Cartesian products in IFS theory

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Historical remarks

- Zadeh 1975 - Cartesian product of fuzzy sets
- Gottwald 1986 - Cartesian product with respect to triangular norms
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Standard properties of the Cartesian product

Let $A, D \subset X, B, C \subset Y$. From the set theory we have the following properties of the Cartesian product (cf. Kuratowski & Mostowski 1976)
1. Nonemptiness (null property)

$$(A \neq \emptyset \& B \neq \emptyset) \Leftrightarrow (A \times B \neq \emptyset),$$

$$(A \times B = \emptyset) \Leftrightarrow (A = \emptyset \text{ or } B = \emptyset).$$

2. Monotonicity
If $A \neq \emptyset$, then

$$(B \subset C) \Leftrightarrow (A \times B \subset A \times C) \Leftrightarrow (B \times A \subset C \times A).$$

If $A \times B \neq \emptyset$, then

$$(A \times B \subset D \times C) \Leftrightarrow (A \subset D,\ B \subset C).$$

3. Compatibility with projections
If $A \neq \emptyset$, $B \neq \emptyset$, then

$$P_1(A \times B) \times P_2(A \times B) = A \times B,$$

where for $R \subset X \times Y$ we use projections

$$P_1(R) = \{x \in X | \underset{y \in Y}{\exists} (x, y) \in R\},$$

$$P_2(R) = \{y \in Y | \underset{x \in X}{\exists} (x, y) \in R\}.$$

4. Distributivity over set operations

$$A \times (B \cup C) = (A \times B) \cup (A \times C),$$

$$(B \cup C) \times A = (B \times A) \cup (C \times A),$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$

$$(B \cap C) \times A = (B \times A) \cap (C \times A).$$

The $*$-product

Let us consider a bounded lattice $(L, \vee, \wedge, 0, 1)$ with a binary operation $* : L \times L \to L$.

Definition 1. *By a $*$-product of L-fuzzy sets $A : X \to L$, $B : Y \to L$ we call an L-fuzzy set $A \times_* B : X \times Y \to L$, where*

$$(A \times_* B)(x, y) = A(x) * B(y), (x, y) \in X \times Y.$$

We examine additional properties of the operation $*$ necessary for the above listed properties of the $*$-product.

Nonemptiness

Theorem 1. *Let operation* $*: L \times L \to L$ *has the zero element* 0. *The* $*$*-product has nonemptiness property, i.e.*

$$(A \times_* B = \emptyset) \Leftrightarrow (A = \emptyset \text{ or } B = \emptyset)$$

iff the operation $*$ *does not have zero divisors.*

Corollary 1. *Let* $A \in L(X)$. *If operation* $*: L \times L \to L$ *has the zero element* 0, *then the* $*$*-product has zero element* $\emptyset$, *i.e.*

$$A \times_* \emptyset = \emptyset \times_* A = \emptyset.$$

Corollary 2. *If* $* \leqslant T_{Ł}$ *with the Łukasiewicz triangular norm* $T_{Ł}$ *in* $L = [0, 1]$, *then the* $*$*-product does not have nonemptiness property.*

Monotonicity

Theorem 2. *If operation* $* : L \times L \to L$ *is isotone, then the* $*$*-product is also isotone, i.e.*

$$(B \leqslant C) \Rightarrow (A \times_* B \leqslant A \times_* C,\ B \times_* A \leqslant C \times_* A).$$

If additionally the operation $*$ *has the neutral element 1, then the* $*$*-product fulfils the following monotonicity property:*

$$\exists_{s \in X} A(s) = 1 \Rightarrow \{(B \leqslant C)$$

$$\Leftrightarrow (A \times_* B \leqslant A \times_* C) \ \Leftrightarrow (B \times_* A \leqslant C \times_* A)\}.$$

Corollary 3. *If operation* $*$ *in* $L = [0, 1]$ *is a triangular norm, then the* $*$*-product fulfils the above monotonicity property.*

Compatibility with projections

Definition 2. *Let L be a complete lattice. Projections of L-fuzzy relation R are defined by*

$$R_1(x) = P_1(R)(x) = \sup_{y \in Y} R(x, y),$$

$$R_2(y) = P_1(R)(y) = \sup_{x \in X} R(x, y).$$

We say that the $*$-product is compatible with projections if

$$(supA = supB = 1) \Rightarrow$$

$$(P_1(A \times_* B) = A,\ P_2(A \times_* B) = B).$$

Theorem 3. *Let L be a complete lattice with a binary operation $*$ infinitely distributive over supremum. The $*$-product is compatible with projections iff the operation $*$ has the neutral element 1.*

Corollary 4. *If operation $*$ in $L = [0, 1]$ is a left continuous triangular norm, then the $*$-product of fuzzy sets is compatible with projections.*

Distributivity over set operations

Set operations on L-fuzzy sets can be based on the lattice operations or on additional operations in lattice ordered semigroups (generalized set operations).

Theorem 4. *If operation $*$ is distributive with respect to lattice operations, then the $*$-product is distributive with respect to lattice based set operations.*

Lemma 1 ((Drewniak 1983))**.** *If operation $*$ with the neutral element 1 is distributive with respect to operation $\circ$ and $1 \circ 1 = 1$, then the operation $\circ$ is idempotent.*

Because of the above lemma we have no distributivity of the $*$-product in the case of generalized set operations, because ordered semigroups as triangular norms, uninorms or nullnorms have idempotent elements in boundary points 0 and 1.

In particular, we get

Corollary 5. *If operation* $*$ *in* $L = [0, 1]$ *has the neutral element* 1 *and generalized set operations on fuzzy sets are defined by triangular norms and conorms different from* $\vee$ *and* $\wedge$*, then the* $*$*-product is not distributive with respect to these set operations.*

In such situation we look for another idempotent multivalued conjunctions and disjunction different from the lattice operations. Recently such investigations concern conjuctive and disjunctive idempotent uninorms in $L = [0, 1]$ with a characterization of operations $*$ distributive over them (cf. Drewniak, Drygaś, Rak 2008).

Consequences for products of L-sets and fuzzy sets

Because of standard properties of the Cartesian product we can join assumptions from the above theorems and we get

Theorem 5. *Let L be a complete lattice with a binary operation $*$ infinitely distributive over supremum. If operation $*$ has the neutral element 1 and does not have zero divisors, then the $*$-product of L-fuzzy sets has nonemptiness and monotonicity properties, is compatible with projections and distributive with respect to lattice based set operations.*

In the case of $L = [0, 1]$, we have

Theorem 6. *If operation $*$ is a left-continuous seminorm without zero divisors, then the $*$-product of fuzzy sets has nonemptiness and monotonicity properties, is compatible with projections and distributive with respect to lattice based set operations.*

Consequences for products of IFS

In the case of IFS we have the triangle lattice

$$L^* = \{(p,q)|p,q \in [0,1], p+q \leqslant 1\}.$$

It is a complete lattice with lattice operations based on min and max. In this lattice we can consider representable intuitionistic triangular norms $*$ based on triangular norm T and triangular conorm S, where $S \leqslant 1 - T$ (cf. Deschrijver et al. 2004):

$$(p,q) * (r,s) = (T(p,r), S(q,s)),\ (p,q),(r,s) \in L^*.$$

Theorem 7. *Let L^* be the triangle lattice with additional representable operation $*$ defined above. If a triangular norm T and triangular conorm S are strict and $S \leqslant 1 - T$, then the $*$-product of IFS has nonemptiness and monotonicity properties, is compatible with projections and distributive with respect to lattice based set operations.*

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