\left(\underset{t\in T}{\cup}A_t\right)\right) &= \widetilde{\pi}\left(\underset{t\in T}{\cup}A_t\right)=\\ \underset{t\in T}{\vee_{\mathcal{L}}}\widetilde{\pi}(A_t) &= \left(\underset{t\in T}{\vee}\pi_1(A_t),\underset{t\in T}{\wedge}\pi_2(A_t)\right),\end{aligned}$$

we have

$$\pi_1\left(\underset{t\in T}{\cup}A_t\right)=\underset{t\in T}{\vee}\pi_1(A_t)$$

and

$$\begin{aligned}\pi_2^{\sim}\left(\underset{t\in T}{\cup}A_t\right) &= 1-\pi_2\left(\underset{t\in T}{\cup}A_t\right)=1-\underset{t\in T}{\wedge}\pi_2(A_t)\\ &= \underset{t\in T}{\vee}(1-\pi_2(A_t))=\underset{t\in T}{\vee}\pi_2^{\sim}(A_t).\end{aligned}$$

■

Theorem 2.4 *Any intuitionistic fuzzy-valued possibility measure is continuous from below.*

Proof. If $\widetilde{\pi} = (\pi_1, \pi_2)$ is an intuitionistic fuzzy-valued possibility measure then π_1 and $\pi_2^{\sim}$ in Theorem 2.3 are continuous from below according to Theorem 1.6. We obtain π_1 and π_2 continuous from below, that is $\widetilde{\pi}$ continuous from below (see Definition 1.2). ■

Example 2.5 *Let $X = \mathbb{Q}$ and $\widetilde{\pi} : \mathcal{P}(X) \to \mathcal{L}, \widetilde{\pi} = (\pi_1, \pi_2)$ defined by*

$$\widetilde{\pi}(A) = \begin{cases} (1,0), & \text{if } A \neq \emptyset \\ (0,1), & \text{if } A = \emptyset. \end{cases}$$

It is obvious that $\widetilde{\pi}(X) = (1,0), \widetilde{\pi}\left(\underset{t\in T}{\cup} A_t\right) = \underset{t\in T}{\vee_{\mathcal{L}}} \widetilde{\pi}(A_t) = (1,0)$ if there exists $t_0 \in T$ such that $A_{t_0} \neq \emptyset$ and $\widetilde{\pi}\left(\underset{t\in T}{\cup} A_t\right) = \underset{t\in T}{\vee_{\mathcal{L}}} \widetilde{\pi}(A_t) = (0,1)$ contrariwise, therefore $\widetilde{\pi}$ is an intuitionistic fuzzy-valued possibility measure. If we take $A_n = \left(0, \frac{1}{n}\right), n \geq 2$ then $(A_n)_{n\in\mathbb{Q}}$ is decreasing, $\underset{n\geq 2}{\cap} A_n = \emptyset, \widetilde{\pi}(A_n) = (1,0), \forall n \in \mathbb{Q}, n \geq 2$, we obtain

$$\pi_1\left(\underset{n\geq 2}{\cap} A_n\right) = \pi_1(\emptyset) = 0 \neq 1 = \lim_{n\to\infty} \pi_1(A_n),$$

therefore $\widetilde{\pi}$ is not continuous from above.

The density functions help us to construct intuitionistic fuzzy-valued possibility measures.

Definition 2.6 *If $\widetilde{\pi} : \mathcal{P}(X) \to \mathcal{L}$ is an intuitionistic fuzzy-valued possibility measure then the function $\widetilde{f} : X \to \mathcal{L}$ defined by*

$$\widetilde{f}(x) = \widetilde{\pi}(\{x\})$$

is called its density function.

Theorem 2.7 *If $\widetilde{f}$ is the density function of an intuitionistic fuzzy-valued possibility measure $\widetilde{\pi} : \mathcal{P}(X) \to \mathcal{L}$ then*

$$\underset{x\in X}{\vee_{\mathcal{L}}} \widetilde{f}(x) = (1,0).$$

Conversely, if $\widetilde{f} : X \to \mathcal{L}$ satisfies $\underset{x\in X}{\vee_{\mathcal{L}}} \widetilde{f}(x) = (1,0)$ then there exists an intuitionistic fuzzy-valued possibility measure $\widetilde{\pi}$ such that $\widetilde{f}$ is the density function of $\widetilde{\pi}$.

Proof. If $\widetilde{f}$ is the density function of $\widetilde{\pi}$ then we get

$$\begin{aligned} \underset{x\in X}{\vee_{\mathcal{L}}} \widetilde{f}(x) &= \underset{x\in X}{\vee_{\mathcal{L}}} \widetilde{\pi}(\{x\}) = \widetilde{\pi}\left(\underset{x\in X}{\cup} \{x\}\right) \\ &= \widetilde{\pi}(X) = (1,0). \end{aligned}$$

Conversely, taking

$$\widetilde{\pi}(A) = \underset{x\in A}{\vee_{\mathcal{L}}} \widetilde{f}(x),$$

for any $A \in \mathcal{P}(X)$, then

$$\widetilde{\pi}(X) = \underset{x\in X}{\vee_{\mathcal{L}}} \widetilde{f}(x) = (1,0)$$

and

$$\widetilde{\pi}\left(\underset{t\in T}{\cup} A_t\right) = \underset{x\in \underset{t\in T}{\cup} A_t}{\vee_{\mathcal{L}}} \widetilde{f}(x) = \underset{t\in T}{\vee_{\mathcal{L}}} \underset{x\in A_t}{\vee_{\mathcal{L}}} \widetilde{f}(x) = \underset{t\in T}{\vee_{\mathcal{L}}} \widetilde{\pi}(A_t),$$

for any subfamily $(A_t)_{t\in T} \subset \mathcal{P}(X)$, where T is an arbitrary index set. If $A = \{x\}$ then we obtain $\widetilde{f}(x) = \widetilde{\pi}(\{x\})$, therefore $\widetilde{f}$ is the density function of $\widetilde{\pi}$. ■

Example 2.8 *Let $X = \{x_1, x_2, x_3, x_4\}$ and $\widetilde{f} : X \to \mathcal{L}$ a function given by $\widetilde{f}(x_1) = (1,0), \widetilde{f}(x_2) = (0.3, 0.4), \widetilde{f}(x_3) = (0.4, 0.6), \widetilde{f}(x_4) = (0.2, 0.1)$. The intuitionistic fuzzy-valued possibility measure $\widetilde{\pi}$ induced by $\widetilde{f}$ is given by*

$$\begin{aligned}
\widetilde{\pi}(\emptyset) &= \underset{x\in\emptyset}{\vee_{\mathcal{L}}} \widetilde{f}(x) = (0,1), \\
\widetilde{\pi}(\{x_1\}) &= \widetilde{\pi}(\{x_1,x_2\}) = \widetilde{\pi}(\{x_1,x_3\}) = \widetilde{\pi}(\{x_1,x_4\}) = (1,0) \\
\widetilde{\pi}(\{x_2,x_3\}) &= (0.3,0.4) \vee_{\mathcal{L}} (0.4,0.6) = (0.4,0.4) \\
\widetilde{\pi}(\{x_2,x_4\}) &= (0.3,0.4) \vee_{\mathcal{L}} (0.2,0.1) = (0.3,0.1) \\
\widetilde{\pi}(\{x_3,x_4\}) &= (0.4,0.6) \vee_{\mathcal{L}} (0.2,0.1) = (0.4,0.1) \\
\widetilde{\pi}(\{x_1,x_2,x_3\}) &= \widetilde{\pi}(\{x_1,x_2,x_4\}) = \widetilde{\pi}(\{x_1,x_3,x_4\}) = (1,0) \\
\widetilde{\pi}(\{x_2,x_3,x_4\}) &= (0.3,0.4) \vee_{\mathcal{L}} (0.4,0.6) \vee_{\mathcal{L}} (0.2,0.1) = (0.4,0.1) \\
\widetilde{\pi}(\{x_1,x_2,x_3,x_4\}) &= (1,0).
\end{aligned}$$

3 Intuitionistic fuzzy-valued necessity measures

The concept of necessity measure on classical sets with intuitionistic fuzzy values is introduced in the following definition.

Definition 3.1 *The dual $\widetilde{\omega} : \mathcal{P}(X) \to \mathcal{L}$ of an intuitionistic fuzzy-valued possibility measure $\widetilde{\pi} : \mathcal{P}(X) \to \mathcal{L}, \widetilde{\pi}(A) = (\pi_1(A), \pi_2(A))$, that is $\widetilde{\omega}(A) = (\pi_2(A^c), \pi_1(A^c))$ for any $A \in \mathcal{P}(X)$, is called an intuitionistic fuzzy-valued necessity measure on $\mathcal{P}(X)$.*

Theorem 3.2 *Any intuitionistic fuzzy-valued necessity measure is continuous from above.*

Proof. Theorem 1.3 and Theorem 2.4 imply the dual of any intuitionistic fuzzy-valued possibility measure is continuous from above. ■

The following theorem characterizes the intuitionistic fuzzy-valued necessity measures.

Theorem 3.3 *An intuitionistic fuzzy-valued set function $\widetilde{\omega} : \mathcal{P}(X) \to \mathcal{L}$ is an intuitionistic fuzzy-valued necessity measure if and only if it satisfies*

$$\widetilde{\omega}\left(\bigcap_{t\in T} A_t\right) = \bigwedge_{\mathcal{L}}{}_{t\in T} \widetilde{\omega}(A_t),$$

for any subfamily $(A_t)_{t\in T} \subset \mathcal{P}(X)$, where T is an arbitrary index set.

Proof. If $\widetilde{\omega}$ is an intuitionistic fuzzy-valued necessity measure then it is the dual of an intuitionistic fuzzy-valued possibility measure $\widetilde{\pi} = (\pi_1, \pi_2)$. Taking into account Theorem 2.3 we obtain

$$\begin{aligned}
\widetilde{\omega}\left(\bigcap_{t\in T} A_t\right) &= \left(\pi_2\left(\left(\bigcap_{t\in T} A_t\right)^c\right), \pi_1\left(\left(\bigcap_{t\in T} A_t\right)^c\right)\right) \\
&= \left(\pi_2\left(\bigcup_{t\in T} A_t^c\right), \pi_1\left(\bigcup_{t\in T} A_t^c\right)\right) \\
&= \left(1 - \pi_2^{\sim}\left(\bigcup_{t\in T} A_t^c\right), \pi_1\left(\bigcup_{t\in T} A_t^c\right)\right) \\
&= \left(1 - \bigvee_{t\in T} \pi_2^{\sim}(A_t^c), \bigvee_{t\in T} \pi_1(A_t^c)\right) \\
&= \left(\bigwedge_{t\in T} (1 - \pi_2^{\sim}(A_t^c)), \bigvee_{t\in T} \pi_1(A_t^c)\right) \\
&= \left(\bigwedge_{t\in T} \pi_2(A_t^c), \bigvee_{t\in T} \pi_1(A_t^c)\right) \\
&= \bigwedge_{\mathcal{L}}{}_{t\in T} (\pi_2(A_t^c), \pi_1(A_t^c)) \\
&= \bigwedge_{\mathcal{L}}{}_{t\in T} \widetilde{\omega}(A_t),
\end{aligned}$$

for any subfamily $(A_t)_{t\in T} \subset \mathcal{P}(X)$, where T is an arbitrary index set and $\pi_2^{\sim}(A) = 1 - \pi_2(A)$, for every $A \in \mathcal{P}(X)$.

Conversely, let $\widetilde{\omega} = (\omega_1, \omega_2)$ an intuitionistic fuzzy-valued set function which satisfies $\widetilde{\omega}\left(\bigcap_{t\in T} A_t\right) = \bigwedge_{\mathcal{L}}{}_{t\in T} \widetilde{\omega}(A_t)$, that is $\omega_1\left(\bigcap_{t\in T} A_t\right) = \bigwedge_{t\in T} \omega_1(A_t)$ and $\omega_2\left(\bigcap_{t\in T} A_t\right) = \bigvee_{t\in T} \omega_2(A_t)$, for any subfamily $(A_t)_{t\in T} \subset \mathcal{P}(X)$, where T is an arbitrary index set. We must prove only that $\widetilde{\pi} = (\pi_1, \pi_2)$ defined by

$$\pi_1(A) = \omega_2(A^c)$$

and

$$\pi_2(A) = \omega_1(A^c)$$

is an intuitionistic fuzzy-valued possibility measure because it is obvious that $\widetilde{\omega}$ is the dual of $\widetilde{\pi}$. We have

$$\widetilde{\pi}(X) = (\pi_1(X), \pi_2(X)) = (\omega_2(\emptyset), \omega_1(\emptyset)) = (1, 0)$$

and

$$
\begin{aligned}
\widetilde{\pi}\left(\bigcup_{t\in T} A_t\right) &= \left(\omega_2\left(\left(\left(\bigcup_{t\in T} A_t\right)^c\right)\right), \omega_1\left(\left(\left(\bigcup_{t\in T} A_t\right)^c\right)\right)\right) \\
&= \left(\omega_2\left(\bigcap_{t\in T} A_t^c\right), \omega_1\left(\bigcap_{t\in T} A_t^c\right)\right) \\
&= \left(\bigvee_{t\in T} \omega_2\left(A_t^c\right), \bigwedge_{t\in T} \omega_1\left(A_t^c\right)\right) \\
&= \bigvee_{t\in T}{}_{\mathcal{L}} \left(\omega_2\left(A_t^c\right), \omega_1\left(A_t^c\right)\right) \\
&= \bigvee_{t\in T}{}_{\mathcal{L}} \left(\pi_1\left(A_t\right), \pi_2\left(A_t\right)\right) \\
&= \bigvee_{t\in T}{}_{\mathcal{L}} \widetilde{\pi}\left(A_t\right).
\end{aligned}
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