# On a Class of d-Fuzzy Sets and d-Intuitionistic Fuzzy Sets 

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#### Abstract

In the paper a class of d-Fuzzy Sets (d-FS) and a class of d-Intuitinistic Fuzzy Sets (d-IFS) are introduced and studied. These classes are generated by specific kinds of norms on $R^{2}$. The investigation follows [1]


## 1 Keywords

Fuzzy Set, d-Fuzzy Set (d-FS), Intuitionistic Fuzzy Set (IFS), d-Intuitionistic Fuzzy Set (d-IFS), metric, norm

## 2 Used Denotations

$R^{2}$ - for the standard 2-dimensional vector space, $R_{+}^{2}$ - for the set of all vectors with non-negative components from $R^{2} ; I$ - for the interval $[0,1] ; \times$ - for Cartesian product of sets; $E$ - for the universe; $\mu: E \rightarrow[0,1], \nu: E \rightarrow[0,1]$ - for the membership and non-membership functions.

## 3 Introduction

In [1] for the first time were introduced the so-called d-IFS sets. These sets are generated by an arbitrary metric on $R^{2}$, and they are a continuation of the classical fuzzy sets (FS) (see [2] and of Atanassov's IFS, considered for the first time in [3], because FS and IFS are received for a particular choice of the metric $d$. Thus in [1] a possibility for a topological point of view on the introduction of FS and IFS has been proposed. In the present paper our aim is to introduce an important classes of d-FS and d-IFS, depending on infinitely many norms $\psi_{\alpha}$ on $R^{2}$, which correspond to the norms $\varphi_{\alpha}$ that were considered in paragraphs 4 and 5 from [1]. Also, the corresponding domains characterising the above-mentioned classes of d-FS and of d-IFS, depending on the norms $\psi_{\alpha}$ are introduced and studied. The investigation is a continuation and furthering of the results in [1].

## $4 \quad d_{\varphi_{\alpha}}-$ FS and $d_{\varphi_{\alpha}}-$ IFS

Below we use some of the results in [1]:
Let $d: R^{2} \times R^{2} \rightarrow[0,+\infty)$ be an arbitrary metric on $R^{2}$ and $\mu: E \rightarrow I, \nu: E \rightarrow I$ be arbitrary mappings. We call the set

$$
\{\mu(x), \nu(x) \mid x \in E\}
$$

$d$-fuzzy set or $d$-FS, if it is fulfilled

$$
\forall x \in E, d((\mu(x), \nu(x)),(0,0))=1
$$

Also we call the set

$$
\{\mu(x), \nu(x) \mid x \in E\}
$$

$d$-Intuitionistic Fuzzy Set or $d$-IFS, if it is fulfilled:

$$
\forall x \in E, d((\mu(x), \nu(x)),(0,0)) \leq 1
$$

Let $\varphi: R^{2} \rightarrow[0,+\infty)$ be an arbitrary norm on $R^{2}$. Then as usual, $\varphi$ represents a metric $d=d_{\varphi}$ on $R^{2}$, that is given by the formula:

$$
\begin{equation*}
d_{\varphi}\left(\left(\mu_{1}, \nu_{1}\right),\left(\mu_{2}, \nu_{2}\right)\right)=\varphi\left(\mu_{1}-\mu_{2}, \nu_{1}-\nu_{2}\right),\left(\forall\left(\mu_{1}, \nu_{1}\right),\left(\mu_{2}, \nu_{2}\right) \in R^{2}\right) \tag{1}
\end{equation*}
$$

Therefore, every norm on $R^{2}$ generates $d_{\varphi}$-FS and $d_{\varphi}$-IFS.
Further we suppose that $\alpha \in(0, \infty)$. Then we introduce the following norms on $R^{2}$ :

$$
\begin{gather*}
\varphi_{\alpha}(\mu, \nu)=\left(|\mu|^{\alpha}+|\nu|^{\alpha}\right)^{1 / \alpha} \\
\varphi_{\infty}(\mu, \nu)=\max (|\mu|,|\nu|) \tag{2}
\end{gather*}
$$

These norms and their corresponding metrics $d_{\varphi_{\alpha}}$ and $d_{\varphi_{\infty}}$ generate $d_{\varphi_{\alpha}}$ - $\mathrm{FS}, d_{\varphi_{\alpha}}$-IFS, $d_{\varphi_{\infty}}$-FS, and $d_{\varphi_{\infty}}-$ IFS, respectively. We must note the following:
Fact1: A set $\mathbf{A}$ is FS, respectively IFS iff $\mathbf{A}$ is $d_{\varphi_{1}}$-FS, respectively $d_{\varphi_{1}}$-IFS i.e. FS and IFS are generated by the well-known Hamming's norm.
Fact2: $d_{\varphi_{\alpha}}$-FS are given by:

$$
\left\{(\mu(x), \nu(x)) \mid x \in E, \mu: E \rightarrow I, \nu: E \rightarrow I \&\left((\mu(x))^{\alpha}+(\nu(x))^{\alpha}\right)=1\right\}
$$

Fact3: $d_{\varphi_{\alpha}}$-IFS are given by:

$$
\left\{(\mu(x), \nu(x)) \mid x \in E, \mu: E \rightarrow I, \nu: E \rightarrow I \&\left((\mu(x))^{\alpha}+(\nu(x))^{\alpha}\right) \leq 1\right\}
$$

Fact4: Any $d_{\varphi_{\infty}}$-FS or $d_{\varphi_{\infty}}$-IFS is a limit of $d_{\varphi_{\alpha}}$-FS or $d_{\varphi_{\alpha}}$-IFS, respectively when $\alpha \rightarrow+\infty$ but $\mu$ and $\nu$ are the same functions.
We call $d_{\varphi_{2}}$-FS and $d_{\varphi_{2}}$-IFS, Euclidean-FS and Euclidean-IFS, respectively.The reason for this definition is that $\varphi_{2}$ is called Euclidean norm.
Let

$$
\begin{aligned}
K_{\alpha} & :=\left\{\left.(\mu, \nu)|\mu, \nu \in[-1,1] \&| \mu\right|^{\alpha}+|\nu|^{\alpha}=1\right\} \\
\tilde{K}_{\alpha} & :=\left\{\left.(\mu, \nu)|\mu, \nu \in[-1,1] \&| \mu\right|^{\alpha}+|\nu|^{\alpha} \leq 1\right\}
\end{aligned}
$$

Then $K_{\alpha}$ and $\tilde{K}_{\alpha}$ are the unit circle and disk, centered at the origin of $R^{2}$ (considered as a plane), with respect to the metric $d_{\varphi_{\alpha}}$, generated by norm $\varphi_{\alpha}$ on $R^{2}$.
We introduce the sets $K_{\alpha}^{*}$ and $\tilde{K}_{\alpha}^{*}$ by:

$$
\begin{aligned}
K_{\alpha}^{*} & =K_{\alpha} \cap(I \times I) ; \\
\tilde{K}_{\alpha}^{*} & =\tilde{K}_{\alpha} \cap(I \times I) ;
\end{aligned}
$$

We introduce the domain $\tilde{K}_{\infty}^{*}$ as $I \times I$.
Below we shall discuss the important question about the connection between $d_{\varphi_{\alpha}}$-IFS and $d_{\varphi_{\beta}}$-IFS, when $\alpha, \beta \in(0,+\infty]$ and $\alpha \neq \beta$.

The key to answering this question is hidden behind the fact that the norm $\varphi_{\alpha}$ is a strictly decreasing function with respect to $\alpha$, on $(0,+\infty)$ and :

$$
\lim _{\alpha \rightarrow+\infty} \varphi_{\alpha}=\varphi_{\infty}
$$

This was proved in [1].
As a result of the above reasonings we conclude that the closed domains $\tilde{K}_{\alpha}^{*}$ grow as $\alpha$ increases in value, where $\alpha \in(0,+\infty)$, i.e.

$$
\tilde{K}_{\alpha}^{*} \subset \tilde{K}_{\beta}^{*}
$$

for $\alpha, \beta \in(0,+\infty), \alpha<\beta$
Moreover, there exists

$$
\lim _{\alpha \rightarrow+\infty} \tilde{K}_{\alpha}^{*}=\tilde{K}_{\infty}^{*}=I \times I
$$

In short we have:

$$
\tilde{K}_{1}^{*} \subset \tilde{K}_{2}^{*} \subset \ldots \subset \tilde{K}_{\infty}^{*}=I \times I
$$

Let $\alpha, \beta \in(0,+\infty)$ and $\alpha<\beta$. From the above considerations we obtain that if A is a $\varphi_{\alpha}$-IFS then A is $\varphi_{\beta}$-IFS.
It is also seen that any IFS is Euclidean-IFS, and moreover, any IFS is $\varphi_{\alpha}$-IFS for $\alpha>1$ but is not $\varphi_{\alpha}$-IFS for $\alpha<1$.

## $5 \quad d_{\psi_{\alpha}}-$ FS and $d_{\psi_{\alpha}}$ - IFS

Let $\alpha \in(-\infty,+\infty)$. We introduce

$$
\begin{equation*}
\psi_{\alpha}(\mu, \nu)=\left(\frac{|\mu|^{\alpha}+|\mu|^{\alpha}}{2}\right)^{\frac{1}{\alpha}} \tag{3}
\end{equation*}
$$

Also we introduce

$$
\begin{equation*}
\psi_{0}(\mu, \nu)=\lim _{\alpha \rightarrow 0} \psi_{\alpha}(\mu, \nu)=\sqrt{|\mu||\nu|} \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi_{+\infty}(\mu, \nu)=\lim _{\alpha \rightarrow+\infty} \psi_{\alpha}(\mu, \nu) \tag{5}
\end{equation*}
$$

Considered for a fixed $\alpha, \psi_{\alpha}(\mu, \nu)$ is known as a Power Mean of order $\alpha$ of the numbers $|\mu|,|\nu|$. More specifically $\psi_{-1}(\mu, \nu), \psi_{0}(\mu, \nu), \psi_{1}(\mu, \nu), \psi_{2}(\mu, \nu)$ are known as Harmonic Mean, Geometric Mean (see (4)) and Root-Mean-Square of the numbers $|\mu|,|\nu|$, respectively (see[4]).

It is obvious that

$$
\begin{equation*}
\psi_{+\infty}(|\mu|,|\nu|)=\varphi_{\infty}(|\mu|,|\nu|)=\max (|\mu|,|\nu|) \tag{6}
\end{equation*}
$$

(see (2) and (5)).
If we put in (1) $\psi_{\alpha}(\mu, \nu)\left(\right.$ from (3)) we are able to generate $d_{\psi_{\alpha}}$ and $d_{\psi_{\infty}}$ - metrics on $R^{2}$ and as a result $d_{\psi_{\alpha}}$-FS, $d_{\psi_{\alpha}}$-IFS, $d_{\psi_{+\infty}}$-FS, and $d_{\psi_{+\infty}}$-IFS are introduced. For this purpose we must assume that $|\mu|$ and $|\nu|$ are the values of two non-complementary functions $\mu(x)$ and $\nu(x)$, such that $\mu: E \rightarrow I, \nu: E \rightarrow I$ (e.g. we may interprete these functions from classical point of view as degree of membership to a certain property and a degree non-membership to a different property).

Let $\alpha \in(0,+\infty]$. In this case we introduce the sets:

$$
B_{\alpha}:=\left\{(\mu, \nu) \mid \mu, \nu \in[-1,1] \& \psi_{\alpha}(\mu+\nu)=1\right\}
$$

and

$$
\tilde{B}_{\alpha}:=\left\{(\mu, \nu) \mid \mu, \nu \in[-1,1] \& \psi_{\alpha}(\mu+\nu) \leq 1\right\}
$$

Then $B_{\alpha}$ and $\tilde{B}_{\alpha}$ are the unit circle and disk, centered at the origin of $R^{2}$ (considered as a plane), with respect to the metric $d_{\psi_{\alpha}}$, generated by norm $\psi_{\alpha}$ on $R^{2}$. Also we introduce the sets $B_{\alpha}^{*}$ and $\tilde{B}_{\alpha}^{*}$ by:

$$
\begin{aligned}
& B_{\alpha}^{*}=B_{\alpha} \cap R_{+}^{2} ; \\
& \tilde{B}_{\alpha}^{*}=\tilde{B}_{\alpha} \cap R_{+}^{2} ;
\end{aligned}
$$

Finally, because of (6), we introduce the domain $\tilde{B}_{\infty}^{*}$ as $I \times I$ (i.e. the domain $\tilde{B}_{\infty}^{*}$ coincides with the domain $\tilde{K}_{\infty}^{*}$ considered in [1]).
Below we shall discuss the important question about the connection between $d_{\varphi_{\alpha}}$-IFS, $d_{\varphi_{\beta}}$, - IFS, $d_{\psi_{\alpha}}$-IFS, $d_{\psi_{\beta}}$-IFS when $\alpha, \beta \in[0,+\infty]$ and $\alpha \neq \beta$.
It is well-known that considered as a function of $\alpha$ only, $\psi_{\alpha}(\mu, \nu)$ is a strictly increasing function on the interval $(-\infty,+\infty]$ (see [5]). Therefore, $\psi_{\alpha}(\mu, \nu)$ is strictly increasing fiunction with respect to $\alpha$ on the interval $[0,+\infty]$. Hence, the closed domains $\tilde{B}_{\alpha}^{*}$ diminishes as $\alpha$ increases in value when $\alpha \in[0,+\infty]$. Therefore, if $\alpha, \beta \in(0,+\infty)$ and $\alpha<\beta$ then it is fulfilled:

$$
\tilde{B}_{\beta}^{*} \subset \tilde{B}_{\alpha}^{*}
$$

Hence,

$$
I \times I \equiv \tilde{B}_{+\infty}^{*} \subset \tilde{B}_{\beta}^{*} \subset \tilde{B}_{\alpha}^{*} \subset \tilde{B}_{0}^{*}
$$

It is easy to see that the closed domain $\tilde{B}_{0}^{*}$ coincides with the closed domain encapsulated between the coordinate rays of the first quadrant in the Cartesian coordinate system $O_{\mu \nu}$ centered at $(0,0)$ and the branch of the hyperbola given by the equation $\mu \nu=1$ located in the first quadrant. This follows from the fact that

$$
\psi_{0}(\mu, \nu)=\sqrt{|\mu||\nu|}
$$

as it was mentioned above. Thus the following chains of inclusions is fulfilled:

$$
\tilde{B}_{0}^{*} \supset \tilde{B}_{1}^{*} \supset \tilde{B}_{2}^{*} \supset \ldots \supset \tilde{B}_{+\infty}^{*} \equiv I \times I
$$

But when $0<\alpha<\beta<+\infty$ we have:

$$
\tilde{K}_{\alpha}^{*} \subset \tilde{K}_{\beta}^{*} \subset \tilde{K}_{\infty}^{*} \equiv I \times I \equiv \tilde{B}_{+\infty}^{*}
$$

Hence for $0<\alpha<\beta<+\infty$ it is fulfilled:

$$
\begin{equation*}
\tilde{K}_{\alpha}^{*} \subset \tilde{K}_{\beta}^{*} \subset \tilde{K}_{\infty}^{*} \equiv I \times I \equiv \tilde{B}_{+\infty}^{*} \subset \tilde{B}_{\beta}^{*} \subset \tilde{B}_{\alpha}^{*} \subset \tilde{B}_{0}^{*} \tag{7}
\end{equation*}
$$

Hence the validity of the double chain of inclusions that is given below

$$
\tilde{K}_{1}^{*} \subset \tilde{K}_{2}^{*} \subset \ldots \subset \tilde{K}_{\infty}^{*} \equiv I \times I \equiv \tilde{B}_{+\infty}^{*} \subset \ldots \subset \tilde{B}_{2}^{*} \subset \tilde{B}_{1}^{*} \subset \tilde{B}_{0}^{*}
$$

is established because of (7). The above considerations give us the complete answer of the question stated before about the connection between the sets: $d_{\varphi_{\alpha}}$-IFS, $d_{\varphi_{\beta}}$-IFS, $d_{\psi_{\alpha}}$-IFS, $d_{\psi_{\beta}}$-IFS. They deliver an answer by providing a proof for the following:
Theorem. Let $0<\alpha<\beta \leq+\infty$, then:
$i_{1}$ : If A is $d_{\varphi_{\alpha}}$-IFS, then A is $d_{\varphi_{\beta}}$-IFS, $d_{\varphi_{\infty}}$-IFS, $d_{\psi_{\beta}}$-IFS, $d_{\psi_{\alpha}}$ - IFS and $\psi_{0}$-IFS, simultaneously.
$i_{2}$ : If A is $d_{\psi_{\beta}}$-IFS, then A is $d_{\psi_{\alpha}}$-IFS and $d_{\psi_{0}}$-IFS too.
$i_{3}: d_{\varphi_{\infty}}$-IFS sets coincide with $d_{\psi_{+\infty}}$-IFS sets.
It is natural to call: $d_{\psi_{0}}$-IFS - Geometric Mean-IFS; $d_{\psi_{1}}$-IFS - Arithmetic Mean-IFS; $d_{\psi_{2}}$-IFS - Root-Mean-Square-IFS. It is clear that every IFS is $d_{\varphi_{\alpha}}$ IFS, too, when $1<\alpha \leq+\infty$. Also, every IFS is $d_{\psi_{\alpha}}$-IFS when $0 \leq \alpha \leq+\infty$. In particular, every IFS is Geometric Mean-IFS, Arithmetic Mean-IFS and Root-Mean-Square-IFS.
Open problem: Propose a suitable definition of the modal operators possibility and necessity for the $d_{\psi_{\alpha}}$-IFS, and investigate their properties.

## References

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