# ON THE MODAL OPERATORS DEFINED OVER THE INTUITIONISTIC FUZZY SETS 

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#### Abstract

Intuitionistic fuzzy set is one of the extensions of fuzzy sets. A lot of operators are defined over the IFSs (modal, topological, level). Some of these operators (e.g., the modal operators) cannot be defined over ordinary fuzzy sets. A short review of the results, related to the extended modal operators over IFSs and new definitions and results unpublished up to now, have been given. Keywords. Intuitionistic fuzzy set (IFS), Negation, Operator


## 1 Introduction

Several operators are defined in the Intuitionistic Fuzzy Sets theory (IFSs, [1]). They are classified in three groups: modal, topological and level operators. The relations between them and the relations between the operators and operations over IFSs were discussed in [1]. New modal operators, that are analogous simultaneously to the first one, as well as to the operation "negation" were introduced and studied in [2].

Here we give a short review of the results related to the different modal type of operators and we will discus their new properties.

## 2 Short remarks on the IFSs

Let a set $E$ be fixed. An IFS $A$ in $E$ is an object of the following form:

$$
A=\left\{\left\langle x, \mu_{A}(x), \nu_{A}(x)\right\rangle \mid x \in E\right\}
$$

where functions $\mu_{A}: E \rightarrow[0,1]$ and $\nu_{A}: E \rightarrow[0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$ it holds that

$$
0 \leq \mu_{A}(x)+\nu_{A}(x) \leq 1
$$

Let for every $x \in E$ :

$$
\pi_{A}(x)=1-\mu_{A}(x)-\nu_{A}(x)
$$

Therefore, function $\pi$ determines the degree of uncertainty.
For every two IFSs $A$ and $B$ a lot of relations and operations are defined (see, e.g. [1]), whereby the important ones, for the present research, are:

$$
\begin{array}{ll}
A \subset B & \text { iff } \quad(\forall x \in E)\left(\mu_{A}(x) \leq \mu_{B}(x) \& \nu_{A}(x) \geq \nu_{B}(x)\right) ; \\
A \supset B \quad & \text { iff } \quad B \subset A ; \\
A=B & \text { iff } \quad(\forall x \in E)\left(\mu_{A}(x)=\mu_{B}(x) \& \nu_{A}(x)=\nu_{B}(x)\right) ; \\
\bar{A} \quad=\quad\left\{\left\langle x, \nu_{A}(x), \mu_{A}(x)\right\rangle \mid x \in E\right\} .
\end{array}
$$

The above operations and relations are defined similarly to these from the fuzzy set theory. More interesting are the modal operators that can be defined over the IFSs. They do not have analogues in fuzzy set theory.

In [1] the following two intuitionistic fuzzy analogues of the modal logic operators "necessity" and "possibility" were defined (see, e.g., [4]):

$$
\begin{aligned}
& \square A=\left\{\left\langle x, \mu_{A}(x), 1-\mu_{A}(x)\right\rangle \mid x \in E\right\} ; \\
& \diamond A=\left\{\left\langle x, 1-\nu_{A}(x), \nu_{A}(x)\right\rangle \mid x \in E\right\}
\end{aligned}
$$

Two analogues of the topological operators can be defined over the IFSs, too (see [1]): operator "closure" $C$ and operator "interior" $I$ :

$$
\begin{aligned}
& \mathcal{C}(A)=\left\{\left\langle x, \sup _{y \in E} \mu_{A}(y), \inf _{y \in E} \nu_{A}(y)\right\rangle \mid x \in E\right\}, \\
& \mathcal{I}(A)=\left\{\left\langle x, \inf _{y \in E} \mu_{A}(y), \sup _{y \in E} \nu_{A}(y)\right\rangle \mid x \in E\right\} .
\end{aligned}
$$

Following $[1,2,3]$ we shall introduce modal operators, that are extensions of the two modal operators above, and that have the forms:

$$
\begin{aligned}
& D_{\alpha}(A)=\left\{\left\langle x, \mu_{A}(x)+\alpha \cdot \pi_{A}(x), \nu_{A}(x)+(1-\alpha) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& F_{\alpha, \beta}(A)=\left\{\left\langle x, \mu_{A}(x)+\alpha \cdot \pi_{A}(x), \nu_{A}(x)+\beta \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& \text { where } \alpha+\beta \leq 1, \\
& G_{\alpha, \beta}(A)=\left\{\left\langle x, \alpha \cdot \mu_{A}(x), \beta \cdot \nu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& H_{\alpha, \beta}(A)=\left\{\left\langle x, \alpha \cdot \mu_{A}(x), \nu_{A}(x)+\beta \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& H_{\alpha, \beta}^{*}(A)=\left\{\left\langle x, \alpha \cdot \mu_{A}(x), \nu_{A}(x)+\beta \cdot\left(1-\alpha \cdot \mu_{A}(x)-\nu_{A}(x)\right)\right\rangle \mid x \in E\right\}, \\
& J_{\alpha, \beta}(A)=\left\{\left\langle x, \mu_{A}(x)+\alpha \cdot \pi_{A}(x), \beta \cdot \nu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& J_{\alpha, \beta}^{*}(A)=\left\{\left\langle x, \mu_{A}(x)+\alpha \cdot\left(1-\mu_{A}(x)-\beta \cdot \nu_{A}(x)\right), \beta \cdot \nu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& d_{\alpha}(A)=\left\{\left\langle x, \nu_{A}(x)+\alpha \cdot \pi_{A}(x), \mu_{A}(x)+(1-\alpha) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& f_{\alpha, \beta}(A)=\left\{\left\langle x, \nu_{A}(x)+\alpha \cdot \pi_{A}(x), \mu_{A}(x)+\beta \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& \text { where } \alpha+\beta \leq 1,
\end{aligned}
$$

$$
\begin{aligned}
& g_{\alpha, \beta}(A)=\left\{\left\langle x, \alpha \cdot \nu_{A}(x), \beta \cdot \mu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& h_{\alpha, \beta}(A)=\left\{\left\langle x, \alpha \cdot \nu_{A}(x), \mu_{A}(x)+\beta \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& h_{\alpha, \beta}^{*}(A)=\left\{\left\langle x, \alpha \cdot \nu_{A}(x), \mu_{A}(x)+\beta \cdot\left(1-\alpha \cdot \nu_{A}(x)-\mu_{A}(x)\right)\right\rangle \mid x \in E\right\}, \\
& j_{\alpha, \beta}(A)=\left\{\left\langle x, \nu_{A}(x)+\alpha \cdot \pi_{A}(x), \beta \cdot \mu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& j_{\alpha, \beta}^{*}(A)=\left\{\left\langle x, \nu_{A}(x)+\alpha \cdot\left(1-\nu_{A}(x)-\beta \cdot \mu_{A}(x)\right), \beta \cdot \mu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& F_{B}(A)=\left\{\left\langle x, \mu_{A}(x)+\mu_{B}(x) \cdot \pi_{A}(x), \nu_{A}(x)+\nu_{B}(x) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& G_{B}(A)=\left\{\left\langle x, \mu_{B}(x) \cdot \mu_{A}(x), \nu_{B}(x) \cdot \nu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& H_{B}(A)=\left\{\left\langle x, \mu_{B}(x) \cdot \mu_{A}(x), \nu_{A}(x)+\nu_{B}(x) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
& H_{B}^{*}(A)=\left\{\left\langlex, \mu_{B}(x) \cdot \mu_{A}(x), \nu_{A}(x)+\nu_{B}(x) \cdot\left(1-\mu_{B}(x) \cdot \mu_{A}(x)\right.\right.\right. \\
&\left.\left.\left.-\nu_{A}(x)\right)\right\rangle \mid x \in E\right\}, \\
& J_{B}(A)=\left\{\left\langle x, \mu_{A}(x)+\mu_{B}(x) \cdot \pi_{A}(x), \nu_{B}(x) \cdot \nu_{A}(x)\right\rangle \mid x \in E\right\}, \\
& J_{B}^{*}(A)=\left\{\left\langlex, \mu_{A}(x)+\mu_{B}(x) \cdot\left(1-\mu_{A}(x)-\nu_{B}(x) \cdot \nu_{A}(x)\right),\right.\right. \\
&\left.\left.\nu_{B}(x) \cdot \nu_{A}(x)\right\rangle \mid x \in E\right\} .
\end{aligned}
$$

Immediately, we see that for each IFS $A$ :

$$
\begin{gather*}
A=F_{0,0}(A)=G_{1,1}(A)=H_{1,0}(A)=H_{1,0}^{*}(A)=J_{0,1}(A)=J_{0,1}^{*}(A) \\
\bar{A}=f_{0,0}(A)=g_{1,1}(A)=h_{1,0}(A)=h_{1,0}^{*}(A)=j_{0,1}(A)=j_{0,1}^{*}(A) . \tag{1}
\end{gather*}
$$

As we showed above, operators $d_{\alpha}$ and $f_{\alpha, \beta}$ are direct extensions of the operators "necessity" and "possibility", while the other new operators have no analogues in the ordinary modal logic. On the other hand, we saw that there is a similarity between the behaviour of operators $d_{\alpha}, f_{\alpha, \beta}, g_{\alpha, \beta}$ and operation "negation". In the present form of this operation in IFS theory [1], it satisfies the equality

$$
\begin{equation*}
\overline{\bar{A}}=A \tag{2}
\end{equation*}
$$

for every IFS $A$.
Therefore, operators $f_{\alpha, \beta}$ (in particular, $d_{\alpha}$ ) and $g_{\alpha, \beta}$ are extensions of the operation "negation".

## 3 Main results

As we showed above, up to now the ordanary modal operators $\square$ and $\diamond$ are extended three times:
Group I: operators $D_{\alpha}, F_{\alpha, \beta}, G_{\alpha, \beta}, H_{\alpha, \beta}, H_{\alpha, \beta}^{*}, J_{\alpha, \beta}, J_{\alpha, \beta}^{*}$;
Group II: operators $d_{\alpha}, f_{\alpha, \beta}, g_{\alpha, \beta}, h_{\alpha, \beta}, h_{\alpha, \beta}^{*}, j_{\alpha, \beta}, j_{\alpha, \beta}^{*}$;
Group III: operators $D_{B}, F_{B}, G_{B}, H_{B}, H_{B}^{*}, J_{B}, J_{B}^{*}$,
where $\alpha, \beta \in[0,1]$ and $B$ is a fixed IFS.

Now, we will introduce for a first time and will study some properties of one more, forth, group of operators, extending operators from Group II and being analogous of the operators from Group III. They have the forms

$$
\begin{aligned}
f_{B}(A)= & \left\{\left\langle x, \nu_{A}(x)+\mu_{B}(x) \cdot \pi_{A}(x), \mu_{A}(x)+\nu_{B}(x) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
g_{B}(A)= & \left\{\left\langle x, \mu_{B}(x) \cdot \nu_{A}(x), \nu_{B}(x) \cdot \mu_{A}(x)\right\rangle \mid x \in E\right\}, \\
h_{B}(A)= & \left\{\left\langle x, \mu_{B}(x) \cdot \nu_{A}(x), \mu_{A}(x)+\nu_{B}(x) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}, \\
h_{B}^{*}(A)= & \left\{\left\langlex, \mu_{B}(x) \cdot \nu_{A}(x),\right.\right. \\
& \left.\left.\mu_{A}(x)+\nu_{B}(x) \cdot\left(1-\mu_{A}(x)-\mu_{B}(x) \cdot \nu_{A}(x)\right)\right\rangle \mid x \in E\right\}, \\
j_{B}(A)= & \left\{\left\langle x, \nu_{A}(x)+\mu_{B}(x) \cdot \pi_{A}(x), \nu_{B}(x) \cdot \mu_{A}(x)\right\rangle \mid x \in E\right\}, \\
j_{B}^{*}(A)= & \left\{\left\langlex, \nu_{A}(x)+\mu_{B}(x) \cdot\left(1-\nu_{B}(x) \cdot \mu_{A}(x)-\nu_{A}(x)\right),\right.\right. \\
& \left.\left.\nu_{B}(x) \cdot \mu_{A}(x)\right\rangle \mid x \in E\right\} .
\end{aligned}
$$

Let

$$
\begin{aligned}
O^{*} & =\{\langle x, 0,1\rangle \mid x \in E\}, \\
E^{*} & =\{\langle x, 1,0\rangle \mid x \in E\} .
\end{aligned}
$$

For these operators the following assertions are valid.
Theorem 1: For every three IFSs $A, B, C$ :
(a) $d_{B}(A), f_{B}(A), g_{B}(A), h_{B}(A), h_{B}^{*}(A), j_{B}(A)$, and $j_{B}^{*}(A)$ are IFSs;
(b) $d_{B}(A)=f_{\square B}(A)$,
(c) $\square A=d_{O^{*}}(A)=f_{O^{*}}(A)$,
(d) $\diamond A=d_{E^{*}}(A)=f_{E^{*}}(A)$,
(e) if $B \subset C$, then $f_{B}(A) \subset f_{C}(A)$,
(f) if $B \subset C$, then $g_{B}(A) \subset g_{C}(A)$,
(g) if $B \subset C$, then $h_{B}(A) \subset h_{C}(A)$,
(h) if $B \subset C$, then $h_{B}^{*}(A) \subset h_{C}^{*}(A)$,
(i) if $B \subset C$, then $j_{B}(A) \subset j_{C}(A)$,
(j) if $B \subset C$, then $j_{B}^{*}(A) \subset j_{C}^{*}(A)$,
(k) $\overline{f_{B}(\bar{A})}=f_{\bar{B}}(A)$,
(l) $\overline{g_{B}(\bar{A})}=g_{\bar{B}}(A)$,
(m) $\overline{h_{B}(\bar{A})}=h_{\bar{B}}(A)$,
(n) $\overline{h_{B}^{*}(\bar{A})}=h_{\bar{B}}^{*}(A)$,
(o) $\overline{j_{B}(\bar{A})}=j_{\bar{B}}(A)$,
(p) $\overline{j_{B}^{*}(\bar{A})}=j_{\bar{B}}^{*}(A)$.

Theorem 2: For every two IFSs $A$ and $B$ :
(a) $\mathcal{C} f_{B}(A) \supset f_{B} \mathcal{C} A$,
(b) $\mathcal{I} f_{B}(A) \subset f_{B} \mathcal{I} A$,
(c) $g_{B}(\mathcal{C} A)=\mathcal{C} g_{B}(A)$,
(d) $g_{B}(\mathcal{I} A)=\mathcal{I} g_{B}(A)$.

Theorem 3: For every three IFSs $A, B, C$ :
(a) $f_{B}\left(f_{C}(A)\right)=f_{f_{B}(C)}(A)$,
(b) $d_{B}\left(d_{C}(A)\right)=d_{\diamond C}(\bar{A})$,
(c) $g_{B}\left(g_{C}(A)\right)=g_{g_{B}(C)}(\bar{A})=g_{C}\left(g_{B}(A)\right)$.

Proof: (a) Let $B, C$ are fixed IFSs. Then:

$$
\begin{aligned}
& f_{B}\left(f_{C}(A)\right) \\
= & f_{B}\left(\left\{\left\langle x, \nu_{A}(x)+\mu_{C}(x) \cdot \pi_{A}(x), \mu_{A}(x)+\nu_{C}(x) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\}\right) \\
= & \left\{\left\langlex, \mu_{A}(x)+\nu_{C}(x) \cdot \pi_{A}(x)\right.\right. \\
& +\mu_{B}(x) \cdot\left(1-\mu_{A}(x)-\mu_{C}(x) \cdot \pi_{A}(x)-\nu_{A}(x)-\nu_{C}(x) \cdot \pi_{A}(x)\right), \\
& \nu_{A}(x)+\mu_{C}(x) \cdot \pi_{A}(x)+\nu_{B}(x) \cdot\left(1-\mu_{A}(x)\right. \\
& \left.\left.\left.-\mu_{C}(x) \cdot \pi_{A}(x)-\nu_{A}(x)-\delta \cdot \pi_{A}(x)\right)\right\rangle \mid x \in E\right\} \\
= & \left\{\left\langlex, \mu_{A}(x)+\left(\nu_{C}(x)+\mu_{B}(x) \cdot \pi_{C}(x)\right) \cdot \pi_{A}(x),\right.\right. \\
& \left.\left.\nu_{A}(x)+\left(\mu_{C}(x)+\nu_{B}(x) \cdot \pi_{C}(x)\right) \cdot \pi_{A}(x)\right\rangle \mid x \in E\right\} \\
= & f_{f_{B}(C)}(A) .
\end{aligned}
$$

Theorem 4: For every three IFSa $A, B, C$ :
(a) $F_{B}\left(f_{C}(A)\right)=f_{F_{B}(C)}(A)=F_{F_{B}(C)}(\bar{A})$,
(b) $f_{B}\left(F_{C}(A)\right)=f_{f_{B}(C)}(A)=f_{F_{B}(\bar{C})}(A)=F_{f_{B}(C)}(\bar{A})=F_{F_{B}(\bar{C})}(\bar{A})$,
(c) $G_{B}\left(g_{C}(A)\right)=g_{G_{B}(C)}(A)=G_{G_{B}(C)}(\bar{A})$,
(d) $g_{B}\left(G_{C}(A)\right)=g_{g_{B}(C)}(A)=g_{G_{B}(\bar{C})}(A)=G_{g_{B}(C)}(\bar{A})=F_{G_{B}(\bar{C})}(\bar{A})$.

## 4 Conclusion

The newly defined modal type of operators over IFSs have not only a self-dependent place in IFSs theory. It shows that the operators in modal logic (see, e.g., [4]) have new and interesting properties that have not been studied up to now as in the frameworks of the ordinary modal logic, as well as in the frameworks of the IFSs and IF logic theories, which largest bibliography by the moment is given in [5].

## References

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