# Solving Problems in Intuitionistic Fuzzy Relational Calculus with Fuzzy Relational Calculus Toolbox 

Ketty Peeva ${ }^{1}$ and Yordan Kyosev ${ }^{2}$<br>${ }^{1}$ Faculty of Applied Mathematics and Informatics, Technical University of Sofia, P. O. Box 384, Sofia 1000, Bulgaria<br>E-mail: kgp@tu - sofia.bg<br>${ }^{2}$ Department of Textile Engineering, Technical University of Sofia, 8 Blvr. Kl. Ohridski, Sofia 1797, Bulgaria<br>E-mail: jkiossev@tu - sofia.bg<br>URL: http : //www.geocities.com/jkiossev


#### Abstract

We develop library for fuzzy relational calculus over the fuzzy algebra ( $[0,1]$, max, $\mathrm{min})$ in MATLAB workspace. It includes various operations and compositions with fuzzy relations, solving direct and inverse problems, applications in artificial intelligence areas.

In this paper we present software for direct and inverse problem resolution in intuitionistic fuzzy relational calculus.


Keywords: Intuitionistic fuzzy relations, software for inverse problem

## 1 Introduction

The existing Fuzzy Toolbox in MATLAB does not propose options for solving direct or inverse problems even in conventional fuzzy relational calculus. Only a few computer codes for fuzzy relational calculus have been reported in the literature [2], [3].

The first attempt for intuitionistic fuzzy relational calculus (IFRC) is described in [4], where the software is a set of functions in MATLAB 6 (R12) environment.

This paper presents next development in this direction. The authors develop in MATLAB workspace a toolbox for fuzzy relational calculus for the fuzzy algebra ( $[0,1]$, max, min). Its integral part includes software for intuitionistic fuzzy relational calculus. We operate with intuitionistic fuzzy relations and intuitionistic fuzzy membership matrices. The package provides options for compositions (direct problem resolution) and for solving intuitionistic
fuzzy relational equations (inverse problem resolution). Solving problems in intuitionistic fuzzy relational calculus requires calling other packages (for max-min, min-max and other operations) from the fuzzy relational toolbox.

In Section 2 we introduce the basic operations in intuitionistic fuzzy relational calculus and formulate direct and inverse problems. In Section 3 we present the software providing the library for intuitionistic fuzzy relational calculus. The exposition is completed with practical examples.

## 2 Basic problems in IFRC

We work over the fuzzy algebra $I=([0,1]$, max, min, 0,1$)$.
Let $E \neq \emptyset$ be a crisp set and $A \subseteq E$. An intuitionistic fuzzy set (IFS) [1] $\hat{A}$ on $E$ is

$$
\hat{A}=\left\{\left(x, \mu_{A}(x), \nu_{A}(x)\right) \mid x \in E\right\}
$$

where for each $x \in E, \mu_{A}: E \rightarrow[0,1]$ defines the degree of membership and $\nu_{A}: E \rightarrow[0,1]$ defines the degree of non-membership, respectively, of the elements $x \in E$ to $\hat{A}$ and for each $x \in E$ holds $0 \leq \mu_{A}(x)+\nu_{A}(x) \leq 1$.

The class of all intuitionistic fuzzy sets over $E$ is denoted by $\operatorname{Int}(E)$.

## Intuitionistic fuzzy relations and matrices

An intutionistic fuzzy relation (IFR) between two nonempty sets $X$ and $Y$ is an intutionistic fuzzy set on $X \times Y$, written $R \in \operatorname{Int}(X \times Y)$. $X \times Y$ is called support of $R$. We write $R \subseteq X \times Y$ for the (conventional) fuzzy relation $R$ between $X$ and $Y$.

Any IFR $R \in \operatorname{Int}(X \times Y)$ is given as follows:

$$
\begin{gathered}
R=\left\{\left((x, y), \mu_{R}(x, y), \nu_{R}(x, y)\right) \mid(x, y) \in X \times Y, \mu_{R}, \nu_{R}: X \times Y \rightarrow[0,1]\right\} \\
0 \leq \mu_{R}(x, y)+\nu_{R}(x, y) \leq 1
\end{gathered}
$$

for each $(x, y) \in X \times Y$.
The matrix $A=\left(\mu_{i j}^{A}, \nu_{i j}^{A}\right)_{m \times n}$ with $\mu_{i j}^{A}, \nu_{i j}^{A} \in[0,1]$ such that

$$
0 \leq \mu_{i j}^{A}+\nu_{i j}^{A} \leq 1 \text { for each } i, j, 1 \leq i \leq m, 1 \leq j \leq n
$$

is called an intuitionistic fuzzy membership matrix (IFM) of type $m \times n$.
When the IFR is over finite support, it is representable by IFM, written for convenience with the same letter. For instance, if the IFR $R \in \operatorname{Int}(X \times Y)$ is over finite support, its representative matrix is stipulated to be the matrix $R=\left(\mu_{x_{i} y_{j}}^{R}, \nu_{x_{i} y_{j}}^{R}\right)_{m \times n}$ such that

$$
\mu_{x_{i} y_{j}}^{R}=\mu_{R}\left(x_{i}, y_{j}\right), \quad \nu_{x_{i} y_{j}}^{R}=\nu_{R}\left(x_{i}, y_{j}\right)
$$

According to this stipulation instead of IFRs we consider intuitionistic fuzzy matrices and operations with them.

For $a, b \in I$ we define

$$
a \alpha b=\left\{\begin{array}{ll}
1, & \text { if } a \leq b \\
b, & \text { if } a>b
\end{array} \quad \text { and } \quad a \varepsilon b=\left\{\begin{array}{ll}
b, & \text { if } a<b \\
0, & \text { if } a \geq b
\end{array} .\right.\right.
$$

Two IFMs $A=\left(\mu_{i j}^{A}, \nu_{i j}^{A}\right)_{m \times p}$ and $B=\left(\mu_{i j}^{B}, \nu_{i j}^{B}\right)_{p \times n}$ are called conformable in this order, if the number of columns in $A$ is equal to the number of rows in $B$.

Definition 1 [3] If $A=\left(\mu_{i j}^{A}, \nu_{i j}^{A}\right)_{m \times p}$ and $B=\left(\mu_{i j}^{B}, \nu_{i j}^{B}\right)_{p \times n}$ are conformable IFMs, then the matrix $C=\left(\mu_{i j}^{C}, \nu_{i j}^{C}\right)_{m \times n}$ is called:
i) $\boldsymbol{S}$ - $\boldsymbol{C}$ product of $A$ and $B$, written $C=A * B$, if for each $i, j, 1 \leq i \leq m, 1 \leq j \leq n$

$$
\mu_{i j}^{C}=\stackrel{p}{k=1}_{p}\left(\mu_{i k}^{A} \wedge \mu_{k j}^{B}\right), \quad \nu_{i j}^{C}=\wedge_{k=1}^{p}\left(\nu_{i k}^{A} \vee \nu_{k j}^{B}\right) ;
$$

ii) $\alpha-\varepsilon$ product of $A$ and $B$, written $C=A \otimes B$, if for each $i, j, 1 \leq i \leq m, \quad 1 \leq j \leq n$

$$
\mu_{i j}^{C}=\wedge_{k=1}^{p}\left(\mu_{i k}^{A} \alpha \mu_{k j}^{B}\right), \quad \nu_{i j}^{C}=\vee_{k=1}^{p}\left(\nu_{i k}^{A} \varepsilon \nu_{k j}^{B}\right) .
$$

The S-C product (from standard-costandard) of $A$ and $B$ is denoted as:

$$
\begin{equation*}
(A \bullet B, A \circ B)=A * B \tag{1}
\end{equation*}
$$

The $\alpha-\varepsilon$ product of $A$ and $B$ is written as:

$$
\begin{equation*}
(A \alpha B, A \varepsilon B)=A \otimes B \tag{2}
\end{equation*}
$$

Here $A \bullet B, A \circ B, A \alpha B$ and $A \varepsilon B$ are determined as follows.
Definition 2 [3] Let $A=\left(a_{i j}\right)_{m \times p}$ and $B=\left(b_{i j}\right)_{p \times n}$ be given conformable finite fuzzy matrices.

The matrix $C=A \bullet B=\left(c_{i j}\right)_{m \times n}$ is called max - min or standard product of $A$ and $B$ if

$$
c_{i j}=\vee_{k=1}^{p}\left(a_{i k} \wedge b_{k j}\right) \text { for each } i, 1 \leq i \leq m, \text { and each } j, 1 \leq j \leq n
$$

The matrix $C=A \circ B=\left(c_{i j}\right)_{m \times n}$ is called min - max or costandard product of $A$ and $B$ if

$$
c_{i j}=\bigwedge_{k=1}^{p}\left(a_{i k} \vee b_{k j}\right) \text { for each } i, 1 \leq i \leq m \text {, and each } j, 1 \leq j \leq n
$$

The matrix $C=A \alpha B=\left(c_{i j}\right)_{m \times n}$ is called $\alpha$-product of $A$ and $B$ if

$$
c_{i j}=\wedge_{k=1}^{p}\left(a_{i k} \alpha b_{k j}\right) \text { for each } i, 1 \leq i \leq m, \text { and each } j, 1 \leq j \leq n .
$$

The matrix $C=A \varepsilon B=\left(c_{i j}\right)_{m \times n}$ is called $\varepsilon$-product of $A$ and $B$ if

$$
c_{i j}=\stackrel{p}{k=1}_{\stackrel{p}{2}}\left(a_{i k} \varepsilon b_{k j}\right) \text { for each } i, 1 \leq i \leq m, \text { and each } j, 1 \leq j \leq n
$$

If $A$ and $B$ are conformable IFMs, then $A * B$ is IFM [3]. This is not valid in general for $A \otimes B$.

## Direct and inverse problems

Definition 4 [3] Let $A$ and $B$ be conformable IFMs.
i) Computing the product $(A \bullet B, A \circ B)=A * B$ or $(A \alpha B, A \varepsilon B)=A \otimes B$ (when it is an IFM) is called direct problem resolution for $*$-composition or for $\otimes$-composition of matrices, respectively.
ii) Let $A$ and $B$ be conformable IFMs and let $C=A * B$. The equation

$$
\begin{equation*}
A * B=C \tag{3}
\end{equation*}
$$

where one of the matrices on the left side is unknown and the other two matrices are given, is called $\boldsymbol{S}$ - $\boldsymbol{C}$ intuitionistic fuzzy matrix equation. Solving (3) for the unknown matrix is called inverse problem resolution.

Theorem 1 [3] Let $A$ and $C$ be IFMs of finite type, and let $\boldsymbol{B}$ be the set of all matrices $B$, such that $(A \bullet B, A \circ B)=A * B=C$. Then:
i) $\boldsymbol{B} \neq \emptyset \quad \Leftrightarrow \quad A^{-1} \otimes C \in \boldsymbol{B}$;
ii) If the equation (3) is solvable for $B$ then $A^{-1} \otimes C$ is its greatest solution.

Here $A^{-1}$ denotes the transpose of $A[3]$.
Theorem 2 For the intuitionistic fuzzy matrix equation $A * B=C$ there exists polynomial time algorithm for establishing its solvability and for computing its greatest solution.

Similar result is valid for intuitionistic fuzzy relational equation $R * Q=T$.

## 3 Software for IFRC

In this section we describe how to solve problems in IFRC using fuzzy relational calculus toolbox.

We create a new class for the intuitionistic matrices. Any intuitionistic fuzzy matrix has two entries in each cell - the degree of membership $\mu$ and degree of non-membership $\nu$. We use structured variable with two fields: a field n - for "non-membership" and a field m - for "membership" degrees, respectively.

For instance for the intuitionistic matrix:

$$
Y=\left(\begin{array}{cc}
<0.8,0.1> & <0.0,0.9> \\
<0.5,0.5> & <0.7,0.2> \\
<0.2,0.7> & <0.6,0.4>
\end{array}\right)
$$

in MATLAB environment we give the membership degrees in the variable Ym:

```
Ym=[lllll}0.80.
    0.5 0.7
    0.2 0.6]
```

then the non-membership degrees in field Yn:

```
Yn=[lllll}0.10.
    0.50.2
    0.7 0.4]
```

In order the manipulation with intuitionistic matrices to be like the manipulation with regular objects, above described structure is used to define new class im as new object. The following command calls im function to construct new intuitionistic matrix object with membership matrix Ym and non-membership matrix Yn:

```
>> Y=im(Ym,Yn)
Y =
<0.80, 0.10> <0.00, 0.90>
<0.50, 0.50> <0.70, 0.20>
<0.20, 0.70> <0.60, 0.40>
```

For this object we developed appropriated functions, as for instance $S-C$ and $\alpha-\varepsilon$. Most of these functions are using operations from the fuzzy relational toolbox, but this remains hidden for the user and he/she can concentrate on the intuitionistic problematic. For instance we realize $S-C$ product calling

```
function Comp=sc_comp(A,B)
```

that is demonstrated in the following example:
Example 1 Compute $S-C$ product of matrices A and Y, where

$$
A=\left(\begin{array}{ccc}
<0.10,0.10> & <0.30,0.20> & <0.40,0.60> \\
<0.00,1.00> & <0.25,0.50> & <0.30,0.30> \\
<0.30,0.70> & <1.00,0.00> & <0.00,0.30> \\
<0.70,0.20> & <0.20,0.40> & <0.90,0.10>
\end{array}\right)
$$

and Y is as given above.

```
>> SC=sc_comp(A,Y)
SC =
<0.30, 0.10> <0.40, 0.20>
<0.25, 0.50> <0.30, 0.40>
<0.50, 0.50> <0.70, 0.20>
<0.70, 0.20> <0.60, 0.40>
```

The $\alpha-\varepsilon$-product is called similarly:

```
AE=ae_comp(A,Y)
```

AE =
<0.20, 0.70> <0.00, 0.90>
<0.20, 0.70> <1.00, 0.40>
<0.50, 0.70> <0.00, 0.90>
<0.20, 0.70> <0.00, 0.90>

Since the $\alpha-\varepsilon$-product of two intuitionistic matrices may be intuitionistic matrix or may not be intuitionistic matrix, we develop a special function, called

```
checkim(A)
```

for checking the result. For each element this function returns $<0,0\rangle$, if the sum of the resulting membership and nonmembership degrees not greater then 1. Otherwise this function returns the ordered pair of computed by $\alpha-\varepsilon-$ operation degrees. For instance checking the last result $A E$ we obtain

```
>> checkim(AE)
ans =
<0.00, 0.00> <0.00, 0.00>
<0.00, 0.00> <1.00, 0.40>
<0.50, 0.70> <0.00, 0.00>
<0.00, 0.00> <0.00, 0.00>
```

The result is not intuitionistic fuzzy matrix - there are two positions $(3,1)$ and $(2,2)$, where the sum of membership and nonmembership degrees is greater then 1.

Example 2 - solving intuitionistic fuzzy relational equations
The eventual greatest solution of the intuitionistic fuzzy relational equation $R * S=T$ for unknown Snew is calculated using $\alpha-\varepsilon$ product.

Let we solve this problem, if $R=A, T=S C$ from above given examples. Here the the notation $R .{ }^{\prime}$ returns the transpose of the matrix $R$.

```
>> Snew=ae_comp(A.',SC)
Snew =
<1.00, 0.00> <0.60, 0.40>
<0.50, 0.50> <0.70, 0.20>
<0.25, 0.50> <0.60, 0.40>
```

The standard-constandard composition of $R$ and Snew is denoted by Tnew:

```
>> T=sc_comp(A,Snew)
T =
<0.30, 0.10> <0.40, 0.20>
<0.25, 0.50> <0.30, 0.40>
<0.50, 0.50> <0.70, 0.20>
<0.70, 0.20> <0.60, 0.40>
```

$\mathrm{T}=\mathrm{SC}$ because:
> Tnew-SC
ans $=$
<0.00, 0.00> <0.00, 0.00>
<0.00, 0.00> <0.00, 0.00>
<0.00, 0.00> <0.00, 0.00>
<0.00, 0.00> <0.00, 0.00>
Hence R * Snew $=\mathrm{SC}=\mathrm{T}=\mathrm{R}$ * S
and the fuzzy intuitionitisc relational equation $R * S=T$ is solvable for $S$ and its greatest solution is Snew .

## 4 Conclusions

In this paper we demonstrate a simple usage of fuzzy relational calculus toolbox for solving problems in intuitionistic fuzzy relational calculus. The toolbox should be considered as a first step in creating library for intuitionistic fuzzy sets.

## References

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