Notes on Intuitionistic Fuzzy Sets
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Relations between some IF modal operators and IF negations

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Abstract: There have been many studies about intuitionistic fuzzy modal operators and intuitionistic fuzzy negations. The relation between some intuitionistic fuzzy modal operators and negations were firstly examined by Hinde and Atanassov [9]. New properties about intuitionistic fuzzy negations $\neg_1, \neg_4, \neg_8, \neg_{20}, \neg_{25}, \neg^{\varepsilon}$ with some intuitionistic fuzzy one type, second type and uni-type modal operators are studied.
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1 Introduction

In 1965, Zadeh [14] introduced the fuzzy sets theory as an extension of crisp sets. The concept of intuitionistic fuzzy sets (IFS) was introduced by Atanassov in 1983 [1], as a form of extension of fuzzy sets by expanding the truth value set to the lattice $[0,1] \times [0,1]$ defined as follows.

Definition 1. *Let* $L = [0,1]$*, then* $L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\}$ *is a lattice with* $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ *"*$x_1 \leq y_1$ *and* $x_2 \geq y_2$*."*

For $(x_1, y_1), (x_2, y_2) \in L^*$, the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as follows:
$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$
$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$

For each $J \subseteq L^*$
$\sup J = (\sup\{x : (x, y \in [0,1]), ((x,y) \in J)\}, \inf\{y :(x, y \in [0,1])((x,y) \in J)\})$ and
$\inf J = (\inf\{x : (x, y \in [0,1])((x,y) \in J)\}, \sup\{y : (x, y \in [0,1])((x,y) \in J)\})$.

Operations defined over fuzzy sets generalized to intuitionistic fuzzy sets under suitable conditions. The operation "negation" is an important one and there are nearly 53 negation forms defined up to now. Some properties of the negations were examined by researchers. The relations between the intuitionistic fuzzy negations and some intuitionistic fuzzy modal operations were studied in following years.

First one type intuitionistic fuzzy modal operators were defined by Atanassov in 1999 and then different authors have made studies on this area [2, 4–8, 10–13]. New intuitionistic fuzzy modal operators have been defined, the relations between these operators, the relations with algebraic structures and different relations have been examined by several authors.

In this study, our aim is to examine the relations between last intuitionistic fuzzy modal operators and some intuitionistic fuzzy negations.

Definition 2 ([1]). *An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\},$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A", $\nu_A(x), (\nu_A : X \to [0,1])$ is called the "degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{ for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by IFS(X).

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 3 ([1]). *An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X$ such that $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4 ([1]). *Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\}$ then the set below is called the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \,|x \in X\}.$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \,|x \in X\}$$
$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \,|x \in X\}$$

The notion of intuitionistic fuzzy modal operators was firstly introduced by Atanassov as follows:

Definition 5 ([1]). *Let X be universal and $A \in$ IFS(X) then*

1. $\Box(A) = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle | x \in X\}$

2. $\Diamond(A) = \{\langle x, 1-\nu_A(x), \nu_A(x)\rangle | x \in X\}$

In the following years, new modal operators were defined and some properties were examined by several authors. The operators we have studied in this paper are as follows. In 2007, Çuvalcıoğlu introduced $E_{\alpha,\beta}$ operator. Then the operators $Z^{\omega}_{\alpha,\beta}$ and $Z^{\omega,\theta}_{\alpha,\beta}$ were defined as extensions of $E_{\alpha,\beta}$ by the same author.

Definition 6 ([5]). *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \in$ IFS(X), $\alpha, \beta \in [0,1]$. We define the following operator:*

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta)\rangle | x \in X\}$$

Definition 7 ([6]). *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \in$ IFS(X), $\alpha, \beta, \omega \in [0,1]$ then*

$$Z^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta)\rangle | x \in X\}$$

Definition 8 ([6]). *Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \in$ IFS(X), $\alpha, \beta, \omega, \theta \in [0,1]$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta)\rangle | x \in X\}$$

The concept of intuitionistic fuzzy uni-type operators was defined by Çuvalcıoğlu. Some of them are as follows.

Definition 9 ([7]). *Let X be a universal, $A \in$ IFS(X) and $\alpha, \beta, \omega \in [0,1]$ then*

1. $\boxplus^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha(\beta\nu_A(x) + \omega - \omega\beta)\rangle | x \in X\}$

2. $\boxtimes^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega\alpha), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle | x \in X\}$

Definition 10 ([7]). *Let X be a set, $A \in$ IFS(X) and $\alpha, \beta \in [0,1]$ then*

1. $B_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle | x \in X\}$

2. $\boxminus_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\beta)\nu_A(x)), \alpha((1-\alpha)\mu_A(x) + \nu_A(x))\rangle | x \in X\}$

The one type modal operators $L^{\omega}_{\alpha,\beta}$ and $K^{\omega}_{\alpha,\beta}$ were studied by Yılmaz and Bal in 2014. After then, the second type modal operators $T_{\alpha,\beta}$ and $S_{\alpha,\beta}$ were defined by Yılmaz and Çuvalcıoğlu.

Definition 11 ([13]). *Let X be a set and $A \in$ IFS$(X), \alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$*

1. $L^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\rangle\, x \in X\}$

2. $K^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha(1-\beta)\mu_A(x) + \alpha\beta(1-\omega), \alpha\nu_A(x) + \omega(1-\alpha)\rangle\, x \in X\}$

Definition 12 ([12]). *Let X be a set and $A \in$ IFS$(X), \alpha, \beta, \alpha + \beta \in [0,1]$.*

1. $T_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha), \alpha(\nu_A(x) + (1-\beta)\mu_A(x))\rangle | x \in X\}$ *where* $\alpha + \beta \in [0,1]$.

2. $S_{\alpha,\beta}(A) = \{\langle x, \alpha(\mu_A(x) + (1-\beta)\nu_A(x)), \beta(\nu_A(x) + (1-\alpha)\mu_A(x) + \alpha)\rangle | x \in X\}$ *where* $\alpha + \beta \in [0,1]$.

The intuitionistic fuzzy negations examined in this study are as follows.

Definition 13 ([3]). *Let X be a set and $A \in IFS(X)$*

1. $\neg_1 A = \{\langle \nu_A(x), \mu_A(x)\rangle \mid x \in E\}$,
2. $\neg_4 A = \{\langle \nu_A(x), 1 - \nu_A(x)\rangle \mid x \in E\}$,
3. $\neg_8 A = \{\langle 1 - \mu_A(x), \mu_A(x)\rangle \mid x \in E\}$,
4. $\neg_{20} A = \{\langle x, \nu_A(x), 0\rangle \mid x \in E\}$,
5. $\neg_{25} A = \{\langle 1 - \nu_A(x), 0\rangle \mid x \in E\}$,
6. $\neg^{\varepsilon} A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \varepsilon)\rangle \mid x \in E\}, \varepsilon \in [0,1]$

2 Main results

Theorem 1. *Let X be a set and $A \in IFS(X)$. If $\alpha, \beta \in [0,1]$ and $\beta \geq \alpha$ then*

1. $\neg_4 (E_{\alpha,\beta}(A)) \sqsubseteq E_{\alpha,\beta}(\neg_4 A)$
2. $\neg_4 \left(Z^{\omega}_{\alpha,\beta}(A)\right) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\neg_4 A)$

Proof. (1) If $\alpha \leq \beta$ then

$$\begin{aligned} \alpha - \alpha\beta &\leq \beta - \alpha\beta \Rightarrow \alpha\beta\nu_A(x) + \alpha - \alpha\beta \leq \alpha\beta\nu_A(x) + \beta - \alpha\beta \\ &\Rightarrow \mu_{\neg_4\left(E_{\alpha,\beta}(A)\right)}(x) \leq \mu_{E_{\alpha,\beta}(\neg_4 A)} \end{aligned}$$

and

$$\begin{aligned} 1 - \alpha &\geq 1 - \beta \Rightarrow 1 - \alpha \geq \alpha(1 - \beta) \\ &\Rightarrow 1 - \alpha + \alpha\beta \geq \alpha \\ &\Rightarrow 1 - \alpha + \alpha\beta + \alpha\beta\nu_A(x) \geq \alpha - \alpha\beta\nu_A(x) \\ &\Rightarrow \nu_{\neg_4\left(E_{\alpha,\beta}(A)\right)} \geq \nu_{E_{\alpha,\beta}(\neg_4 A)} \end{aligned}$$

So, $\neg_4 (E_{\alpha,\beta}(A)) \sqsubseteq E_{\alpha,\beta}(\neg_4 A)$.
(2)

$$\begin{aligned} \beta\omega &\geq \alpha\omega \Rightarrow \beta\omega - \alpha\beta\omega \geq \alpha\omega - \alpha\beta\omega \\ &\Rightarrow \alpha\beta\nu_A(x) + \beta\omega - \alpha\beta\omega \geq \alpha\beta\nu_A(x) + \alpha\omega - \alpha\beta\omega \\ &\Rightarrow \mu_{Z^{\omega}_{\alpha,\beta}(\neg_4 A)}(x) \geq \mu_{\neg_4\left(Z^{\omega}_{\alpha,\beta}(A)\right)}(x) \end{aligned}$$

and

$$\begin{aligned} 1+2\alpha\beta\omega &\geq \alpha\beta+2\alpha\omega \\ &\Rightarrow 1-\alpha\beta\nu_A(x)-\alpha\omega+\alpha\beta\omega \geq \alpha\beta-\alpha\beta\nu_A(x)+\alpha\omega-\alpha\beta\omega \\ &\Rightarrow \nu_{\neg_4\left(Z^{\omega}_{\alpha,\beta}(A)\right)}(x) \geq \nu_{Z^{\omega}_{\alpha,\beta}(\neg_4 A)}(x) \end{aligned}$$

Hence, $\neg_4\left(Z^{\omega}_{\alpha,\beta}(A)\right) \sqsubseteq Z^{\omega}_{\alpha,\beta}(\neg_4 A)$. □

Theorem 2. *Let X be a set and $A \in$ IFS(X). If $\alpha, \beta, \alpha+\beta \in [0,1]$ and $\beta \geq \alpha$ then*

1. $S_{\alpha,\beta}(\neg_4 A) \sqsubseteq T_{\alpha,\beta}(\neg_4 A)$
2. $\neg_4(T_{\alpha,\beta}(A)) \sqsubseteq \neg_4(S_{\alpha,\beta}(A))$

Proof. (1) For $\beta \geq \alpha$,

$$\begin{aligned} \beta &\geq \alpha(1-\beta) \Rightarrow \beta+\alpha\beta\nu_A(x) \geq \alpha-\alpha\beta+\alpha\beta\nu_A(x) \\ &\Rightarrow \mu_{S_{\alpha,\beta}(\neg_4 A)}(x) \geq \mu_{T_{\alpha,\beta}(\neg_4 A)}(x) \end{aligned}$$

and

$$\begin{aligned} \alpha &\leq \beta+\alpha\beta \Rightarrow \alpha-\alpha\beta\nu_A(x) \leq \beta-\alpha\beta\nu_A(x)+\alpha\beta \\ &\Rightarrow \nu_{S_{\alpha,\beta}(\neg_4 A)}(x) \leq \nu_{T_{\alpha,\beta}(\neg_4 A)}(x) \end{aligned}$$

Therefore, $S_{\alpha,\beta}(\neg_4 A) \sqsubseteq T_{\alpha,\beta}(\neg_4 A)$.
(2) Straightforward. □

Theorem 3. *Let X be a set and $A \in$ IFS(X). If $\alpha \in [0,1]$ then*

$$E_{\alpha\alpha}(\neg_8 A) \sqsubseteq \neg_8(E_{\alpha\alpha}(A))$$

Theorem 4. *Let X be a set and $A \in$ IFS(X). If $\alpha, \beta, \alpha+\beta \in [0,1]$ and $\beta \geq \alpha$ then*

1. $S_{\alpha,\beta}(\neg_8 A) \sqsubseteq T_{\alpha,\beta}(\neg_8 A)$
2. $\neg_8(T_{\alpha,\beta}(A)) \sqsubseteq \neg_8(S_{\alpha,\beta}(A))$

Proof. (1) For $\beta \geq \alpha$,

$$\begin{aligned} \beta &\geq \alpha \Rightarrow (1+\alpha)\beta \geq \alpha \\ &\Rightarrow \beta+\alpha\beta-\alpha\beta\mu_A(x) \geq \alpha-\alpha\beta\mu_A(x) \\ &\Rightarrow \mu_{T_{\alpha,\beta}(\neg_8 A)}(x) \geq \mu_{S_{\alpha,\beta}(\neg_8 A)}(x) \end{aligned}$$

and

$$\begin{aligned} \beta &\geq \alpha \Rightarrow \beta \geq \alpha(1-\beta) \\ &\Rightarrow \alpha\beta\mu_A(x)+\beta \geq \alpha\beta\mu_A(x)+\alpha(1-\beta) \\ &\Rightarrow \nu_{S_{\alpha,\beta}(\neg_8 A)}(x) \geq \nu_{T_{\alpha,\beta}(\neg_8 A)}(x) \end{aligned}$$

Hence, $S_{\alpha,\beta}(\neg_8 A) \sqsubseteq T_{\alpha,\beta}(\neg_8 A)$.
(2) Straightforward. □

Theorem 5. *Let X be a set and $A \in$ IFS (X) . If $\alpha, \beta \in [0,1]$ and $\alpha \geq \beta$ then*

$$E_{\alpha,\beta}(\neg_{20}A) \sqsubseteq \neg_{20}\left(E_{\alpha,\beta}(A)\right).$$

Proof. If $\alpha \geq \beta$ then

$$\begin{aligned}
\alpha - \alpha\beta &\geq \beta - \alpha\beta \\
&\Rightarrow \alpha\beta\nu_A(x) + \alpha - \alpha\beta \geq \beta - \alpha\beta + \alpha\beta\nu_A(x) \\
&\Rightarrow \mu_{\neg_{20}\left(E_{\alpha,\beta}(A)\right)}(x) \geq \mu_{E_{\alpha,\beta}(\neg_{20}A)}(x)
\end{aligned}$$

and

$$\begin{aligned}
\alpha &\geq \alpha\beta \Rightarrow \alpha - \alpha\beta \geq 0 \\
&\Rightarrow \nu_{E_{\alpha,\beta}(\neg_{20}A)}(x) \geq \nu_{\neg_{20}\left(E_{\alpha,\beta}(A)\right)}(x)
\end{aligned}$$

So, $E_{\alpha,\beta}(\neg_{20}A) \sqsubseteq \neg_{20}\left(E_{\alpha,\beta}(A)\right)$. □

Theorem 6. *Let X be a set and $A \in$ IFS (X) . If $a, \beta \in [0,1]$ then*

1. $B_{\alpha,\beta}(\neg_{20}A) \sqsubseteq \neg_{20}\left(B_{\alpha,\beta}(A)\right)$.
2. $\boxminus_{\alpha,\beta}(\neg_{20}A) \sqsubseteq \neg_{20}\left(\boxminus_{\alpha,\beta}(A)\right)$.

Theorem 7. *Let X be a set and $\alpha, \beta \in [0,1]$. If $\alpha \geq \beta$ then*

$$\neg_{20}\left(\boxminus_{\alpha,\beta}(A)\right) \sqsubseteq \neg_{20}\left(B_{\alpha,\beta}(A)\right).$$

Theorem 8. *Let X be a set and $\alpha, \beta \in [0,1]$. If $\beta \geq \alpha$ then*

1. $E_{\alpha,\beta}(\neg_{25}A) \sqsubseteq \neg_{25}\left(E_{\alpha,\beta}(A)\right)$.
2. $Z^{\omega}_{\alpha,\beta}(\neg_{25}A) \sqsubseteq \neg_{25}\left(Z^{\omega}_{\alpha,\beta}(A)\right)$.

Proof. It is enough to prove the second property. For $\beta \geq \alpha$,

$$\begin{aligned}
1 &\geq \alpha\beta - 2\alpha\beta\omega + \alpha\omega + \beta\omega \\
&\Rightarrow 1 - \alpha\beta\nu_A(x) - \alpha\omega + \alpha\beta\omega \geq \alpha\beta - \alpha\beta\nu_A(x) + \beta\omega - \alpha\beta\omega \\
&\Rightarrow \mu_{\neg_{25}\left(Z^{\omega}_{\alpha,\beta}(A)\right)}(x) \geq \mu_{Z^{\omega}_{\alpha,\beta}(\neg_{25}A)}(x)
\end{aligned}$$

and

$$\begin{aligned}
\alpha\omega &\geq \alpha\beta\omega \Rightarrow \alpha\omega - \alpha\beta\omega \geq 0 \\
&\Rightarrow \nu_{Z^{\omega}_{\alpha,\beta}(\neg_{25}A)}(x) \geq \nu_{\neg_{25}\left(Z^{\omega}_{\alpha,\beta}(A)\right)}(x)
\end{aligned}$$

So, $Z^{\omega}_{\alpha,\beta}(\neg_{25}A) \sqsubseteq \neg_{25}\left(Z^{\omega}_{\alpha,\beta}(A)\right)$. □

Theorem 9. *Let X be a set and $\alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$.*

1. $K^{\omega}_{\alpha,\beta}(\neg^{\varepsilon}A) \sqsubseteq \neg^{\varepsilon}\left(L^{\omega}_{\alpha,\beta}(A)\right)$.
2. $L^{\omega}_{\alpha,\beta}(\neg^{\varepsilon}A) \sqsubseteq \neg^{\varepsilon}\left(K^{\omega}_{\alpha,\beta}(A)\right)$.

Proof. (1) (i) Let $\alpha(1-\beta)+\alpha\beta(1-\omega)\leq\alpha(1-\beta)\nu_A(x)+\alpha(1-\beta)\varepsilon+\alpha\beta(1-\omega)$ then $1-\nu_A(x)\leq\varepsilon$.

$$\begin{aligned}\alpha(1-\beta) &\leq 1\Rightarrow\alpha(1-\beta)(1-\nu_A(x))\leq\varepsilon\\ &\Rightarrow \alpha(1-\beta)\leq\alpha(1-\beta)\nu_A(x)+\varepsilon\\ &\Rightarrow \alpha(1-\beta)+\alpha\beta(1-\omega)\leq\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega)+\varepsilon\end{aligned}$$

(ii) Let $\alpha(1-\beta)+\alpha\beta(1-\omega)\geq\alpha(1-\beta)\nu_A(x)+\alpha(1-\beta)\varepsilon+\alpha\beta(1-\omega)$ then $1-\nu_A(x)\geq\varepsilon$.

$$\begin{aligned}\alpha(1-\beta)\nu_A(x)+\alpha(1-\beta)\varepsilon+\alpha\beta(1-\omega) &= \alpha(1-\beta)(\nu_A(x)+\varepsilon)+\alpha\beta(1-\omega)\\ &\leq \alpha(1-\beta)+\alpha\beta(1-\omega)\\ &= \alpha-\alpha\beta+\alpha\beta-\alpha\beta\omega=\alpha-\alpha\beta\omega\leq 1\end{aligned}$$

and

$$\begin{aligned}&\alpha(1-\beta)\varepsilon\leq\varepsilon\\ &\Rightarrow\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega)+\alpha(1-\beta)\varepsilon\leq\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega)+\varepsilon\end{aligned}$$

So, $\mu_{K^{\omega}_{\alpha,\beta}(\neg^{\varepsilon}A)}(x)\leq\mu_{\neg^{\varepsilon}\left(L^{\omega}_{\alpha,\beta}(A)\right)}(x)$.
On the other hand,
(i) Let $\omega(1-\alpha)\geq\alpha\mu_A(x)-\alpha\varepsilon+\omega(1-\alpha)$ then $\varepsilon\geq\mu_A(x)$.

$$\alpha\mu_A(x)\leq\varepsilon\Rightarrow\alpha\mu_A(x)+\omega(1-\alpha)-\varepsilon\leq\omega(1-\alpha)$$

and $0\leq\omega(1-\alpha)$.
(ii) Let $\omega(1-\alpha)\leq\alpha\mu_A(x)-\alpha\varepsilon+\omega(1-\alpha)$ then $\varepsilon\leq\mu_A(x)$.

$$-\varepsilon\leq-\alpha\varepsilon\Rightarrow\alpha\mu_{A(x)}+\omega(1-\alpha)-\varepsilon\leq\alpha\mu_A(x)-\alpha\varepsilon+\omega(1-\alpha)$$

and $0\leq\alpha\mu_A(x)-\alpha\varepsilon+\omega(1-\alpha)$.
Hence, $\nu_{K^{\omega}_{\alpha,\beta}(\neg^{\varepsilon}A)}(x)\geq\nu_{\neg^{\varepsilon}\left(L^{\omega}_{\alpha,\beta}(A)\right)}(x)$.
(2)
(i) Let $\alpha+\omega(1-\alpha)\leq\alpha\nu_A(x)+\alpha\varepsilon+\omega(1-\alpha)$ then $1-\nu_A(x)\leq\varepsilon$.

$$\alpha(1-\nu_A(x))\leq\varepsilon\Rightarrow\alpha+\omega(1-\alpha)\leq\alpha\nu_A(x)+\omega(1-\alpha)+\varepsilon$$

and $\alpha+\omega(1-\alpha)\leq 1$.
(ii) Let $\alpha+\omega(1-\alpha)\geq\alpha\nu_A(x)+\alpha\varepsilon+\omega(1-\alpha)$ then $1-\nu_A(x)\geq\varepsilon$.

$$\begin{aligned}\alpha\nu_A(x)+\alpha\varepsilon+\omega(1-\alpha) &\leq \alpha(1-\varepsilon)+\alpha\varepsilon+\omega(1-\alpha)\\ &= \alpha+\omega-\alpha\omega\leq 1\end{aligned}$$

and it is clear that

$$\alpha\nu_A(x)+\omega(1-\alpha)+\alpha\varepsilon\leq\alpha\nu_A(x)+\omega(1-\alpha)+\varepsilon$$

So, $\mu_{L^{\omega}_{\alpha,\beta}(\neg^{\varepsilon}A)}\left(x\right) \leq \mu_{\neg^{\varepsilon}\left(K^{\omega}_{\alpha,\beta}(A)\right)}\left(x\right).$
Conversely,
(i) Let $\alpha\beta\left(1-\omega\right) \geq \alpha\left(1-\beta\right)\mu_A\left(x\right) - \alpha\left(1-\beta\right)\varepsilon + \alpha\beta\left(1-\omega\right)$ then $\varepsilon \geq \mu_A\left(x\right).$

$$\begin{aligned} \varepsilon &\geq \alpha\left(1-\beta\right)\mu_A\left(x\right) \\ &\Rightarrow \alpha\beta\left(1-\omega\right) \geq \alpha\left(1-\beta\right)\mu_A\left(x\right) + \alpha\beta\left(1-\omega\right) - \varepsilon \end{aligned}$$

and $\alpha\beta\left(1-\omega\right) \geq 0.$
(ii) Let $\alpha\beta\left(1-\omega\right) \leq \alpha\left(1-\beta\right)\mu_A\left(x\right) - \alpha\left(1-\beta\right)\varepsilon + \alpha\beta\left(1-\omega\right)$ then $\varepsilon \leq \mu_A\left(x\right).$

$$\begin{aligned} \alpha\left(1-\beta\right)\mu_A\left(x\right) - \alpha\left(1-\beta\right)\varepsilon + \alpha\beta\left(1-\omega\right) &\geq \alpha\left(1-\beta\right) + \alpha\beta\left(1-\omega\right) \\ &\geq 0 \end{aligned}$$

and it is clear that

$$\alpha\left(1-\beta\right)\mu_A\left(x\right) - \alpha\left(1-\beta\right)\varepsilon + \alpha\beta\left(1-\omega\right) \geq \alpha\left(1-\beta\right)\mu_A\left(x\right) + \alpha\beta\left(1-\omega\right) - \varepsilon$$

Hence, $\nu_{K^{\omega}_{\alpha,\beta}(\neg^{\varepsilon}A)}\left(x\right) \leq \nu_{\neg^{\varepsilon}\left(L^{\omega}_{\alpha,\beta}(A)\right)}\left(x\right).$ $\square$

Corollary 1. *Let X be a set and $A \in$ IFS$\left(X\right)$. If $\alpha, \beta \in [0,1]$ then*

1. $\alpha = \beta = 1 \Longrightarrow \neg_4\left(E_{\alpha,\beta}(A)\right) = E_{\alpha,\beta}(\neg_4 A)$
2. $\alpha = \beta = 1 \Longrightarrow \neg_{20}E_{\alpha,\beta}(A) = E_{\alpha,\beta}(\neg_{20}A) = \neg_{20}A$

Corollary 2. *Let X be a set and $\alpha, \beta, \alpha+\beta \in [0,1]$.*

1. $T_{\alpha,\beta}(\neg_1 A) = S_{\alpha,\beta}(\neg_1 A)$
2. $T_{\alpha,\beta}(\neg_1 A) = \neg_1\left(S_{\alpha,\beta}(A)\right)$

Corollary 3. *Let X be a set and $\alpha, \beta, \omega \in [0,1]$ then*

1. $\neg_{20}\left(\boxtimes^{\omega}_{\alpha,\beta}(A)\right) = \neg_{20}\left(\neg_{20}\left(B_{\alpha,\beta}(A)\right)\right)$
2. $\neg_{20}\left(Z^{\omega}_{\alpha,\beta}(A)\right) = \neg_{20}\left(\boxplus^{\omega}_{\alpha,\beta}(A)\right)$

Corollary 4. *Let X be a set and $\alpha, \beta, \omega \in [0,1]$ and $\alpha+\beta \leq 1$ then*

1. $L^{\omega}_{\alpha,\beta}(\neg_1 A) = \neg_1\left(K^{\omega}_{\alpha,\beta}(A)\right)$
2. $L^{\omega}_{\alpha,\beta}(\neg_1 A) = K^{\omega}_{\alpha,\beta}(\neg_1 A)$

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