Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy interpretation of a classical propositional logic formula

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Abstract: The formula

$$\neg A = (A \to ((A \to A) \wedge \neg(A \to A)))$$

is a tautology in the classical propositional logic. In this paper, we determine all intuitionistic fuzzy implications that satisfy this formula together with the classical intuitionistic fuzzy negation or with the negation generated by this implication.

Keywords: Intuitionistic fuzzy implication, Intuitionistic fuzzy negation, Tautology.
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1 Introduction

After the first—classical—and the numerous non-classical intuitionistic fuzzy implications, introduced firstly in [6] (see [7]), more than 200 different intuitionistic fuzzy implications were introduced. Each of these implications generates an intuitionistic fuzzy negation, and by the moment there have been more than 50 unique, different intuitionistic fuzzy negations.

In the series of papers [1–5, 9–13, 15] and book [7], the properties of these implications and negations have been studied by the authors and other researchers.

In [14], the following formula is discussed:

$$\neg A = (A \to ((A \to A) \wedge \neg(A \to A))). \qquad (*)$$

Obviously, formula $(*)$ is a tautology in the classical propositional logic.

In the present research, we will check which pf the existing intuitionistic fuzzy implications and negations satisfy $(*)$ when the evaluation of the variable A is an intuitionistic fuzzy pair, i.e., this evaluation has the form af an intuitionistic fuzzy pair (see, [7, 8]):

$$V(A) = \langle a, b \rangle,$$

where $a, b \in [0, 1]$ and $a + b \leq 1$.

We will use two different pairs of operations "implication" and "negation". In the first case, the implications will be different, while the negation will be only the classical intuitionistic fuzzy negation, that is defined by

$$\neg\langle a, b\rangle = \neg_1\langle a, b\rangle = \langle b, a\rangle.$$

In the second case, the implications will be different and the negations will be generated for the respective implication by the formula

$$\neg_{\varphi(i)}\langle a, b\rangle = \langle a, b\rangle \to_i \langle 0, 1\rangle,$$

where $\varphi(i)$ is the number of the negation generated by the i-th intuitionistic fuzzy implication.

The pair of an intuitionistic fuzzy implication and the thereby generated specific negation on its own turn generates an intuitionistic fuzzy conjunction, as follows:

$$\begin{aligned}
\langle a, b\rangle \wedge_{i,1} \langle c, d\rangle &= \neg(\langle a, b\rangle \to_i \neg\langle c, d\rangle),\\
\langle a, b\rangle \wedge_{i,2} \langle c, d\rangle &= \neg_{\varphi(i)}(\langle a, b\rangle \to_i \neg_{\varphi(i)}\langle c, d\rangle),\\
\langle a, b\rangle \wedge_{i,3} \langle c, d\rangle &= \neg_{\varphi(i)}(\neg_{\varphi(i)}\neg_{\varphi(i)}\langle a, b\rangle \to_i \neg_{\varphi(i)}\langle c, d\rangle).
\end{aligned}$$

The third cases are different from the second cases when the intuitionistic fuzzy negation does not satisfy the De Morgan's laws.

In the Appendix 1 and Appendix 2, all intuitionistic fuzzy implications and negations, listed in the Theorems, are respectively given. In Appendix 3, the relationships between these negations and implications are given.

2 Main results

Below, we will formulate and prove three assertions.

Theorem 1. *For $i = 1, 4, 7, 10, 19, 61, 63, 67, 68, 69, 70, 73, 166, 186, 192$, formula $(*)$ is valid for the intuitionistic fuzzy implication $\rightarrow_i$, the classical negation $\neg_1$ and the conjunction $\wedge_{i,1}$.*

Proof. Below, we will prove the validity of the assertion, when $i = 1$. This first implication is an intuitionistic fuzzy modification of Zadeh's implication and it has the form:

$$\langle a, b\rangle \rightarrow_1 \langle c, d\rangle = \langle \max(b, \min(a, c)), \min(a, d)\rangle.$$

This intuitionistic fuzzy implication generates the classical intuitionistic fuzzy negation $\neg_1$ or (as usual, for brevity) $\neg$. From this fact it follows that the three forms of the disjunction and conjunction coincide. They have the forms:

$$\langle a, b\rangle \vee_1 \langle c, d\rangle = \langle \max(a, \min(b, c)), \min(b, d)\rangle,$$

$$\langle a, b\rangle \wedge_1 \langle c, d\rangle = \langle \min(a, c), \max(b, \min(a, d))\rangle.$$

Now, the left-hand side of $(*)$ has the form:

$$V(\neg A) = \langle b, a\rangle,$$

while the right side of $(*)$ has the form:

$$\begin{aligned}
&V(A \rightarrow_1 ((A \rightarrow_1 A) \wedge_{1,1} \neg_1(A \rightarrow_1 A)))\\
&= \langle a, b\rangle \rightarrow_1 (((\langle a, b\rangle \rightarrow_1 \langle a, b\rangle) \wedge_{1,1} \neg_1(\langle a, b\rangle \rightarrow_1 \langle a, b\rangle))\\
&= \langle a, b\rangle \rightarrow_1 (\langle \max(b, \min(a, a)), \min(a, b)\rangle \wedge_{1,1} \neg_1\langle \max(b, \min(a, a)), \min(a, b)\rangle)\\
&= \langle a, b\rangle \rightarrow_1 (\langle \max(a, b), \min(a, b)\rangle \wedge_{1,1} \neg_1\langle \max(a, b), \min(a, b)\rangle)\\
&= \langle a, b\rangle \rightarrow_1 (\langle \max(a, b), \min(a, b)\rangle \wedge_{1,1} \langle \min(a, b), \max(a, b)\rangle)\\
&= \langle a, b\rangle \rightarrow_1 \langle \min(\max(a, b), \min(a, b)), \max(\min(a, b), \min(\max(a, b), \max(a, b)))\rangle\\
&= \langle a, b\rangle \rightarrow_1 \langle \min(a, b), \max(\min(a, b), \max(a, b))\rangle\\
&= \langle a, b\rangle \rightarrow_1 (\langle \min(a, b), \max(a, b)\rangle\\
&= \langle \max(b, \min(a, \min(a, b))), \min(a, \max(a, b))\rangle\\
&= \langle \max(b, \min(a, b)), a\rangle\\
&= \langle b, a\rangle,
\end{aligned}$$

i.e., the left-hand side and the right-hand side of $(*)$ coincide and the equality is valid.
The checks of the remaining equalities are similar. □

We can mention that formula $(*)$ will also be valid, if we use implication $\rightarrow_1$, negation $\neg$ and the most popular form of the conjunction:

$$\langle a, b\rangle \wedge \langle c, d\rangle = \langle \min(a, c), \max(b, d)\rangle.$$

In this case, the check is the following:

$$
\begin{aligned}
& V(A \to_1 ((A \to_1 A) \wedge \neg(A \to_1 A))) \\
& = \langle a, b\rangle \to_1 (((\langle a, b\rangle \to_1 \langle a, b\rangle) \wedge \neg(\langle a, b\rangle \to_1 \langle a, b\rangle)) \\
& = \langle a, b\rangle \to_1 (\langle \max(b, \min(a, a)), \min(a, b)\rangle \wedge \neg\langle \max(b, \min(a, a)), \min(a, b)\rangle) \\
& = \langle a, b\rangle \to_1 (\langle \max(a, b), \min(a, b)\rangle \wedge \neg\langle \max(a, b), \min(a, b)\rangle) \\
& = \langle a, b\rangle \to_1 (\langle \max(a, b), \min(a, b)\rangle \wedge \langle \min(a, b), \max(a, b)\rangle) \\
& = \langle a, b\rangle \to_1 \langle \min(\max(a, b), \min(a, b)), \max(\min(a, b), \max(a, b))\rangle \\
& = \langle a, b\rangle \to_1 \langle \min(a, b), \max(a, b)\rangle \\
& = \langle \max(b, \min(a, \min(a, b))), \min(a, \max(a, b))\rangle \\
& = \langle \max(b, \min(a, b))), a\rangle \\
& = \langle b, a\rangle.
\end{aligned}
$$

Let us define for each $x \in [0, 1]$:

$$
\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},
$$

and

$$
\overline{\mathrm{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}.
$$

Theorem 2. *For* $i = 1, 2, 3, 4, 7, 8, 10, 11, 14, 15, 16, 18, 19, 20, 22, 23, 24, 26, 27, 30, 31, 32, 33, 34, 37, 39, 40, 41, 42, 43, 44, 45, 47, 48, 49, 52, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 67, 68, 69, 70, 73, 74, 76, 77, 79, 81, 83, 84, 87, 88, 89, 90, 92, 93, 96, 97, 99, 100, 101, 102, 104, 105, 107, 108, 109, 114, 166, 167, 168, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 194, 195, 198, 199, 200, 201, 202, 203, 205, 206,$ *formula* $(*)$ *is valid for the intuitionistic fuzzy implication* $\to_i$*, the negation* $\neg_{\varphi(i)}$ *and the conjunction* $\wedge_{i,2}$.

Proof. Below, we will prove the validity of the assertion, when $i = 2$. This second implication is an intuitionistic fuzzy modification of Gaines–Rescher's implication and it has the form:

$$
\langle a, b\rangle \to_2 \langle c, d\rangle = \langle \overline{\mathrm{sg}}(a - c), d\mathrm{sg}(a - c)\rangle.
$$

It generates the intuitionistic fuzzy negation $\neg_2$ that has the form:

$$
\neg_2\langle a, b\rangle = \langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle.
$$

Then, having in mind that for each $x \in [0, 1]$ we can directly check that:

$$
\mathrm{sg}(\overline{\mathrm{sg}}(x)) = \overline{\mathrm{sg}}(x)
$$

and

$$
\overline{\mathrm{sg}}(\overline{\mathrm{sg}}(x)) = \mathrm{sg}(x),
$$

for the three forms of the second conjunction, we obtain:

$$\begin{aligned}
\langle a,b\rangle \wedge_{2,1} \langle c,d\rangle &= \neg(\langle a,b\rangle \rightarrow_2 \neg\langle c,d\rangle)\\
&= \neg\langle \overline{\mathrm{sg}}(a-c), d\mathrm{sg}(a-c)\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a-c)), \mathrm{sg}(\overline{\mathrm{sg}}(a-c))\rangle\\
&= \langle \mathrm{sg}(a-c), d\overline{\mathrm{sg}}(a-c)\rangle
\end{aligned}$$

$$\begin{aligned}
\langle a,b\rangle \wedge_{2,2} \langle c,d\rangle &= \neg_2(\langle a,b\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\langle a,b\rangle \rightarrow_2 \langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)\\
&= \neg_2\langle \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c)), \mathrm{sg}(c)\mathrm{sg}(a-\overline{\mathrm{sg}}(c))\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))), \mathrm{sg}(\overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c)))\rangle\\
&= \langle \mathrm{sg}(a-\overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(a-\overline{\mathrm{sg}}(c))\rangle.
\end{aligned}$$

$$\begin{aligned}
\langle a,b\rangle \wedge_{2,3} \langle c,d\rangle &= \neg_2(\neg_2\neg_2\langle a,b\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\neg_2\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(a)), \mathrm{sg}(\overline{\mathrm{sg}}(a))\rangle \rightarrow_2 \neg_2\langle c,d\rangle)\\
&= \neg_2(\langle \mathrm{sg}(a), \overline{\mathrm{sg}}(a)\rangle \rightarrow_2 \neg_2\langle \overline{\mathrm{sg}}(c), \mathrm{sg}(c)\rangle)\\
&= \neg_2\langle \overline{\mathrm{sg}}(\mathrm{sg}(a)-\overline{\mathrm{sg}}(c)), \mathrm{sg}(c)\mathrm{sg}(\mathrm{sg}(a)-\overline{\mathrm{sg}}(c))\rangle\\
&= \langle \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(\mathrm{sg}(a)-\overline{\mathrm{sg}}(c))), \mathrm{sg}(\overline{\mathrm{sg}}(\mathrm{sg}(a)-\overline{\mathrm{sg}}(c)))\rangle\\
&= \langle \mathrm{sg}(\mathrm{sg}(a)-\overline{\mathrm{sg}}(c)), \overline{\mathrm{sg}}(\mathrm{sg}(a)-\overline{\mathrm{sg}}(c))\rangle.
\end{aligned}$$

In the present proof, we need only the form of conjunction $\wedge_{2,2}$, but we have shown the three forms that can be used in the detailed proofs of the three theorems. The same will be done in the proof of Theorem 3.

Now, the left-hand side of $(*)$ has the form:

$$V(\neg A) = \langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle,$$

while the right-hand side of $(*)$ has the form:

$$\begin{aligned}
&V(A \rightarrow ((A \rightarrow A) \wedge \neg(A \rightarrow A)))\\
&= \langle a,b\rangle \rightarrow (((\langle a,b\rangle \rightarrow \langle a,b\rangle) \wedge \neg(\langle a,b\rangle \rightarrow \langle a,b\rangle))\\
&= \langle a,b\rangle \rightarrow ((\langle \overline{\mathrm{sg}}(a-a), b\mathrm{sg}(a-a)\rangle \wedge \neg(\langle \overline{\mathrm{sg}}(a-a), b\mathrm{sg}(a-a)\rangle))\\
&= \langle a,b\rangle \rightarrow ((\langle \overline{\mathrm{sg}}(0), b\mathrm{sg}(0)\rangle \wedge \neg\langle \overline{\mathrm{sg}}(0), b\mathrm{sg}(0)\rangle)\\
&= \langle a,b\rangle \rightarrow ((\langle 1,0\rangle \wedge \neg\langle 1,0\rangle)\\
&= \langle a,b\rangle \rightarrow ((\langle 1,0\rangle \wedge \langle 0,1\rangle)\\
&= \langle a,b\rangle \rightarrow \langle \mathrm{sg}(1-\overline{\mathrm{sg}}(0)), \overline{\mathrm{sg}}(1-\overline{\mathrm{sg}}(0))\rangle\\
&= \langle a,b\rangle \rightarrow \langle 0,1\rangle\\
&= \langle \mathrm{sg}(a-0), \overline{\mathrm{sg}}(a-0)\rangle\\
&= \langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a)\rangle.
\end{aligned}$$

Therefore, both sides of $(*)$ coincide. The checks of the remaining equalities are proved in the same manner. $\square$

We must note that none of the intuitionistic fuzzy implications ($\rightarrow_9, \rightarrow_{17}, \rightarrow_{21}$) related to the intuitionistic fuzzy negation $\neg_3$ satisfy $(*)$. Really, they are defined by

$$\begin{aligned}
&\langle a, b\rangle \rightarrow_9 \langle c, d\rangle = \langle b + a^2c, ab + a^2d\rangle \\
&\langle a, b\rangle \rightarrow_{17} \langle c, d\rangle = \langle \max(b, c), \min(ab + a^2, d)\rangle \\
&\langle a, b\rangle \rightarrow_{21} \langle c, d\rangle = \langle \max(b, c(c + d)), \min(a(a + b), d(c^2 + d + cd))\rangle.
\end{aligned}$$

and the negation, generated by each of them, $\neg_3$ is

$$\neg_3\langle a, b\rangle = \langle b, ab + a^2\rangle$$

and

$$\langle a, b\rangle \wedge_{9,1} \langle c, d\rangle = \neg(\langle a, b\rangle \rightarrow_9 \neg\langle c, d\rangle) = \langle ab + a^2d, b + a^2c\rangle.$$

Now, for

$$V(A) = \left\langle \frac{1}{4}, 0 \right\rangle$$

we obtain

$$V(\neg A) = \left\langle 0, \frac{1}{4} \times 0 + \left(\frac{1}{4}\right)^2 \right\rangle = \left\langle 0, \frac{1}{16} \right\rangle,$$

while, for example for $\rightarrow_9$ we have

$$\begin{aligned}
&V(A \rightarrow_9 ((A \rightarrow_9 A) \wedge_{9,1} \neg_3(A \rightarrow_9 A))) \\
&= \langle a, b\rangle \rightarrow_9 ((\langle a, b\rangle \rightarrow_9 \langle a, b\rangle) \wedge_{9,1} \neg_3(\langle a, b\rangle \rightarrow_9 \langle a, b\rangle)) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \rightarrow_9 ((\left\langle \tfrac{1}{4}, 0\right\rangle \rightarrow_9 \left\langle \tfrac{1}{4}, 0\right\rangle) \wedge_{9,1} \neg_3(\left\langle \tfrac{1}{4}, 0\right\rangle \rightarrow_9 \left\langle \tfrac{1}{4}, 0\right\rangle)) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \rightarrow_9 (\left\langle \tfrac{1}{64}, 0\right\rangle \wedge_{9,1} \neg_3 \left\langle \tfrac{1}{64}, 0\right\rangle) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \rightarrow_9 (\left\langle \tfrac{1}{64}, 0\right\rangle \wedge_{9,1} \left\langle 0, \tfrac{1}{4096}\right\rangle) \\
&= \left\langle \tfrac{1}{4}, 0\right\rangle \rightarrow_9 \left\langle 0, \tfrac{1}{16777216}\right\rangle \\
&= \left\langle 0, \tfrac{1}{268435456}\right\rangle \neq \left\langle 0, \tfrac{1}{16}\right\rangle.
\end{aligned}$$

Theorem 3. *For* $i = 1, 2, 3, 4, 7, 8, 10, 11, 14, 15, 16, 18, 19, 20, 22, 23, 24, 26, 27, 30, 31, 32, 33, 34, 37, 39, 40, 41, 42, 43, 44, 45, 47, 48, 49, 52, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 67, 68, 69, 70, 73, 74, 76, 77, 79, 81, 83, 84, 87, 88, 89, 90, 92, 93, 96, 97, 99, 100, 101, 102, 104, 105, 107, 108, 109, 114, 117, 166, 167, 168, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 194, 195, 198, 199, 200, 201, 202, 203, 205, 206$*, formula* $(*)$ *is valid for the intuitionistic fuzzy implication* $\rightarrow_i$*, the negation* $\neg_{\varphi(i)}$ *and the conjunction* $\wedge_{i,3}$*.*

Proof. Below, we will prove the validity of the assertion, when $i = 18$. This implication has the form:

$$\langle a, b\rangle \rightarrow_{18} \langle c, d\rangle = \langle \max(b, c), \min(1 - b, d)\rangle.$$

It generates the intuitionistic fuzzy negation $\neg_4$ (because $\varphi(18) = 4$) that has the form:

$$\neg_4\langle a, b\rangle = \langle b, 1 - b\rangle.$$

In this case,

$$\begin{aligned}
\langle a,b\rangle \wedge_{18,1} \langle c,d\rangle &= \neg(\langle a,b\rangle \to_{18} \neg\langle c,d\rangle)\\
&= \neg(\langle a,b\rangle \to_{18} \langle d,1-d\rangle)\\
&= \neg\langle \max(b,d), \min(1-b,1-d)\rangle\\
&= \langle \min(1-b,1-d), 1-\min(1-b,1-d)\rangle\\
&= \langle 1-\max(b,d), \max(b,d)\rangle,
\end{aligned}$$

$$\begin{aligned}
\langle a,b\rangle \wedge_{18,2} \langle c,d\rangle &= \neg_4(\langle a,b\rangle \to_{18} \neg_4\langle c,d\rangle)\\
&= \neg_4(\langle a,b\rangle \to_{18} \langle d,1-d\rangle)\\
&= \neg_4\langle \max(b,d), \min(1-b,1-d)\rangle\\
&= \neg_4\langle \max(b,d), 1-\max(b,d)\rangle\\
&= \langle 1-\max(b,d), \max(b,d)\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\langle a,b\rangle \wedge_{18,3} \langle c,d\rangle &= \neg_4(\neg_4\neg_4\langle a,b\rangle \to_{18} \neg_4\langle c,d\rangle)\\
&= \neg_4(\neg_4\langle b,1-b\rangle \to_{18} \neg_4\langle c,d\rangle)\\
&= \neg_4(\langle 1-b,b\rangle \to_{18} \langle d,1-d\rangle)\\
&= \neg_4\langle \max(b,d), \min(1-b,1-d)\rangle\\
&= \neg_4\langle \max(b,d), 1-\max(b,d)\rangle\\
&= \langle 1-\max(b,d), \max(b,d)\rangle.
\end{aligned}$$

Now, the left-hand side of $(*)$ has the form:

$$V(\neg A) = \langle b, 1-b\rangle,$$

while the right-hand side of $(*)$ has the form:

$$\begin{aligned}
&(A \to_{18} ((A \to_{18} A) \wedge_{18,3} \neg_4(A \to_{18} A)))\\
&= \langle a,b\rangle \to_{18} ((\langle a,b\rangle \to_{18} \langle a,b\rangle) \wedge_{18,3} \neg_4(\langle a,b\rangle \to_{18} \langle a,b\rangle))\\
&= \langle a,b\rangle \to_{18} (\langle \max(a,b), \min(1-b,b)\rangle \wedge_{18,3} \neg_4(\langle \max(a,b), \min(1-b,b)\rangle))\\
&= \langle a,b\rangle \to_{18} (\langle \max(a,b), \min(1-b,b)\rangle \wedge_{18,3} \langle \min(1-b,b), 1-\min(1-b,b)\rangle)\\
&= \langle a,b\rangle \to_{18} (\langle \max(a,b), \min(1-b,b)\rangle \wedge_{18,3} \langle \min(1-b,b), \max(1-b,b)\rangle)\\
&= \langle a,b\rangle \to_{18} \langle 1-\max(\min(1-b,b), \max(1-b,b)), \max(1-\min(1-b,b), \max(1-b,b))\rangle\\
&= \langle a,b\rangle \to_{18} \langle 1-\max(1-b,b), \max(1-b,b)\rangle\\
&= \langle \max(b, \min(1-b,b)), \min(1-b, \max(1-b,b))\rangle\\
&= \langle b, 1-b\rangle.
\end{aligned}$$

Therefore, both sides of $(*)$ coincide.

The checks of the remaining equalities are proved in the same manner. □

3 Conclusion

The introduction of this logical formula establishes an additional criterion for evaluating the correctness of the implications we have defined. Among the set of good implications (see [**?**]), those numbered by $5, 9, 13, 17, 28, 71, 110, 112, 125$ do not satisfy the formula under discussion in at least one of the cases.

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Appendix 1: List of the intuitionistic fuzzy implications cited in the theorems

$\rightarrow_1$	$\langle \max(b, \min(a, c)), \min(a, d) \rangle$
$\rightarrow_2$	$\langle \overline{\text{sg}}(a - c), d\text{sg}(a - c) \rangle$
$\rightarrow_3$	$\langle 1 - (1 - c)\text{sg}(a - c)), d\text{sg}(a - c) \rangle$
$\rightarrow_4$	$\langle \max(b, c), \min(a, d) \rangle$
$\rightarrow_7$	$\langle \min(\max(b, c), \max(a, b), \max(c, d)),$ $\max(\min(a, d), \min(a, b), \min(c, d)) \rangle$
$\rightarrow_8$	$\langle 1 - (1 - \min(b, c))\text{sg}(a - c), \max(a, d)\text{sg}(a - c), \text{sg}(d - b) \rangle$
$\rightarrow_{10}$	$\langle c\overline{\text{sg}}(1 - a) + \text{sg}(1 - a)(\overline{\text{sg}}(1 - c) + b\text{sg}(1 - c)),$ $d\overline{\text{sg}}(1 - a) + a\text{sg}(1 - a)\text{sg}(1 - c) \rangle$
$\rightarrow_{11}$	$\langle 1 - (1 - c)\text{sg}(a - c), d\text{sg}(a - c)\text{sg}(d - b) \rangle$
$\rightarrow_{14}$	$\langle 1 - (1 - c)\text{sg}(a - c) - d\overline{\text{sg}}(a - c)\text{sg}(d - b), d\text{sg}(d - b) \rangle$
$\rightarrow_{15}$	$\langle 1 - (1 - \min(b, c))\text{sg}(\text{sg}(a - c) + \text{sg}(d - b))$ $- \min(b, c)\text{sg}(a - c)\text{sg}(d - b),\ 1 - (1 - \max(a, d))\text{sg}(\overline{\text{sg}}(a - c)$ $+ \overline{\text{sg}}(d - b)) - \max(a, d)\overline{\text{sg}}(a - c)\overline{\text{sg}}(d - b) \rangle$
$\rightarrow_{16}$	$\langle \max(\overline{\text{sg}}(a), c), \min(\text{sg}(a), d) \rangle$
$\rightarrow_{18}$	$\langle \max(b, c), \min(1 - b, d) \rangle$
$\rightarrow_{19}$	$\langle \max(1 - \text{sg}(\text{sg}(a) + \text{sg}(1 - b)), c), \min(\text{sg}(1 - b), d) \rangle$
$\rightarrow_{20}$	$\langle \max(\overline{\text{sg}}(a), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(c)) \rangle$
$\rightarrow_{22}$	$\langle \max(b, 1 - d), 1 - \max(b, 1 - d) \rangle$
$\rightarrow_{23}$	$\langle 1 - \min(\text{sg}(1 - b), \overline{\text{sg}}(1 - d)), \min(\text{sg}(1 - b), \overline{\text{sg}}(1 - d)) \rangle$
$\rightarrow_{24}$	$\langle \overline{\text{sg}}(a - c)\overline{\text{sg}}(d - b), \text{sg}(a - c)\text{sg}(d - b) \rangle$

$\rightarrow_{26}$	$\langle \max(\overline{\text{sg}}(1-b), c), \min(\text{sg}(a), d)\rangle$
$\rightarrow_{27}$	$\langle \max(\overline{\text{sg}}(1-b), \text{sg}(c)), \min(\text{sg}(a), \overline{\text{sg}}(1-d))\rangle$
$\rightarrow_{30}$	$\langle \max(1-a, \min(a, 1-d)), \min(a, d)\rangle$
$\rightarrow_{31}$	$\langle \overline{\text{sg}}(a+d-1), d\text{sg}(a+d-1)\rangle$
$\rightarrow_{32}$	$\langle 1-d\text{sg}(a+d-1), d\text{sg}(a+d-1)\rangle$
$\rightarrow_{33}$	$\langle 1-\min(a, d), \min(a, d)\rangle$
$\rightarrow_{34}$	$\langle \min(1, 2-a-d), \max(0, a+d-1)\rangle$
$\rightarrow_{37}$	$\langle 1-\max(a, d)\text{sg}(a+d-1), \max(a, d)\text{sg}(a+d-1)\rangle$
$\rightarrow_{39}$	$\langle (1-d)\overline{\text{sg}}(1-a) + \text{sg}(1-a)(\overline{\text{sg}}(d) + (1-a)\text{sg}(d)),$ $d\overline{\text{sg}}(1-a) + a\text{sg}(1-a)\text{sg}(d)\rangle$
$\rightarrow_{40}$	$\langle 1-\text{sg}(a+d-1), 1-\overline{\text{sg}}(a+d-1)\rangle$
$\rightarrow_{41}$	$\langle \max(\overline{\text{sg}}(a), 1-d), \min(\text{sg}(a), d)\rangle$
$\rightarrow_{42}$	$\langle \max(\overline{\text{sg}}(a), \text{sg}(1-d)), \min(\text{sg}(a), \overline{\text{sg}}(1-d))\rangle$
$\rightarrow_{43}$	$\langle \max(\overline{\text{sg}}(a), 1-d), \min(\text{sg}(a), d)\rangle$
$\rightarrow_{44}$	$\langle \max(\overline{\text{sg}}(a), 1-d), \min(a, d)\rangle$
$\rightarrow_{45}$	$\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(a, \overline{\text{sg}}(1-d))\rangle$
$\rightarrow_{47}$	$\langle \overline{\text{sg}}(1-b-c), (1-c)\text{sg}(1-b-c)\rangle$
$\rightarrow_{48}$	$\langle 1-(1-c)\text{sg}(1-b-c), (1-c)\text{sg}(1-b-c)\rangle$
$\rightarrow_{49}$	$\langle \min(1, b+c), \max(0, 1-b-c)\rangle$
$\rightarrow_{52}$	$\langle 1-(1-\min(b, c))\text{sg}(1-b-c), 1-\min(b, c)\text{sg}(1-b-c)\rangle$
$\rightarrow_{55}$	$\langle 1-\text{sg}(1-b-c), 1-\overline{\text{sg}}(1-b-c)\rangle$
$\rightarrow_{56}$	$\langle \max(\overline{\text{sg}}(1-b), c), \min(\text{sg}(1-b), 1-c)\rangle$
$\rightarrow_{57}$	$\langle \max(\overline{\text{sg}}(1-b), \text{sg}(c)), \min(\text{sg}(1-b), \overline{\text{sg}}(c))\rangle$
$\rightarrow_{58}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), 1-\max(b, c)\rangle$
$\rightarrow_{59}$	$\langle \max(\overline{\text{sg}}(1-b), c), 1-\max(b, c)\rangle$
$\rightarrow_{60}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(1-b, \overline{\text{sg}}(c))\rangle$
$\rightarrow_{61}$	$\langle \max(c, \min(b, d)), \min(a, d)\rangle$
$\rightarrow_{62}$	$\langle \overline{\text{sg}}(d-b), a\text{sg}(d-b)\rangle$
$\rightarrow_{63}$	$\langle 1-(1-b)\text{sg}(d-b), a\text{sg}(d-b)\rangle$
$\rightarrow_{65}$	$\langle 1-(1-\min(b, c))\text{sg}(d-b), \max(a, d)\text{sg}(d-b)\text{sg}(a-c)\rangle$
$\rightarrow_{67}$	$\langle b\overline{\text{sg}}(1-d) + \text{sg}(1-d)(\overline{\text{sg}}(1-b) + c\text{sg}(1-b)),$ $a\overline{\text{sg}}(1-d) + d\text{sg}(1-d)\text{sg}(1-b)\rangle$
$\rightarrow_{68}$	$\langle 1-(1-b)\text{sg}(d-b), a\text{sg}(d-b)\text{sg}(a-c)\rangle$
$\rightarrow_{69}$	$\langle 1-(1-b)\text{sg}(d-b) - a\overline{\text{sg}}(d-b)\text{sg}(a-c), a\text{sg}(a-c)\rangle$
$\rightarrow_{70}$	$\langle \max(\overline{\text{sg}}(d), b), \min(\text{sg}(d), a)\rangle$
$\rightarrow_{73}$	$\langle \max(1-\max(\text{sg}(d), \text{sg}(1-c)), b), \min(\text{sg}(1-c), a)\rangle$
$\rightarrow_{74}$	$\langle \max(\text{sg}(b), \overline{\text{sg}}(d)), \min(\overline{\text{sg}}(b), \text{sg}(d))\rangle$
$\rightarrow_{76}$	$\langle \max(c, 1-a), \min(1-c, a)\rangle$
$\rightarrow_{77}$	$\langle (1-\min(\overline{\text{sg}}(1-a), \text{sg}(1-c))), \min(\overline{\text{sg}}(1-a), \text{sg}(1-c))\rangle$
$\rightarrow_{79}$	$\langle \max(\overline{\text{sg}}(1-c), \text{sg}(b)), \min(\text{sg}(d), \overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{81}$	$\langle \max(\overline{\text{sg}}(1-b), \overline{\text{sg}}(1-c)), \min(d, \overline{\text{sg}}(1-a))\rangle$

$\rightarrow_{83}$	$\langle \overline{\text{sg}}(a+d-1), a\text{sg}(a+d-1)\rangle$
$\rightarrow_{84}$	$\langle 1-a\text{sg}(a+d-1), a\text{sg}(a+d-1)\rangle$
$\rightarrow_{87}$	$\langle \max(\overline{\text{sg}}(d), 1-a), \min(\text{sg}(d), a)\rangle$
$\rightarrow_{88}$	$\langle \max(\overline{\text{sg}}(d), \text{sg}(1-a)), \min(\text{sg}(d), \overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{89}$	$\langle \max(\overline{\text{sg}}(d), 1-a), \min(d, a)\rangle$
$\rightarrow_{90}$	$\langle \max(\overline{\text{sg}}(a), \overline{\text{sg}}(d)), \min(d, \overline{\text{sg}}(1-a))\rangle$
$\rightarrow_{92}$	$\langle \overline{\text{sg}}(1-b-c), \min(1-b, \text{sg}(1-b-c))\rangle$
$\rightarrow_{93}$	$\langle (1-\min(1-b, \text{sg}(1-b-c)), \min(1-b, \text{sg}(1-b-c))\rangle$
$\rightarrow_{96}$	$\langle \max(\overline{\text{sg}}(1-c), b), \min(\text{sg}(1-b), 1-c)\rangle$
$\rightarrow_{97}$	$\langle \max(\overline{\text{sg}}(1-c), \text{sg}(b)), \min(\text{sg}(1-c), \overline{\text{sg}}(b))\rangle$
$\rightarrow_{99}$	$\langle \max(\overline{\text{sg}}(1-c), \overline{\text{sg}}(1-b)), \min(1-c, \overline{\text{sg}}(b))\rangle$
$\rightarrow_{100}$	$\langle \max(b\text{sg}(a), c), \min(a\text{sg}(b), d)\rangle$
$\rightarrow_{101}$	$\langle \max(b\text{sg}(a), c\text{sg}(d)), \min(a\text{sg}(b), \text{sg}(c)d)\rangle$
$\rightarrow_{102}$	$\langle \max(b, c\text{sg}(d)), \min(a, \text{sg}(c)d)\rangle$
$\rightarrow_{104}$	$\langle \max(\min(1-a, \text{sg}(a)), \min(1-d, \text{sg}(d))),$ $\min(a, \text{sg}(1-a), d, \text{sg}(1-d))\rangle$
$\rightarrow_{105}$	$\langle \max(1-a, \min(1-d, \text{sg}(d))), \min(a, d, \text{sg}(1-d))\rangle$
$\rightarrow_{107}$	$\langle \max(\min(b, \text{sg}(1-b)), \min(c, \text{sg}(1-c))),$ $\min(1-b, \text{sg}(b), 1-c, \text{sg}(c))\rangle$
$\rightarrow_{108}$	$\langle \max(b, \min(c, \text{sg}(1-c))), \min(1-b, 1-c, \text{sg}(c))\rangle$
$\rightarrow_{109}$	$\langle b+\min(\overline{\text{sg}}(1-a), c), ab+\min(\overline{\text{sg}}(1-a), d))\rangle$
$\rightarrow_{114}$	$\langle 1-a+\min(\overline{\text{sg}}(1-a), 1-d), a(1-a)+\min(\overline{\text{sg}}(1-a), d)\rangle$
$\rightarrow_{117}$	$\langle 1-a-d+ad, (a(1-a)+\overline{\text{sg}}(1-a))d\rangle$
$\rightarrow_{166}$	$\langle \max(b, \min(a, c)), \min(a, \max(b, d))\rangle$
$\rightarrow_{167}$	$\langle \max(1-a, \min(a, c)), \min(a, 1-\min(a, c))\rangle$
$\rightarrow_{168}$	$\langle \max(1-a, \min(a, 1-d)), 1-\max(1-a, \min(a, 1-d))\rangle$
$\rightarrow_{170}$	$\langle \max(b, \min(1-b, 1-d)), 1-\max(b, \min(1-b, 1-d))\rangle$
$\rightarrow_{171}$	$\langle \overline{\text{sg}}(\max(a, d)-\max(b, c)), \text{sg}(\max(a, d)-\max(b, c))\rangle$
$\rightarrow_{172}$	$\langle \overline{\text{sg}}(a-c), \text{sg}(a-c)\rangle$
$\rightarrow_{173}$	$\langle \overline{\text{sg}}(a+d-1), \text{sg}(a+d-1)\rangle$
$\rightarrow_{174}$	$\langle \overline{\text{sg}}(1-b-c), \text{sg}(1-b-c)\rangle$
$\rightarrow_{175}$	$\langle \overline{\text{sg}}(d-b), \text{sg}(d-b)\rangle$
$\rightarrow_{176}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\max(b, c), \text{sg}(a-c)\min(a, d)\rangle$
$\rightarrow_{177}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\max(1-a, c), \text{sg}(a-c)\min(a, 1-c)\rangle$
$\rightarrow_{178}$	$\langle \overline{\text{sg}}(a-1+d)+\text{sg}(a-1+d)(1-\min(a, d)),$ $\text{sg}(a-1+d)\min(a, d)\rangle$
$\rightarrow_{179}$	$\langle \overline{\text{sg}}(1-b-c)+\text{sg}(1-b-c)\max(b, c),$ $\text{sg}(1-b-c)(1-\max(b, c))\rangle$
$\rightarrow_{180}$	$\langle \overline{\text{sg}}(d-b)+\text{sg}(d-b)\max(b, 1-d), \text{sg}(d-b)\min(1-b, d)\rangle$
$\rightarrow_{181}$	$\langle 1-\text{sg}(a).(1-c), d.\text{sg}(a)\rangle$
$\rightarrow_{182}$	$\langle 1-\text{sg}(a).(1-c), (1-c).\text{sg}(a)\rangle$

$\rightarrow_{183}$	$\langle 1-\text{sg}(a).d, d.\text{sg}(a)\rangle$
$\rightarrow_{184}$	$\langle 1-\text{sg}(1-b).d, d.\text{sg}(1-b)\rangle$
$\rightarrow_{185}$	$\langle 1-\text{sg}(1-b).(1-c), (1-c).\text{sg}(1-b)\rangle$
$\rightarrow_{186}$	$\langle \overline{\text{sg}}(d-b)+\text{sg}(d-b)\max(b,c), \text{sg}(d-b)\min(a,d)\rangle$
$\rightarrow_{187}$	$\langle \overline{\text{sg}}(d-b)+\text{sg}(d-b)\max(1-d,b), \text{sg}(d-b)\min(d,1-b)\rangle$
$\rightarrow_{188}$	$\langle \overline{\text{sg}}(a-1+d)+\text{sg}(a-1+d)(1-\min(a,d)), \text{sg}(a-1+d)\min(a,d)\rangle$
$\rightarrow_{189}$	$\langle \overline{\text{sg}}(1-b-c)+\text{sg}(1-b-c)\max(b,c), \text{sg}(1-b-c)(1-\max(b,c))\rangle$
$\rightarrow_{190}$	$\langle \overline{\text{sg}}(d-b)+\text{sg}(d-b)\max(b,1-d), \text{sg}(d-b)\min(1-b,d)\rangle$
$\rightarrow_{191}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\frac{c}{1-b}, \text{sg}(a-c)\frac{a-c}{1-b}\rangle$
$\rightarrow_{192}$	$\langle \max(c,\min(b,d)), \min(d,\max(a,c))\rangle$
$\rightarrow_{193}$	$\langle \max(c,\min(b,d)), 1-\max(c,\min(b,d))\rangle$
$\rightarrow_{194}$	$\langle 1-\min(d,\max(a,c)), \min(d,\max(a,c))\rangle$
$\rightarrow_{195}$	$\langle \max(c,1-\max(a,c)), 1-\max(c,1-\max(a,c))\rangle$
$\rightarrow_{196}$	$\langle 1-\min(d,\max(a,1-d)), \min(d,\max(a,1-d))\rangle$
$\rightarrow_{197}$	$\langle \max(c,\min(b,1-c)), 1-\max(c,\min(b,1-c))\rangle$
$\rightarrow_{198}$	$\langle \max(1-d,\min(b,d)), 1-\max(1-d,\min(b,d))\rangle$
$\rightarrow_{199}$	$\langle \overline{\text{sg}}(a-c)+\text{sg}(a-c)\frac{c}{a}, \text{sg}(a-c)\frac{a-c}{a}\rangle$
$\rightarrow_{200}$	$\langle \overline{\text{sg}}(a+d-1)+\text{sg}(a+d-1)\frac{1-d}{a}, \text{sg}(a+d-1)\frac{a+d-1}{a}\rangle$
$\rightarrow_{201}$	$\langle \overline{\text{sg}}(d-b)+\text{sg}(d-b)\frac{1-d}{1-b}, \text{sg}(d-b)\frac{d-b}{1-b}\rangle$
$\rightarrow_{202}$	$\langle \overline{\text{sg}}(1-b-c)+\text{sg}(1-b-c)\frac{c}{1-b}, \text{sg}(1-b-c)\frac{1-b-c}{1-b}\rangle$
$\rightarrow_{203}$	$\langle \overline{\text{sg}}(a-c)+sg(a-c)\frac{c}{1-b},\ sg(a-c)\frac{a-c}{1-b}\rangle$
$\rightarrow_{204}$	$\langle \overline{\text{sg}}(a-c)+sg(a-c)\frac{c+1}{2-b},\ sg(a-c)\frac{a-c}{2-b}\rangle$
$\rightarrow_{205}$	$\langle \frac{c}{2-b}, \frac{a+1-c}{2-b}\rangle$
$\rightarrow_{206}$	$\langle \max(\text{sg}(1-a),c), 0\rangle$

Appendix 2: List of the intuitionistic fuzzy negations, generated by the implications cited in the theorems

$\neg_1$	$\langle b,a\rangle$
$\neg_2$	$\langle \overline{\text{sg}}(a), \text{sg}(a)\rangle$
$\neg_4$	$\langle b, 1-b\rangle$
$\neg_5$	$\langle \overline{\text{sg}}(1-b), \text{sg}(1-b)\rangle$
$\neg_6$	$\langle \overline{\text{sg}}(1-b), \text{sg}(a)\rangle$
$\neg_7$	$\langle \overline{\text{sg}}(1-b), a\rangle$
$\neg_8$	$\langle 1-a, a\rangle$
$\neg_9$	$\langle \overline{\text{sg}}(a), a\rangle$
$\neg_{10}$	$\langle \overline{\text{sg}}(1-b), 1-b\rangle$
$\neg_{11}$	$\langle \text{sg}(b), \overline{\text{sg}}(b)\rangle$
$\neg_{13}$	$\langle \text{sg}(1-a), \overline{\text{sg}}(1-a)\rangle$
$\neg_{14}$	$\langle \text{sg}(b), \overline{\text{sg}}(1-a)\rangle$

$\neg_{15}$	$\langle \overline{\text{sg}}(1-b), \overline{\text{sg}}(1-a)\rangle$
$\neg_{16}$	$\langle \overline{\text{sg}}(a), \overline{\text{sg}}(1-a)\rangle$
$\neg_{17}$	$\langle \overline{\text{sg}}(1-b), \overline{\text{sg}}(b)\rangle$
$\neg_{18}$	$\langle b.\text{sg}(a), a.\text{sg}(b)\rangle$
$\neg_{19}$	$\langle b.\text{sg}(a), 0\rangle$
$\neg_{20}$	$\langle b, 0\rangle$
$\neg_{22}$	$\langle \min(1-a, \text{sg}(a)), 0\rangle$
$\neg_{23}$	$\langle 1-a, 0\rangle$
$\neg_{25}$	$\langle \min(b, \text{sg}(1-b)), 0\rangle$
$\neg_{26}$	$\langle b, a.b + \overline{\text{sg}}(1-a)\rangle$
$\neg_{27}$	$\langle 1-a, a.(1-a) + \overline{\text{sg}}(1-a)\rangle$
$\neg_{52}$	$\langle b, a\text{sg}(1-b)\rangle$
$\neg_{53}$	$\langle \overline{\text{sg}}(a) + \text{sg}(a)b, a\rangle$
$\neg_{54}$	$\langle \overline{\text{sg}}(a), \text{sg}(a)\frac{a}{1-b}\rangle$
$\neg_{55}$	$\langle \overline{\text{sg}}(a) + \text{sg}(a)\frac{1}{2-b}, \text{sg}(a)\frac{a}{P}2-b\rangle$
$\neg_{56}$	$\langle 0, \frac{a+1}{2-b}\rangle$
$\neg_{57}$	$\langle \text{sg}(1-a), 0\rangle$

Appendix 3: Relationships between intuitionistic fuzzy negations and implications

$\neg_1$	$\rightarrow_1, \rightarrow_4, \rightarrow_5, \rightarrow_6, \rightarrow_7, \rightarrow_{10}, \rightarrow_{13}, \rightarrow_{61}, \rightarrow_{63}, \rightarrow_{64}, \rightarrow_{66}, \rightarrow_{67}, \rightarrow_{68}, \rightarrow_{69}, \rightarrow_{70}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{73}, \rightarrow_{78}, \rightarrow_{80}, \rightarrow_{124}, \rightarrow_{125}, \rightarrow_{127}, \rightarrow_{166}, \rightarrow_{192}$
$\neg_2$	$\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}, \rightarrow_{16}, \rightarrow_{20}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{37}, \rightarrow_{40}, \rightarrow_{41}, \rightarrow_{42}$ $\rightarrow_{172}, \rightarrow_{173}, \rightarrow_{181}, \rightarrow_{182}, \rightarrow_{183}, \rightarrow_{199}, \rightarrow_{200}$
$\neg_4$	$\rightarrow_{12}, \rightarrow_{18}, \rightarrow_{22}, \rightarrow_{46}, \rightarrow_{49}, \rightarrow_{50}, \rightarrow_{51}, \rightarrow_{53}, \rightarrow_{54}, \rightarrow_{91}, \rightarrow_{93}, \rightarrow_{94}, \rightarrow_{95}, \rightarrow_{96}, \rightarrow_{98}, \rightarrow_{134}, \rightarrow_{135}, \rightarrow_{137}, \rightarrow_{169}, \rightarrow_{170}, \rightarrow_{179}, \rightarrow_{180}, \rightarrow_{187}, \rightarrow_{189}, \rightarrow_{190}, \rightarrow_{193}, \rightarrow_{197}, \rightarrow_{198}$
$\neg_5$	$\rightarrow_{14}, \rightarrow_{15}, \rightarrow_{19}, \rightarrow_{23}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \rightarrow_{56}, \rightarrow_{57}, \rightarrow_{171}, \rightarrow_{174}, \rightarrow_{175}, \rightarrow_{184}, \rightarrow_{185}, \rightarrow_{201}, \rightarrow_{202}$
$\neg_6$	$\rightarrow_{24}, \rightarrow_{26}, \rightarrow_{27}, \rightarrow_{65}$
$\neg_7$	$\rightarrow_{25}, \rightarrow_{28}, \rightarrow_{29}, \rightarrow_{62}$
$\neg_8$	$\rightarrow_{30}, \rightarrow_{33}, \rightarrow_{34}, \rightarrow_{35}, \rightarrow_{36}, \rightarrow_{38}, \rightarrow_{39}, \rightarrow_{76}, \rightarrow_{82}, \rightarrow_{84}, \rightarrow_{85}, \rightarrow_{86}, \rightarrow_{87}, \rightarrow_{89}, \rightarrow_{129}, \rightarrow_{130}, \rightarrow_{132}, \rightarrow_{167}, \rightarrow_{168}, \rightarrow_{177}, \rightarrow_{178}, \rightarrow_{188}, \rightarrow_{194}, \rightarrow_{195}, \rightarrow_{196}$
$\neg_9$	$\rightarrow_{43}, \rightarrow_{44}, \rightarrow_{45}, \rightarrow_{83}$
$\neg_{10}$	$\rightarrow_{58}, \rightarrow_{59}, \rightarrow_{60}, \rightarrow_{92}$
$\neg_{11}$	$\rightarrow_{74}, \rightarrow_{97}$
$\neg_{13}$	$\rightarrow_{77}, \rightarrow_{88}$
$\neg_{14}$	$\rightarrow_{79}$

$\neg_{15}$	$\rightarrow_{81}$
$\neg_{16}$	$\rightarrow_{90}$
$\neg_{17}$	$\rightarrow_{99}$
$\neg_{18}$	$\rightarrow_{100}$
$\neg_{19}$	$\rightarrow_{101}$
$\neg_{20}$	$\rightarrow_{102}$, $\rightarrow_{108}$
$\neg_{22}$	$\rightarrow_{104}$
$\neg_{23}$	$\rightarrow_{105}$
$\neg_{25}$	$\rightarrow_{107}$
$\neg_{26}$	$\rightarrow_{109}$, $\rightarrow_{110}$, $\rightarrow_{111}$, $\rightarrow_{112}$, $\rightarrow_{113}$
$\neg_{27}$	$\rightarrow_{114}$, $\rightarrow_{115}$, $\rightarrow_{116}$, $\rightarrow_{117}$, $\rightarrow_{118}$
$\neg_{52}$	$\rightarrow_{186}$
$\neg_{53}$	$\rightarrow_{176}$
$\neg_{54}$	$\rightarrow_{191}$, $\rightarrow_{203}$
$\neg_{55}$	$\rightarrow_{204}$
$\neg_{56}$	$\rightarrow_{205}$
$\neg_{57}$	$\rightarrow_{206}$