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Extended Choquet Integrals and Intuitionistic Fuzzy Sets

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Let m be a positive monotone set function defined on D ,i.e,(X, D, m) is a monotone measure space, and f non-negative function on X.By Proposition 15 we can define Choque's integral as follows:

DEFINITION 1 Let (X, D, m) be a monotone measure space and $g : [A, B] \longrightarrow [0, \infty]$. Extended Choque's integral of measurable function f with respect to m, denoted by $(\overline{C_g}) \int_A^B f\, dm$,is defined by

$$(\overline{C_g}) \int_A^B f\, dm = \begin{cases} \int_A^B m(\{x|f(x) > g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|f(x) \geq g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

Proposition 1 Let (X, D, m) be a monotone measure space and f a measurable function. If, for every $r \in [A, B]$

$$G(g(r)) = \begin{cases} -m(\{x|f(x) > g(r)\}) \text{ if } f \text{ is o-measurable} \\ -m(\{x|f(x) \geq g(r)\}) \text{ if } f \text{ is c-measurable} \end{cases}$$

and

$$\lim_{r \to B} g(r) = \infty$$

then

$$(\overline{C_g}) \int_A^B f\, dm = \int_A^B (r - A)\, dG(g(r))$$

The integral on the right side is Stieltjes integral.
Proof. By property Stieltjes integral we have

$$\int_A^B (r - A)\, dG(g(r)) = (r - A) \cdot G(g(r))|_A^B - \int_A^B G(g(r))\, dr = (\overline{C_g}) \int_A^B f\, dm$$

Proposition 2 Let (X, D, m) be a monotone measure space. Let f and p measurable functions and $\{f_n\}$ a sequence of measurable functions.
(1)$f \leq p$ implies $(\overline{C_g}) \int_A^B f\, dm \leq (\overline{C_g}) \int_A^B p\, dm$
$(1')g_1 \leq g_2$ implies $(\overline{C_{g_2}}) \int_A^B f\, dm \leq (\overline{C_{g_1}}) \int_A^B f\, dm$
(2) If $g(a \cdot t) = a \cdot g(t)$ for every $a > 0$ (when $t > 0$ and g is continuous then $g(t) = t \cdot g(1)$) then

$(\overline{C_g}) \int_A^B a \cdot f\, dm = a \cdot (\overline{C_g}) \int_{A/a}^{B/a} f\, dm$ for every $a > 0$
$(2')$ Let $Z = \{r|g(r) = 0\}$ and Z is measurable then

$$(\overline{C_g}) \int_A^B 0 \cdot f\, dm = \begin{cases} \int_A^B m(\{x|0 > g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|0 \geq g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$= \begin{cases} 0 \text{if } f \text{ is o-measurable} \\ m(X) \cdot m_1(Z) \text{if } f \text{ is c-measurable} \end{cases}$$

Where m_1 is ordinary measure.
(2")$(\overline{C_g}) \int_A^B a \cdot f\, dm = a \cdot (\overline{C_g}) \int_A^B f\, dm$ for every $a = 0$ and $m_1(Z) = 0$
(3)If m is an ordinary measure, then Choquet's integral coincider with Lebesgue integral.
(4)Let $F = \{r|g(r) \leq 1\}$ and F is measurable then

$$(\overline{C_g}) \int_A^B \chi_A\, dm = m(A) \cdot m_1(F)\, (A \in D)$$

(5) If m is continuous from below, then $f_n \uparrow f$ implies that

$$(\overline{C_g}) \int_A^B f_n\, dm \uparrow (\overline{C_g}) \int_A^B f\, dm$$

(5') If m is continuous from below, then $g_n \uparrow g$ implies that

$$(\overline{C_{g_n}}) \int_A^B f\, dm \uparrow (\overline{C_g}) \int_A^B f\, dm$$

(6) Let m be continuous from above. If $f_n \uparrow f$ and if for at least one value of n the function f_n is bounded and $m(\{f_n > inf_{r \in [A,B]} g(r)\})$, then

$$(\overline{C_g}) \int_A^B f_n\, dm \uparrow (\overline{C_g}) \int_A^B f\, dm$$

(6') Let m be continuous from above. If $g_n \uparrow g$ and if for at least one value of n the function g_n is bounded and $m(\{f > inf_{r \in [A,B]} g_n(r)\})$, then

$$(\overline{C_{g_n}}) \int_A^B f\, dm \uparrow (\overline{C_g}) \int_A^B f\, dm$$

Proof. (1) When $f \leq p$ we have $m(\{x|f(x) > g(r)\}) \leq m(\{x|p(x) > g(r)\})$ if f and p is o-measurable. When f and p is c-measurable we have $m(\{x|f(x) \geq g(r)\}) \leq m(\{x|p(x) \geq g(r)\})$

$$(\overline{C_g}) \int_A^B f\, dm = \begin{cases} \int_A^B m(\{x|f(x) > g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|f(x) \geq g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$\leq \begin{cases} \int_A^B m(\{x|p(x) > g(r)\})\, dr \text{if } p \text{ is o-measurable} \\ \int_A^B m(\{x|p(x) \geq g(r)\})\, dr \text{if } p \text{ is c-measurable} \end{cases}$$

$$= (\overline{C_g}) \int_A^B p\, dm$$

When f and p aren't simultaneously o-measurable or c-measurable we use Proposition 7.1 from [2] and the reasoning above.

(1')When $g_1 \leq g_2$ and f is o-measurable we have $m(\{x|f(x) > g_2(r)\}) \leq m(\{x|p(x) > g_1(r)\})$. When f is c-measurable we have $m(\{x|f(x) \geq g_2(r)\}) \leq m(\{x|p(x) \geq g_1(r)\})$

$$(\overline{C_{g_2}}) \int_A^B f\, dm = \begin{cases} \int_A^B m(\{x|f(x) > g_2(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|f(x) \geq g_2(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$\leq \begin{cases} \int_A^B m(\{x|f(x) > g_1(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|f(x) \geq g_1(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$= (\overline{C_{g_1}}) \int_A^B f\, dm$
For (2) we obtain

$$(\overline{C_g}) \int_A^B a \cdot f\, dm = \begin{cases} \int_A^B m(\{x|a \cdot f(x) > g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|a \cdot f(x) \geq g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$= \begin{cases} \int_A^B m(\{x|f(x) > 1/a \cdot g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|f(x) \geq 1/a \cdot g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$= \begin{cases} \int_A^B m(\{x|f(x) > g(1/a \cdot r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_A^B m(\{x|f(x) \geq g(1/a \cdot r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$= \begin{cases} a \cdot \int_A^B 1/a \cdot m(\{x|f(x) > g(1/a \cdot r)\})\, dr \text{if } f \text{ is o-measurable} \\ a \cdot \int_A^B 1/a \cdot m(\{x|f(x) \geq g(1/a \cdot r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$= a \cdot \begin{cases} \int_{A/a}^{B/a} m(\{x|f(x) > g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_{A/a}^{B/a} m(\{x|f(x) \geq g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$a \cdot (\overline{C_g}) \int_{A/a}^{B/a} f\, dm$
(2') and (2"),(3) are obviously.
(4)$(\overline{C_g}) \int_A^B \chi_A\, dm = (\overline{C_g}) \int_{[A,B]-F} \chi_A\, dm + (\overline{C_g}) \int_F \chi_A\, dm = (\overline{C_g}) \int_F \chi_A\, dm = m(A) \cdot m_1(F)$
(5)Let f_n and$f$ are c and o -measurable .From $f_n \nearrow f$ we have $\{x|f_i > g(r)\} \subset \{x|f_{i+1} > g(r)\}$ for every i and r. m is continuous from below consequently $m(\{x|f > g(r)\}) = m(\bigcup\{x|f_i > g(r)\}) = \lim_{n\to\infty} m(\{x|f_i > g(r)\})$. We use that $\{x|f > g(r)\} = \bigcup\{x|f_i > g(r)\}$. Let $q_i(r) = m(\{x|f_i > g(r)\}) = m(\{x|f_i \geq g(r)\})$ and $q(r) = m(\{x|f > g(r)\}) = m(\{x|f \geq g(r)\})$.

Let $f_n \nearrow f$,f_n and$f$ are c and o -measurable .If m and g are continuous function obviously that $q_i(r)$and $q(r)$ are continuous functions.From Dynni's criterion of uniformly converges we have $q_i(r)$ uniformly converges on $q(r)$ in $[A, B]$.Therefore

$$(\overline{C_g}) \int_A^B f_n\, dm \uparrow (\overline{C_g}) \int_A^B f\, dm$$

because $[A, B]$ is compact set.
Proposition 3 Let (X, D, m) be a monotone measure space.Every simple function f on X can be represented as

$$f = \sum_{i=1}^{n} (a_i - a_{i-1}) \cdot \chi_{A_i}, \quad (12)$$

where $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ and $A_1 \supset A_2 \supset \cdots \supset A_n$. Extended Choquet's integral of a simple function f writen as (12) with respect to m can be represented as

$$(\overline{C_g}) \int_A^B f\, dm = \sum_{i=1}^{n} (a_i - a_{i-1}) \cdot m(A_i) \cdot m_1(F_i) \quad (13)$$

where $g(a \cdot t) = a \cdot g(t)$, $F = \{r | g(r) \leq 1\}$ and $F_i = F \bigcap [A/(a_i - a_{i-1}), B/(a_i - a_{i-1})]$If a monotone set function m is continuous from below, then we can define the integral like Lebesgue integral: for a simple function f defined as (12), we define the integral of f as (13), and for a measurable function f, we define the integral of f as

$$(\overline{C_g}) \int_A^B f\, dm = \lim_{n \to \infty} (\overline{C_g}) \int_A^B f_n\, dm$$

where $\{f_n\}$ is a sequence of simple functions such that for every $x \in X$, $f_n \uparrow f$.

Since monotone set function generally is non-additive, also Extended Choquet's integral is generally non-additive. Concerning the additivity of Extended Choquet's integral, the next theorem holds.

DEFINITION 2 Let (X, D, m) be a monotone measure space,$X = [0, \infty]$ $L = \{g : [0, \infty] \longrightarrow [0, \infty] | g(t \cdot r) = t \cdot g(r), t \in R^+\}$ and $g \in L$

$$< f, g >= (\overline{C_g}) \int_0^\infty f\, dm$$

Lemma 1 Let $\alpha \in R^+$ then

$$< \alpha \cdot f, g >= \alpha \cdot < f, g >$$

Proof. It is obviously from (2)Proposition 2.
Lemma 2 Let $\alpha \in R^+$ then

$$< f, \alpha \cdot g >= 1/\alpha \cdot < f, g >$$

Proof. $< f, \alpha \cdot g >= (\overline{C_{\alpha \cdot g}}) \int_0^\infty f\, dm =$

$$(\overline{C_{\alpha \cdot g}}) \int_0^\infty f\, dm = \begin{cases} \int_0^\infty m(\{x | f(x) > \alpha \cdot g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_0^\infty m(\{x | f(x) \geq \alpha \cdot g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$$= \begin{cases} 1/\alpha \cdot \int_0^\infty m(\{x | f(x) > g(\alpha \cdot r)\})\, d(\alpha \cdot r) \text{if } f \text{ is o-measurable} \\ 1/\alpha \cdot \int_0^\infty m(\{x | f(x) \geq g(\alpha \cdot r)\})\, d(\alpha \cdot r) \text{if } f \text{ is c-measurable} \end{cases}$$

$$= 1/\alpha \cdot \begin{cases} \int_0^\infty m(\{x | f(x) > g(z)\})\, dz \text{if } f \text{ is o-measurable} \\ \int_0^\infty m(\{x | f(x) \geq g(z)\})\, dz \text{if } f \text{ is c-measurable} \end{cases}$$

$$= 1/\alpha \cdot \begin{cases} \int_0^\infty m(\{x | f(x) > g(r)\})\, dr \text{if } f \text{ is o-measurable} \\ \int_0^\infty m(\{x | f(x) \geq g(r)\})\, dr \text{if } f \text{ is c-measurable} \end{cases}$$

$= 1/\alpha \cdot (\overline{C_g}) \int_0^\infty f\, dm = 1/\alpha \cdot < f, g >$.
Lemma 3 Let $m(A \bigcup B) + m(A \bigcap B) \leq m(A) + m(B)$ $(A \in D, B \in D)$ then $< f, g_1 + g_2 > \leq < f, g_1 > + < f, g_2 >$

Proof. We have $m(\{x|f(x) \geq (g_1+g_2)(r)\}) = m(\{x|f(x) \geq g_1(r)\} \bigcap \{x|f(x) \geq g_2(r)\}) \leq m(\{x|f(x) \geq g_1(r)\}) + m(\{x|f(x) \geq g_2(r)\})$ and $m(\{x|f(x) > (g_1+g_2)(r)\}) = m(\{x|f(x) > g_1(r)\} \bigcap \{x|f(x) > g_2(r)\}) \leq m(\{x|f(x) > g_1(r)\}) + m(\{x|f(x) > g_2(r)\})$ By property ordinary integrals we conclude $< f, g_1 + g_2 > \leq < f, g_1 > + < f, g_2 >$

Theorem 1 Let (X, D, m) be a monotone measure space ,

$$m(A \bigcup B) + m(A \bigcap B) \leq m(A) + m(B) \ \ (A \in D, B \in D)$$

,$X = [0, \infty] \alpha_1, \alpha_2, \beta_1, \beta_2 \in R^+$, $g_1, g_2 \in L$ then
$< \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2, \beta_1 \cdot g_1 + \beta_2 \cdot g_2 > \leq \alpha_1/\beta_1 \cdot < f_1, g_1 > + \alpha_1/\beta_2 \cdot < f_1, g_2 > + \alpha_2/\beta_1 \cdot < f_2, g_1 > + \alpha_2/\beta_2 \cdot < f_2, g_2 >$.

Proof. $< \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2, \beta_1 \cdot g_1 + \beta_2 \cdot g_2 > \leq \alpha_1 \cdot < f_1, \beta_1 \cdot g_1 + \beta_2 \cdot g_2 > + \alpha_2 \cdot < f_2, \beta_1 \cdot g_1 + \beta_2 \cdot g_2 > \leq \alpha_1 \cdot < f_1, \beta_1 \cdot g_1 > + \alpha_1 \cdot < f_1, \beta_2 \cdot g_2 > + \alpha_2 \cdot < f_2, \beta_1 \cdot g_1 > + \alpha_2 \cdot < f_2, \beta_2 \cdot g_2 > \leq \alpha_1/\beta_1 \cdot < f_1, g_1 > + \alpha_1/\beta_2 \cdot < f_1, g_2 > + \alpha_2/\beta_1 \cdot < f_2, g_1 > + \alpha_2/\beta_2 \cdot < f_2, g_2 >$

Example 1 Let $X = \{x_1, x_2\}$ and m a monotone set function on $P(X)$ given by $m(\{x_1\}) = 0, m(\{x_2\}) = 0$, and $m(X) = 1$.Let $f(x_1) = 0$, $f(x_2) = 1$, $p(x_1) = 2$, $p(x_2) = 1$. Then $f = p\, a.e.$ in the ordinary sense ,i.e.,$m(f \neq p) = m(x_1) = 0$. But for Chequest's integrals we have $C \int f\, dm = 0$ and $C \int p\, dm = 1$ When we have $g(x) = 1 + x$ then $(\overline{C_{1+x}}) \int_0^1 f\, dm = 0$, $(\overline{C_{1+x}}) \int_0^1 p\, dm = 0$.When we have $g(x) = 1 - x$ then

$$(\overline{C_{1-x}}) \int_0^1 f\, dm = 0,\ (\overline{C_{1-x}}) \int_0^1 p\, dm = 1.$$

References

[1] — Atanassov. K., Intuitionistic Fuzzy Sets,Physica Verlag,1999.

[2] — Pap. Endre., Null-Additive Set Functions,Kluwer,1995.