Notes on Intuitionistic Fuzzy Sets
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t-Lower level set and t-upper level set of an intuitionistic fuzzy set

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Abstract: In this paper we give the definitions of t-lower level set $(L_t(A))$ and t-upper level set $(U_t(A))$ of an Intuitionistic fuzzy set. [1] A t-lower level set is defined by giving a lower boundary on $\mu_A(x) + \nu_A(x)$. A t-upper level set is defined by giving an upper boundary on $\mu_A(x) + \nu_A(x)$. If A is an Intuitionistic fuzzy set of X, then $L_t(A)$ and $U_t(A)$ are subsets of X. In this paper we give some theorems by using t-lower level sets and t-upper level sets and prove them.

Keywords: Intuitionistic fuzzy sets, t-lower level sets, t-upper level sets, Cut of intuitionistic fuzzy set.
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1 Introduction

L. A. Zadeh [7] gave the definition of fuzzy sets in 1965. With this definition fuzzy logic bases was formed. A lot of researchers study onto this area. A. Rosenfeld [4] gave the definition of fuzzy group in 1971 and fuzzy set definition was moved to algebra. In 1983, K. T. Atanassov [1] gave $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ as the definition of intuitionistic fuzzy set thus impoving the fuzzy set definition. There $\mu_A(x)$ is the degree of membership and $\nu_A(x)$ is the degree of non-membership where $1 \geq \mu_A(x) + \nu_A(x) \geq 0$. Many researchers make studies with this definition. Some of them are G. Çuvalcıoğlu [2], F. Tuğrul, M. Çitil [6] and P. K.

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Sharma [5]. P. K. Sharma gave in that paper a classic set $C_{\alpha,\beta}(A) = \{x : x \in X$ such that $\mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}, C(\alpha, \beta)$-cut of intuitionistic fuzzy set where $\alpha, \beta \in [0, 1]$ with $\alpha+\beta \leq 1$ by using intuitionistic fuzzy set definition and gave some properties of $C_{\alpha,\beta}(A)$.

In this study by using intuitionistic fuzzy set definition is given classic sets $U_t(A)$ t-upper Level Set of Intuitionistic Fuzzy Set and $L_t(A)$ t-lower Level Set of Intuitionistic Fuzzy Set definition. In this study are shown some properties of $U_t(A)$ and $L_t(A)$.

2 Preliminaries

Definition 2.1. [1] Let X be a fixed non-empty set. An Intuitionistic fuzzy set (IFS) A of X is an object of the following form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in X$, respectively, and for any $x \in X$, we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Remark 2.2. When $\mu_A(x)+\nu_A(x) = 1$, i.e., when $\nu_A(x) = 1-\mu_A(x) = \mu^c_A(x)$, then A is called fuzzy set [7].

Definition 2.3. [1] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$ be any two IFS's of X, then

i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,

ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$,

iii) $A \cap B = \{\langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cap \nu_B)(x)\rangle : x \in X\}$ where
$(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$ and
$(\nu_A \cap \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$,

iv) $A \cup B = \{\langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cup \nu_B)(x)\rangle : x \in X\}$ where
$(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$ and
$(\nu_A \cup \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$.

Definition 2.4. [5] Let A be an intuitionistic fuzzy set of a universe set X. Then (α, β)-cut of A is a crisp subset $C_{\alpha,\beta}(A)$ of the IFS A is given by

$$C_{\alpha,\beta}(A) = \{x : x \in X \text{such that } \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

Proposition 1. [5] If A and B be two IFS's of a universe set X, then the following statements hold:

i) $C_{\alpha,\beta}(A) \subseteq C_{\delta,\theta}(A)$ if $\alpha \geq \delta$ and $\beta \leq \theta$,

ii) $C_{1-\beta,\beta}(A) \subseteq C_{\alpha,\beta}(A) \subseteq C_{\alpha,1-\alpha}(A)$,

iii) $A \subseteq B$ implies $C_{\alpha,\beta}(A) \subseteq C_{\alpha,\beta}(B)$,

iv) $C_{\alpha,\beta}(A \cap B) = C_{\alpha,\beta}(A) \cap C_{\alpha,\beta}(B)$,

v) $C_{\alpha,\beta}(A \cup B) \supseteq C_{\alpha,\beta}(A) \cup C_{\alpha,\beta}(B)$ equality hold if $\alpha + \beta = 1$,

vi) $C_{\alpha,\beta}(\cap A_i) = \cap C_{\alpha,\beta}(A_i)$,

vii) $C_{0,1}(A) = X$.

3 t-Lower level set of intuitionistic fuzzy sets

Definition 3.1. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an Intuitionistic fuzzy set of X. Then t-lower level set of A $(L_t(A))$ is given by $L_t(A) = \{x \in X : 1 \geq \mu_A(x) + \nu_A(x) \geq t\}$, where $t \in [0, 1]$.*

Proposition 2. *Let A and B be two Intuitionistic fuzzy sets of X, then following holds:*

i) $L_t(A) \subseteq L_k(A)$ *if* $t \geq k$,

ii) $L_t(A) \cap L_t(B) \subseteq L_t(A \cap B)$,

iii) $L_0(A) = X$.

Proof. i) Let $x \in L_t(A)$ then $1 \geq \mu_A(x) + \nu_A(x) \geq t$. Because of $t \geq k$ then $1 \geq \mu_A(x) + \nu_A(x) \geq t \geq k$. So $x \in L_k(A)$. Hence $L_t(A) \subseteq L_k(A)$.

ii) Let $x \in L_t(A) \cap L_t(B)$ then $x \in L_t(A)$ and $x \in L_t(B)$. So $1 \geq \mu_A(x) + \nu_A(x) \geq t$ and $1 \geq \mu_B(x) + \nu_B(x) \geq t$.
Let us take $\mu_A(x) = \alpha$ and $\mu_B(x) = \beta$ where $\alpha, \beta \in [0, 1]$ and $\alpha \geq \beta$.
Then $\mu_A(x) \wedge \mu_B(x) = \beta$. Since $1 - \alpha \geq \nu_A(x) \geq t - \alpha$ and $1 - \beta \geq \nu_B(x) \geq t - \beta \geq t - \alpha$, then $1 - \beta \geq \nu_A(x) \vee \nu_B(x)$ and $\nu_A(x) \vee \nu_B(x) \geq t - \beta$ So $1 \geq \beta + 1 - \beta \geq \mu_A(x) \wedge \mu_B(x) + \nu_A(x) \vee \nu_B(x) \geq \beta + t - \beta \geq t$. Hence $1 \geq (\mu_A \cap \mu_B)(x) + (\nu_A \cap \nu_B)(x) \geq t$, then $x \in L_t(A \cap B)$. Hence $L_t(A) \cap L_t(B) \subseteq L_t(A \cap B)$.

iii) $L_0(A) = \{x \in X : 1 \geq \mu_A(x) + \nu_A(x) \geq 0\} = X$. □

Proposition 3. *Let A_i be an intuitionistic fuzzy set of X for all $i \in I$. Then $L_t(\bigcap_{i\in I} A_i) \supseteq \bigcap_{i\in I} L_t(A_i)$.*

Proof. Let us take $x \in \bigcap_{i\in I} L_t(A_i)$. Then for all $i \in I, x \in L_t(A_i)$. Then $1 \geq \mu_{A_i}(x) + \nu_{A_i}(x) \geq t$. For all $i \in I$ let us take $\mu_{A_i}(x) = \alpha_i$ and $\bigwedge_{i\in I} \alpha_i = \alpha'$. For all $i \in I, \alpha_i \geq \alpha'$. Since for all $i \in I, 1 - \alpha' \geq 1 - \alpha_i \geq \nu_{A_i}(x) \geq t - \alpha_i$ then $1 - \alpha' \geq \bigvee_{i\in I} \nu_{A_i}(x) \geq t - \alpha'$.
So $1 \geq \alpha' + 1 - \alpha' \geq \bigwedge_{i\in I} \mu_{A_i}(x) + \bigvee_{i\in I} \nu_{A_i}(x) \geq \alpha + t - \alpha'$. Hence $1 \geq \bigcap_{i\in I} \mu_{A_i}(x) + \bigcup_{i\in I} \nu_{A_i}(x) \geq t$ then $x \in L_t \bigcap_{i\in I} A_i$. So $L_t(\bigcap_{i\in I} A_i) \supseteq \bigcap_{i\in I} L_t(A_i)$. □

Theorem 3.2. *Let A be an intuitionistic fuzzy set of X. Let $\{t_i : i \in I\}$ be a non-empty subset of $[0, 1]$. Let $\bigwedge_{i\in I} t_i = b$ and $\bigvee_{i\in I} t_i = c$, then:*

i) $\bigcup_{i\in I} L_{t_i}(A) \subseteq L_b(A)$,

ii) $\bigcap_{i\in I} L_{t_i}(A) = L_c(A)$.

Proof. i) Let $x \in \bigcup_{i\in I} L_{t_i}(A)$ then $\exists i \in I, x \in L_{t_i}(A)$. So $1 \geq \mu_A(x) + \nu_A(x) \geq t_i \geq b$. Therefore $x \in L_b(A)$. Hence $\bigcup_{i\in I} L_{t_i}(A) \subseteq L_b(A)$.

ii) Let $x \in \bigcap_{i \in I} L_{t_i}(A)$. Then for all $i \in I, x \in L_{t_i}(A)$ and so $1 \geq \mu_A(x) + \nu_A(x) \geq t_i$. By $\bigvee_{i \in I} t_i = c,\ 1 \geq \mu_A(x) + \nu_A(x) \geq c$. Therefore $x \in L_c(A)$ that is to say

$$\bigcap_{i \in I} L_{t_i}(A) \subseteq L_c(A). \tag{*}$$

Contrary, let $x \in L_c(A)$, then $1 \geq \mu_A(x) + \nu_A(x) \geq c = \bigvee_{i \in I} t_i$. Since $c \geq t_i$ for all $i \in I,\ 1 \geq \mu_A(x) + \nu_A(x) \geq t_i$. Thereby for all $i \in I$ $x \in L_{t_i}(A)$. Then $x \in \bigcap_{i \in I} L_{t_i}(A)$. Hence

$$L_c(A) \subseteq \bigcap_{i \in I} L_{t_i}(A). \tag{**}$$

From (*) and (**) $\bigcap_{i \in I} L_{t_i}(A) = L_c(A)$. □

4 t-Upper level set of intuitionistic fuzzy sets

Definition 4.1. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an Intuitionistic fuzzy set of X. Then the t-upper level set of A, $(U_t(A))$, is given by $U_t(A) = \{x \in X : t \geq \mu_A(x) + \nu_A(x) \geq 0\}$, where $t \in [0, 1]$.*

Proposition 4. *Let A and B be two Intuitionistic fuzzy sets of X then the following statements hold:*

i) $U_t(A) \subseteq U_k(A)$ *if* $k \geq t$,

ii) $U_t(A) \cap U_t(B) \subseteq U_t(A \cap B)$,

iii) $U_1(A) = X$.

Proof. i) Let $x \in U_t(A)$ then $t \geq \mu_A(x) + \nu_A(x) \geq 0$. Because of $k \geq t$ then $k \geq t \geq \mu_A(x) + \nu_A(x) \geq 0$. So $x \in U_k(A)$. Hence $U_t(A) \subseteq U_k(A)$.

ii) Let $x \in U_t(A) \cap U_t(B)$ then $x \in U_t(A)$ and $x \in U_t(B)$. So $t \geq \mu_A(x) + \nu_A(x) \geq 0$ and $t \geq \mu_B(x) + \nu_B(x) \geq 0$.
Let us take $\mu_A(x) = \alpha$ and $\mu_B(x) = \beta$ where $\alpha, \beta \in [0, 1]$ and $\alpha \geq \beta$.
Since $t - \beta \geq t - \alpha \geq \nu_A(x) \geq 0$ and $t - \beta \geq \nu_B(x) \geq 0$ then $t - \beta \geq \nu_A(x) \vee \nu_B(x) \geq 0$.
So $t \geq t - \beta + \beta \geq \mu_A(x) \wedge \mu_B(x) + \nu_A(x) \vee \nu_B(x) \geq \beta \geq 0$.
Hence $t \geq (\mu_A \cap \mu_B)(x) + (\nu_A \cap \nu_B)(x) \geq 0$ then $x \in U_t(A \cap B)$.
Hence $U_t(A) \cap U_t(B) \subseteq U_t(A \cap B)$.

iii) $U_1(A) = \{x \in X : 1 \geq \mu_A(x) + \nu_A(x) \geq 0\} = X$. □

Proposition 5. *Let A_i be an intuitionistic fuzzy set of X for all $i \in I$. Then $U_t(\bigcap_{i \in I} A_i) \supseteq \bigcap_{i \in I} U_t(A_i)$.*

Proof. Let us take $x \in \bigcap_{i\in I} U_t(A_i)$. Then for all $i \in I, x \in U_t(A_i)$. Then $t \geq \mu_{A_i}(x) + \nu_{A_i}(x) \geq 0$. For all $i \in I$ lets take $\mu_{A_i}(x) = \alpha_i$ and $\bigwedge_{i\in I} \mu_{A_i}(x) = \bigwedge_{i\in I} \alpha_i = \alpha'$. For all $i \in I, \alpha_i \geq \alpha'$. Since for all $i \in I, t - \alpha' \geq t - \alpha_i \geq \nu_{A_i}(x) \geq 0$ then $t - \alpha' \geq \bigvee_{i\in I} \nu_{A_i}(x) \geq 0$.
So $t \geq \alpha' + t - \alpha' \geq \bigwedge_{i\in I} \mu_{A_i}(x) + \bigvee_{i\in I} \nu_{A_i}(x) \geq 0$. Hence $t \geq \bigcap_{i\in I} \mu_{A_i}(x) + \bigcup_{i\in I} \nu_{A_i}(x) \geq 0$ then $x \in U_t \bigcap_{i\in I} A_i$. So $U_t(\bigcap_{i\in I} A_i) \supseteq \bigcap_{i\in I} U_t(A_i)$. □

Theorem 4.2. *Let A be an intuitionistic fuzzy set of X. Let $\{t_i : i \in I\}$ be a non-empty subset of $[0, 1]$. Let $\bigwedge_{i\in I} t_i = b$ and $\bigvee_{i\in I} t_i = c$ then:*

i) $\bigcup_{i\in I} U_{t_i}(A) \subseteq U_c(A)$,

ii) $\bigcap_{i\in I} U_{t_i}(A) = U_b(A)$.

Proof. i) Let us $x \in \bigcup_{i\in I} U_{t_i}(A)$ then $\exists i \in I, x \in U_{t_i}(A)$. So $c \geq t_i \geq \mu_A(x) + \nu_A(x) \geq 0$. Therefore $x \in U_c(A)$. Hence $\bigcup_{i\in I} U_{t_i}(A) \subseteq U_c(A)$.

ii) Let us $x \in \bigcap_{i\in I} U_{t_i}(A)$. Then for all $i \in I, x \in U_{t_i}(A)$ and so $t_i \geq \mu_A(x) + \nu_A(x) \geq 0$. By $\bigwedge_{i\in I} t_i = b,\ b \geq \mu_A(x) + \nu_A(x) \geq 0$. Therefore $x \in U_b(A)$ that is to say

$$\bigcap_{i\in I} U_{t_i}(A) \subseteq U_b(A). \tag{*}$$

Contrary lets $x \in U_b(A)$ then $\bigwedge_{i\in I} t_i = b \geq \mu_A(x) + \nu_A(x) \geq 0$. Since $t_i \geq b$ for all $i \in I,\ t_i \geq \mu_A(x) + \nu_A(x) \geq 0$. Thereby for all $i \in I$ $x \in U_{t_i}(A)$. Then $x \in \bigcap_{i\in I} U_{t_i}(A)$. Hence

$$U_b(A) \subseteq \bigcap_{i\in I} U_{t_i}(A). \tag{**}$$

From (*) and (**) $\bigcap_{i\in I} L_{t_i}(A) = L_b(A)$. □

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