Notes on Intuitionistic Fuzzy Sets
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On intuitionistic fuzzy numbers and compact intervals

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Abstract: In this paper, we investigate the relationship between intuitionistic fuzzy numbers and compact intervals. We give necessary and sufficient conditions for a family of intervals to define an intuitionistic fuzzy number. We also give a method for constructing non-linear

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intuitionistic fuzzy numbers. Our approach provides a new standpoint for understanding the structure of intuitionistic fuzzy numbers and offers a tool that may simplify further analysis in intuitionistic fuzzy calculus, particularly non-linear intuitionistic fuzzy integral and differential equations. Clarifying examples are also presented to illustrate the key findings.
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1 Introduction

Since their introduction by Krassimir Atanassov in 1983 [2], intuitionistic fuzzy sets (IFSs) have provided a natural extension of the fuzzy set theory introduced by Lotfi Zadeh [12]. By combining membership and non-membership functions with the induced hesitation degree, IFSs offer a refined framework for modeling incomplete and inconsistent information. In recent years, this theory has been developed from both theoretical and applied perspectives, including decision-making models [7], optimization problems [6], and fuzzy differential and integral equations [11].

Intuitionistic fuzzy numbers (IFNs), as special forms of IFSs on the real line, play a central role in intuitionistic fuzzy analysis. Their algebraic operations, order relations, and metric structures have been investigated to develop an analytical framework comparable to that of classical fuzzy numbers. Atanassov introduced early foundational aspects of intuitionistic fuzzy sets [2,3], while subsequent studies addressed arithmetic operations and structural properties of intuitionistic fuzzy numbers. In addition, metric and topological structures on spaces of intuitionistic fuzzy numbers were investigated in [9], contributing to the development of functional analysis in the intuitionistic fuzzy setting.

A notable step in this direction was taken by Negoiţă and Ralescu [10], who established a necessary and sufficient condition ensuring that a parameterized family of compact intervals generates a fuzzy number, and in 2013 Barnabas Bede [4, pp. 54–58] provided a detailed proof of this result. In 2015, Melliani and his coauthors established a sufficient condition to construct an intuitionistic fuzzy number from a nested family of compact intervals [9]. This construction has since been employed in the study of intuitionistic fuzzy differential equations and related models [1,5]. However, the existing result does not provide a complete structural characterization as it only yields sufficient conditions. Consequently, the precise relationship between IFNs and ordered families of compact subsets of $\mathbb{R}$ remains only partially clarified. This gap also impacts the systematic treatment of non-linear operations, particularly multiplication and division, whose well-posedness is closely linked to the underlying interval representation. Another issue is that the study of non-linear intuitionistic fuzzy differential and integral equations is still insufficient because of the problem of non-linearity in an intuitionistic fuzzy environment.

To partially handle this situation, many researchers try to extend the operation of multiplication to the intuitionistic fuzzy environment, using the Zadeh extension principle. However, the resulting operation is not distributive with respect to the addition [4, pp. 54–58], which restricts the

investigation of non-linear intuitionistic fuzzy problems. In 2007, Xu and Yager [8] defined an intuitionistic fuzzy number as an ordered pair of non-negative numbers (not functions) with a sum less than unity. Based on this point of view, they built a theoretical framework for intuitionistic fuzzy calculus. Nevertheless, this study does not consider intuitionistic fuzzy numbers in their general form as non-negative real functions, reducing their applicability in real situations.

The aim of this paper is to fill this gap by establishing necessary and sufficient conditions under which a parameterized ordered family of compact intervals defines an intuitionistic fuzzy number. We generalize the construction proposed in [9] and provide a characterization theorem that describes IFNs in terms of interval families. This result strengthens the structural foundations of intuitionistic fuzzy analysis and provides a rigorous framework for the construction of non-linear intuitionistic fuzzy functions employing the α-cuts without extending the operation of multiplication to the intuitionistic fuzzy number space. The study has potential applications to non-linear intuitionistic fuzzy differential and integral equations.

The paper is organized as follows. Section 2 recalls the necessary definitions and preliminary results. Section 3 contains the main characterization theorem and other results with their proofs and presents illustrative examples. Section 4 presents concluding remarks.

2 Preliminaries

We recall only the notions needed in the sequel.

Definition 1 ([2]). *Let S be a given non-empty set, then an intuitionistic fuzzy set A on S is a collection of the form: $A = \{\langle s, p(s), q(s)\rangle |\ s \in S\}$, each element of which is represented by a membership function $p : S \to [0,1]$, and a non-membership function $q : S \to [0,1]$, and for all $s \in S : (p(s) + q(s)) \in [0,1]$.*

If $S = \mathbb{R}$, we put

$$\mathbb{IF}(\mathbb{R}) = \{\langle p, q\rangle : \mathbb{R} \to [0,1]^2 : 0 \leq p(s) + q(s) \leq 1\}$$

Definition 2 ([9]). *An intuitionistic fuzzy number (IFN) is an element $\langle p, q\rangle$ of $\mathbb{IF}(\mathbb{R})$, verifying conditions below:*

1. *$\langle p, q\rangle$ is normal, i.e.: $\exists (s_0, s_1) \in \mathbb{R}^2 : p(s_0) = q(s_1) = 1$;*
2. *p is fuzzy convex, i.e.: $\forall (s_1, s_2, \lambda) \in \mathbb{R}^2 \times [0,1] : p(\lambda s_1 + (1-\lambda)s_2) \geq \min(p(s_1), p(s_2))$;*
3. *q is fuzzy concave, i.e.: $\forall (s_1, s_2, \lambda) \in \mathbb{R}^2 \times [0,1] : q(\lambda s_1 + (1-\lambda)s_2) \leq \max(q(s_1), q(s_2))$;*
4. *p and q are upper semi-continuous, and lower semi-continuous respectively;*
5. *$supp\,\langle p, q\rangle = \mathrm{cl}\Big(\{s \in \mathbb{R} : q(s) < 1\}\Big)$ is bounded.*

We denote the set of all intuitionistic fuzzy numbers on $\mathbb{R}$ by $\mathbb{IF}_1$.

Definition 3. *Let $\alpha \in (0,1]$, the upper and lower α-cuts of an IFN $\langle p,q\rangle$, are defined respectively as $[\langle p,q\rangle]^\alpha = \{\varsigma \in \mathbb{R} : q(\varsigma) \leq 1-\alpha\}$ and $[\langle p,q\rangle]_\alpha = \{\varsigma \in \mathbb{R} : p(\varsigma) \geq \alpha\}$ with $[\langle p,q\rangle]^0 = \mathrm{cl}\Big(\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\}\Big)$ and $[\langle p,q\rangle]_0 = \mathrm{cl}\Big(\{\varsigma \in \mathbb{R} : p(\varsigma) > 0\}\Big)$.*

Remark 1. *The two α-cuts defined above are closed, bounded intervals. For $\langle p,q\rangle \in \mathbb{IF}_1$ and $\alpha \in [0,1]$, we put:*

$$p_l(\alpha) = \inf[\langle p,q\rangle]_\alpha \qquad p_r(\alpha) = \sup[\langle p,q\rangle]_\alpha$$
$$q_l(\alpha) = \inf[\langle p,q\rangle]^\alpha \qquad q_r(\alpha) = \sup[\langle p,q\rangle]^\alpha$$

then $[\langle p,q\rangle]^\alpha = [q_l(\alpha), q_r(\alpha)]$, and $[\langle p,q\rangle]_\alpha = [p_l(\alpha), p_r(\alpha)]$.

Proposition 1 ([9])**.** *Let $\alpha, \beta \in [0,1]$, and $\langle p,q\rangle \in \mathbb{IF}_1$, we have:*

1. $[\langle p,q\rangle]_\alpha \subseteq [\langle p,q\rangle]^\alpha$;
2. $[\langle p,q\rangle]_\alpha$ *and* $[\langle p,q\rangle]^\alpha$ *are non-empty compact convex subsets of* $\mathbb{R}$;
3. *If* $\alpha \leq \beta$, *then* $[\langle p,q\rangle]_\beta \subseteq [\langle p,q\rangle]_\alpha$, *and* $[\langle p,q\rangle]^\beta \subseteq [\langle p,q\rangle]^\alpha$;
4. *If* $\alpha_n \nearrow \alpha$, *then* $[\langle p,q\rangle]_\alpha = \bigcap_n [\langle p,q\rangle]_{\alpha_n}$ *and* $[\langle p,q\rangle]^\alpha = \bigcap_n [\langle p,q\rangle]^{\alpha_n}$.

3 Main results

Proposition 2. *Let $\langle p,q\rangle \in \mathbb{IF}_1$, then $supp\Big(\langle p,q\rangle\Big) = \mathrm{cl}\Big(\bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha\Big)$.*

Proof. It suffices to demonstrate that $\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\} = \bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha$. Let $\varsigma \in \mathbb{R}$ with $q(\varsigma) < 1$, then $\delta = 1 - q(\varsigma) > 0$ and $q(\varsigma) = 1 - \delta \leq 1 - \frac{\delta}{2}$, thus $\varsigma \in [\langle p,q\rangle]^{\frac{\delta}{2}} \subseteq \bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha$, i.e., $\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\} \subseteq \bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha$.

Conversely, if $\varsigma \in \bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha$, then there exists $\alpha_0 \in (0,1]$ such that $\varsigma \in [\langle p,q\rangle]^{\alpha_0}$, therefore $q(\varsigma) \leq 1 - \alpha_0 < 1$, hence $\bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha \subseteq \{\varsigma \in \mathbb{R} : q(\varsigma) < 1\}$. Finally, $\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\} = \bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha$ and then $\mathrm{cl}\Big(\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\}\Big) = \mathrm{cl}\Big(\bigcup_{\alpha>0} [\langle p,q\rangle]^\alpha\Big)$. □

Theorem 1. *Let N_α and N^α be two families of subsets of $\mathbb{R}$ with $\alpha \in [0,1]$, such that:*

1. $N_\alpha \subseteq N^\alpha$;
2. N_α *and* N^α *are non-empty compact convex subsets of* $\mathbb{R}$;
3. *If* $\alpha \leq \beta$, *then* $N_\beta \subseteq N_\alpha$ *and* $N^\beta \subseteq N^\alpha$;
4. *If* $\alpha_n \nearrow \alpha$, *then* $N_\alpha = \bigcap_{n\geq 0} N_{\alpha_n}$ *and* $N^\alpha = \bigcap_{n\geq 0} N^{\alpha_n}$;
5. $\mathrm{cl}\Big(\bigcup_{\alpha>0} N^\alpha\Big) = N^0$.

Then there exists a unique $\langle p,q\rangle \in \mathbb{IF}_1$ such that $[\langle p,q\rangle]_\alpha = N_\alpha$ and $[\langle p,q\rangle]^\alpha = N^\alpha$, for any $\alpha \in [0,1]$.

Proof. Define

$$p(\varsigma) = \begin{cases} 0, & \text{if } \varsigma \notin N_0 \\ \sup\{\alpha \in [0,1] : \varsigma \in N_\alpha\}, & \text{if } \varsigma \in N_0 \end{cases}$$

$$q(\varsigma) = \begin{cases} 1, & \text{if } \varsigma \notin N^0 \\ 1 - \sup\{\alpha \in [0,1] : \varsigma \in N^\alpha\}, & \text{if } \varsigma \in N^0 \end{cases}$$

- It is clear that for $\varsigma \in \mathbb{R}$: $p(\varsigma), q(\varsigma) \in [0,1]$. Moreover:
  - Case 1: $\varsigma \in N_0 (\subseteq N^0)$, then

    $$p(\varsigma) + q(\varsigma) = 1 + \sup\{\alpha \in [0,1] : \varsigma \in N_\alpha\} - \sup\{\alpha \in [0,1] : \varsigma \in N^\alpha\}.$$

    Since $\{\alpha \in [0,1] : \varsigma \in N_\alpha\} \subseteq \{\alpha \in [0,1] : \varsigma \in N^\alpha\}$, then:

    $$0 \leq \sup\{\alpha \in [0,1] : \varsigma \in N_\alpha\} \leq \sup\{\alpha \in [0,1] : \varsigma \in N^\alpha\} \leq 1$$

    Thus $0 \leq 1 + \sup\{\alpha \in [0,1] : \varsigma \in N_\alpha\} - \sup\{\alpha \in [0,1] : \varsigma \in N^\alpha\} \leq 1$, i.e., $0 \leq p(\varsigma) + q(\varsigma) \leq 1$.
  - Case 2: $\varsigma \in N^0 \setminus N_0$, therefore $p(\varsigma) + q(\varsigma) = 1 - \sup\{\alpha \in [0,1] : \varsigma \in N^\alpha\} \in [0,1]$.
  - Case 3: $\varsigma \notin N^0$, therefore $\varsigma \notin N_0$. In this case $p(\varsigma) + q(\varsigma) = 1 \in [0,1]$.

  Finally, $(p(\varsigma) + q(\varsigma)) \in [0,1]$ for any $\varsigma \in \mathbb{R}$.
- Since N_1 is non-empty, there exists $\varsigma_0 \in N_1 \subseteq N_0$, therefore $p(\varsigma_0) = 1$. On the other hand, N^0 is compact; therefore, $\mathbb{R} \setminus N^0 \neq \emptyset$, let $\varsigma_1 \in \mathbb{R} \setminus N^0$, then $q(\varsigma_1) = 1$.
- We demonstrate that q is fuzzy concave and, analogously, p is fuzzy convex. Let $a, b \in \mathbb{R}$:
  - Case 1: $a, b \in N^0$, let $q(a) = 1 - \alpha_a$ and $q(b) = 1 - \alpha_b$. Since $\alpha_a = \sup\{\alpha \in [0,1] : a \in N^\alpha\}$, then there exists $\alpha_n \in [0,1]$ with $a \in N^{\alpha_n}$ for any $n \in \mathbb{N}$, such that $\alpha_n \nearrow \alpha_a$, therefore $a \in \bigcap_{n \geq 0} N^{\alpha_n} = N^{\alpha_a}$, and we demonstrate analogously that $b \in N^{\alpha_b}$.
    - Case 1.1: $q(a) \leq q(b)$, i.e., $\alpha_a \geq \alpha_b$, therefore $a \in N^{\alpha_a} \subseteq N^{\alpha_b}$, then $a, b \in N^{\alpha_b}$, which is convex, then $\lambda a + (1-\lambda) b \in N^{\alpha_b}$ for any $\lambda \in [0,1]$, thus $\sup\{\alpha \in [0,1] : \lambda a + (1-\lambda) b \in N^\alpha\} \geq \alpha_b$, then

      $$1 - \sup\{\alpha \in [0,1] : \lambda a + (1-\lambda) b \in N^\alpha\} \leq 1 - \alpha_b,$$

      i.e., $q(\lambda a + (1-\lambda) b) \leq q(b) = \max(q(a), q(b))$.
    - Case 1.2: $q(a) \geq q(b)$ is similar to case 1.1.
  - Case 2: $a \in N^0$ and $b \notin N^0$, then $q(b) = 1 = \max(q(a), q(b))$, then for any $\lambda \in [0,1]$: $q(\lambda a + (1-\lambda) b) \leq 1 = \max(q(a), q(b))$
  - Case 3: $a \notin N^0$ and $b \in N^0$, this case is similar to Case 2.

– Case 4: $a \notin N^0$ and $b \notin N^0$, then $q(a) = q(b) = 1 \geq q(\lambda a + (1-\lambda)b)$ for any $\lambda \in [0,1]$.

Finally, for any $a, b \in \mathbb{R}$ and $\lambda \in [0,1]$: $q(\lambda a + (1-\lambda)b) \leq \max(q(a), q(b))$

- Now, we demonstrate that for any $\alpha \in (0,1]$ $N_\alpha = \{\varsigma \in \mathbb{R} : p(\varsigma) \geq \alpha\}$ and $N^\alpha = \{\varsigma \in \mathbb{R} : q(\varsigma) \leq 1-\alpha\}$. Let $0 < \beta \leq 1$ and $\varsigma \in N_\beta$ then $p(\varsigma) = \sup\{\alpha \in [0,1] : \varsigma \in N_\alpha\} \geq \beta$ as a result $N_\beta \subseteq \{\varsigma \in \mathbb{R} : p(\varsigma) \geq \beta\}$. Conversely, let $\varsigma \in \mathbb{R}$ with $p(\varsigma) \geq \beta$:

  – Case 1: $p(\varsigma) > \beta$, i.e., $\sup\{\alpha \in [0,1] : \varsigma \in N_\alpha\} > \beta$, then there exists $\tilde{\beta} > \beta$ such that $\varsigma \in N_{\tilde{\beta}} \subseteq N_\beta$. In this case $\{\varsigma \in \mathbb{R} : p(\varsigma) > \beta\} \subseteq N_\beta$.

  – Case 2: $p(\varsigma) := \sup\{\alpha \in [0,1] : \varsigma \in N_\alpha\} = \beta$, then there exists $\beta_n \nearrow \beta$ such that $\varsigma \in N_{\beta_n}$ for any $n \in \mathbb{N}$, thus, by Assumption 4 we have $\varsigma \in \bigcap_{n \geq 0} N_{\beta_n} = N_\beta$.

  We conclude from the two cases that $\{\varsigma \in \mathbb{R} : p(\varsigma) \geq \beta\} \subseteq N_\beta$. Finally, $N_\beta = \{\varsigma \in \mathbb{R} : p(\varsigma) \geq \beta\}$. We demonstrate analogously that $N^\alpha = \{\varsigma \in \mathbb{R} : q(\varsigma) \leq 1-\alpha\}$ for any $\alpha \in (0,1]$.

- Since $\{\varsigma \in \mathbb{R} : p(\varsigma) \geq \alpha\} = N_\alpha$ is closed, then $\mathbb{R} \setminus N_\alpha = \{\varsigma \in \mathbb{R} : p(\varsigma) < \alpha\}$ is open, i.e., p is upper semi-continuous. On the other hand, $\{\varsigma \in \mathbb{R} : q(\varsigma) \leq 1-\alpha\} = N^\alpha$ is closed, then $\mathbb{R} \setminus N^\alpha = \{\varsigma \in \mathbb{R} : q(\varsigma) > 1-\alpha\}$ is open, i.e., q is lower semi-continuous.

- To demonstrate that $supp(\langle p, q\rangle) = \mathrm{cl}\Big(\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\}\Big)$ is bounded, we first demonstrate that $\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\} = \bigcup_{\alpha>0} N^\alpha$. Indeed, if $\varsigma \in \bigcup_{\alpha>0} N^\alpha$, then there exists $\alpha_0 > 0$ such that $\varsigma \in N^{\alpha_0}$, i.e., $q(\varsigma) \leq 1-\alpha_0 < 1$. Conversely, if $\varsigma \in \mathbb{R}$ with $q(\varsigma) < 1$, then there exists $\alpha_0 > 0$ such that $q(\varsigma) \leq 1-\alpha_0$ and $\varsigma \in N^{\alpha_0} \subseteq \bigcup_{\alpha>0} N^\alpha$. Eventually $\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\} = \bigcup_{\alpha>0} N^\alpha$, then $\mathrm{cl}\Big(\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\}\Big) = \mathrm{cl}\Big(\bigcup_{\alpha>0} N^\alpha\Big) = N^0$ (Assumption 5). Since N^0 is compact then $supp(\langle p, q\rangle) = \mathrm{cl}\Big(\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\}\Big)$ is bounded.

All these arguments guarantee that $\langle p, q\rangle$ is an intuitionistic fuzzy number. Uniqueness is trivial because two intuitionistic fuzzy numbers are equal if and only if they have the same α-cuts. □

Proposition 3. *Let $p, q : \mathbb{R} \to [0,1]$ be two functions such that $p(\varsigma) + q(\varsigma) \leq 1$, then p and q define an IFN if and only if*

$$N_\alpha = \{\varsigma \in \mathbb{R} : p(\varsigma) \geq \alpha\} \quad \textit{and} \quad N^\alpha = \{\varsigma \in \mathbb{R} : q(\varsigma) \leq 1-\alpha\}$$

verify the assumptions of Theorem 1 for any $\alpha \in (0,1]$, with

$$N_0 = \mathrm{cl}\{\varsigma \in \mathbb{R} : p(\varsigma) > 0\}, \qquad N^0 = \mathrm{cl}\{\varsigma \in \mathbb{R} : q(\varsigma) < 1\}.$$

Proof. It is a direct consequence of Proposition 1, Proposition 2, and Theorem 1. □

Proposition 4. *Let $\alpha \in [0,1]$, and $I_\alpha = [f_l(\alpha), f_r(\alpha)]$ and $I^\alpha = [g_l(\alpha), g_r(\alpha)]$ be two segments of $\mathbb{R}$. If*

1. *$g_l(\alpha) \leq f_l(\alpha) \leq f_r(\alpha) \leq g_r(\alpha)$ for any $\alpha \in [0,1]$;*
2. *f_l and g_l are non-decreasing and f_r and g_r are non-increasing for $\alpha \in [0,1]$;*
3. *f_l, f_r, g_l and g_r are continuous.*

Then, I_α and I^α define an intuitionistic fuzzy number.

Proof. It is clear that I_α and I^α are non-empty compact convex subsets of $\mathbb{R}$, with $I_\alpha \subseteq I^\alpha$. Moreover, by Assumption 2, if $\alpha \leq \beta$ then $f_l(\alpha) \leq f_l(\beta) \leq f_r(\beta) \leq f_r(\alpha)$ and $g_l(\alpha) \leq g_l(\beta) \leq g_r(\beta) \leq g_r(\alpha)$, which gives $I_\beta \subseteq I_\alpha$ and $I^\beta \subseteq I^\alpha$. Now, let $\alpha_n \nearrow \alpha$, then for any $n \in \mathbb{N}$, $I_\alpha \subseteq I_{\alpha_n}$, therefore $I_\alpha \subseteq \bigcap_n I_{\alpha_n}$, conversely, if $\varsigma \in \bigcap_n I_{\alpha_n}$, then $f_l(\alpha_n) \leq \varsigma \leq f_r(\alpha_n)$ for any $n \in \mathbb{N}$, thus by the continuity of f_l and f_r we conclude that $\varsigma \in I_\alpha$. Finally $I_\alpha = \bigcap_n I_{\alpha_n}$ and analogously we obtain $I^\alpha = \bigcap_n I^{\alpha_n}$. Furthermore, $I^\alpha \subseteq I^0$ for any $\alpha > 0$, thus $\bigcup_{\alpha\in(0,1]} I^\alpha \subseteq I^0$, hence $\mathrm{cl}\Big(\bigcup_{\alpha\in(0,1]} I^\alpha\Big) \subseteq I^0$. On the other hand, we have, $\lim_{\alpha\to 0^+} g_l(\alpha) = g_l(0)$ and $\lim_{\alpha\to 0^+} g_r(\alpha) = g_r(0)$, then every interior point of I^0 belongs to I^α for all sufficiently small positive α, while the endpoints of I^0 are limits of corresponding endpoints of I^α.

Finally, $\mathrm{cl}\Big(\bigcup_{\alpha\in(0,1]} I^\alpha\Big) = I^0$. Theorem 1 ensures that I_α and I^α define an intuitionistic fuzzy number. □

Proposition 5. *Let $\alpha \in [0,1]$, and $\langle p, q\rangle \in \mathbb{IF}_1$ and put $I_\alpha(\langle p, q\rangle) = [f(\alpha)H(p_l(\alpha)), g(\alpha)H(p_r(\alpha))]$ and $I^\alpha(\langle p, q\rangle) = [f(\alpha)H(q_l(\alpha)), g(\alpha)H(q_r(\alpha))]$ as two segments of $\mathbb{R}$. If*

1. *$0 \leq f(\alpha) \leq g(\alpha)$ for any $\alpha \in [0,1]$;*
2. *f and g are, respectively, non-decreasing and non-increasing continuous functions for $\alpha \in [0,1]$;*
3. *$H : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing positive function.*

Then I_α and I^α define an intuitionistic fuzzy number.

Proof. Let $\alpha, \beta \in [0,1]$, with $\alpha \leq \beta$ since H is non-decreasing and $\langle p, q\rangle \in \mathbb{IF}_1$, then

$$0 \leq H(q_l(\alpha)) \leq H(p_l(\alpha)) \leq H(p_r(\alpha)) \leq H(q_r(\alpha))$$
$$0 \leq H(p_l(\alpha)) \leq H(p_l(\beta)) \leq H(p_r(\beta)) \leq H(p_r(\alpha))$$

Moreover,

$$0 \leq f(\alpha) \leq f(\beta) \leq g(\beta) \leq g(\alpha)$$

This guarantees that I_α is a non-empty closed interval of $\mathbb{R}$ and $I_\beta \subseteq I_\alpha$. In the same way, we demonstrate that I^α is a non-empty closed interval of $\mathbb{R}$ and $I^\beta \subseteq I^\alpha$.
We demonstrate straightforwardly that the other assumptions of Theorem 1 hold as in the proof of Proposition 4, which allows us to conclude that $I_\alpha(\langle p, q\rangle)$ and $I^\alpha(\langle p, q\rangle)$ define a unique IFN. □

Remark 2. The case where $g(\alpha) = 0$ for any $\alpha \in [0,1]$ is not important because in this case $I_\alpha(\langle p,q\rangle) = I^\alpha(\langle p,q\rangle) = \{0\}$. For this reason we suppose that there exists $\alpha_0 \in [0,1]$ such that $g(\alpha_0) > 0$ and by the monotonicity of g we can choose $\alpha_0 < 1$. Moreover, if $\langle p,q\rangle, \langle \tilde{p},\tilde{q}\rangle \in \mathbb{IF}_1$, then:

$$\Big[\mathcal{H}(\langle p,q\rangle \oplus \langle \tilde{p},\tilde{q}\rangle)\Big]^\alpha = \Big[\mathcal{H}(\langle u,v\rangle)\Big]^\alpha = \Big[f(\alpha)H(v_l(\alpha)), g(\alpha)H(v_r(\alpha))\Big]$$

Since $\big[\langle p,q\rangle \oplus \langle \tilde{p},\tilde{q}\rangle\big]^\alpha = \big[\langle p,q\rangle\big]^\alpha + \big[\langle \tilde{p},\tilde{q}\rangle\big]^\alpha$ and the three sets are bounded, then $v_l(\alpha) = q_l(\alpha) + \tilde{q}_l(\alpha)$ and $v_r(\alpha) = q_r(\alpha) + \tilde{q}_r(\alpha)$, thus

$$\Big[\mathcal{H}(\langle p,q\rangle \oplus \langle \tilde{p},\tilde{q}\rangle)\Big]^\alpha = \Big[f(\alpha)H(q_l(\alpha) + \tilde{q}_l(\alpha)), g(\alpha)H(q_r(\alpha) + \tilde{q}_r(\alpha))\Big]$$

On the other hand,

$$\begin{aligned}\Big[\mathcal{H}(\langle p,q\rangle) \oplus \mathcal{H}(\langle \tilde{p},\tilde{q}\rangle)\Big]^\alpha &= \Big[\mathcal{H}(\langle p,q\rangle)\Big]^\alpha + \Big[\mathcal{H}(\langle \tilde{p},\tilde{q}\rangle)\Big]^\alpha \\ &= \Big[f(\alpha)H(q_l(\alpha)), g(\alpha)H(q_r(\alpha))\Big] + \Big[f(\alpha)H(\tilde{q}_l(\alpha)), g(\alpha)H(\tilde{q}_r(\alpha))\Big] \\ &= \Big[f(\alpha)\Big(H(q_l(\alpha)) + H(\tilde{q}_l(\alpha))\Big), g(\alpha)\Big(H(q_r(\alpha)) + H(\tilde{q}_r(\alpha))\Big)\Big]\end{aligned}$$

Hence, if H is non-linear, then there exist $x, y, \lambda \in \mathbb{R}$, such that: $H(\lambda\, x+y) \neq \lambda\, H(x)+H(y)$ and if we consider the following triangular IFNs:

$$\begin{aligned}\langle p,q\rangle &= (\lambda\, x - 2, \lambda\, x - 1, \lambda\, x; \lambda\, x - 3, \lambda\, x - 1, \lambda\, x + \frac{\alpha_0}{1-\alpha_0}) \\ \langle \tilde{p},\tilde{q}\rangle &= (y-2, y-1, y; y-3, y-1, y + \frac{\alpha_0}{1-\alpha_0})\end{aligned}$$

We obtain $q_r(\alpha_0) = \lambda\, x$ and $\tilde{q}_r(\alpha_0) = y$, thus,

$$\begin{aligned}\Big[\mathcal{H}(\langle p,q\rangle \oplus \langle \tilde{p},\tilde{q}\rangle)\Big]^{\alpha=\alpha_0} &= \Big[f(\alpha_0)H(q_l(\alpha_0) + \tilde{q}_l(\alpha_0)), g(\alpha_0)H(q_r(\alpha_0) + \tilde{q}_r(\alpha_0))\Big] \\ &= \Big[f(\alpha_0)H(q_l(\alpha_0) + \tilde{q}_l(\alpha_0)), g(\alpha_0)H(\lambda\, x + y)\Big]\end{aligned}$$

and

$$\begin{aligned}\Big[\mathcal{H}(\langle p,q\rangle) \oplus \mathcal{H}(\langle \tilde{p},\tilde{q}\rangle)\Big]^{\alpha=\alpha_0} &= \Big[f(\alpha_0)\Big(H(q_l(\alpha_0)) + H(\tilde{q}_l(\alpha_0))\Big), g(\alpha_0)\Big(H(q_r(\alpha_0)) + H(\tilde{q}_r(\alpha_0))\Big)\Big] \\ &= \Big[f(\alpha_0)\Big(H(q_l(\alpha_0)) + H(\tilde{q}_l(\alpha_0))\Big), g(\alpha_0)\Big(H(\lambda\, x) + H(y)\Big)\Big]\end{aligned}$$

If $H(\lambda\, x) = \lambda\, H(x)$, then $H(\lambda\, x) + H(y) = \lambda\, H(x) + H(y) \neq H(\lambda\, x + y)$, else, if $H(\lambda\, x) \neq \lambda\, H(x)$, then $H(\lambda\, x) + H(y) \neq \lambda\, H(x) + H(y)$, therefore

$$\Big[\mathcal{H}(\langle p,q\rangle) \oplus \mathcal{H}(\langle \tilde{p},\tilde{q}\rangle)\Big]^{\alpha=\alpha_0} \neq \Big[f(\alpha_0)\Big(H(q_l(\alpha_0)) + H(\tilde{q}_l(\alpha_0))\Big), g(\alpha_0)H(\lambda\, x + y)\Big]$$

Thus $\Big[\mathcal{H}(\langle p,q\rangle) \oplus \mathcal{H}(\langle \tilde{p},\tilde{q}\rangle)\Big]^{\alpha=\alpha_0} \neq \Big[\mathcal{H}(\langle p,q\rangle \oplus \langle \tilde{p},\tilde{q}\rangle)\Big]^{\alpha=\alpha_0}$, which means that

$$\mathcal{H}(\langle p,q\rangle \oplus \langle \tilde{p},\tilde{q}\rangle) \neq \mathcal{H}(\langle p,q\rangle) \oplus \mathcal{H}(\langle \tilde{p},\tilde{q}\rangle)$$

Therefore, Proposition 5 presents an approach to construct non-linear intuitionistic fuzzy functions.

Example 1. *For* $\alpha \in [0,1]$, $\alpha \mapsto \frac{\alpha}{4}$, $\alpha \mapsto \frac{2-\alpha}{4}$, *and* $\varsigma \mapsto \tanh(\varsigma)+1$ *satisfy the assumptions of Proposition 5, thus for* $\langle p,q\rangle \in \mathbb{IF}_1$, *the two parameterized intervals*

$$I_\alpha(\langle p,q\rangle) = \left[\frac{\alpha}{4}(\tanh(p_l(\alpha))+1), \frac{2-\alpha}{4}(\tanh(p_r(\alpha))+1)\right]$$
$$I^\alpha(\langle p,q\rangle) = \left[\frac{\alpha}{4}(\tanh(q_l(\alpha))+1), \frac{2-\alpha}{4}(\tanh(q_r(\alpha))+1)\right]$$

define a unique intuitionistic fuzzy function $\mathcal{H}: \mathbb{IF}_1 \to \mathbb{IF}_1$ *with*

$$\Big[\mathcal{H}(\langle p,q\rangle)\Big]_\alpha = \left[\frac{\alpha}{4}(\tanh(p_l(\alpha))+1), \frac{2-\alpha}{4}(\tanh(p_r(\alpha))+1)\right]$$
$$\Big[\mathcal{H}(\langle p,q\rangle)\Big]^\alpha = \left[\frac{\alpha}{4}(\tanh(q_l(\alpha))+1), \frac{2-\alpha}{4}(\tanh(q_r(\alpha))+1)\right]$$

Moreover, since $\varsigma \mapsto \tanh(\varsigma)+1$ *is non-linear, then* $\mathcal{H}$ *is non-linear also.*

Example 2. *According to Proposition 5, the function* $\varsigma \mapsto \frac{\exp(\varsigma)}{1+\exp(\varsigma)}$ *induces a non-linear intuitionistic fuzzy function* $\mathcal{H}: \mathbb{IF}_1 \to \mathbb{IF}_1$ *defined for any* $\langle p,q\rangle \in \mathbb{IF}_1$ *as*

$$[\mathcal{H}(\langle p,q\rangle)]_\alpha = \left[\frac{\exp(p_l(\alpha))}{1+\exp(p_l(\alpha))}, \frac{\exp(p_r(\alpha))}{1+\exp(p_r(\alpha))}\right]$$
$$[\mathcal{H}(\langle p,q\rangle)]^\alpha = \left[\frac{\exp(q_l(\alpha))}{1+\exp(q_l(\alpha))}, \frac{\exp(q_r(\alpha))}{1+\exp(q_r(\alpha))}\right]$$

4 Conclusion

In this paper, we established the necessary and sufficient conditions for a family of intervals to be exactly the α-cuts of an IFN. This study inspired us to construct non-linear intuitionistic fuzzy functions, which can open the door to further analytical investigations, especially in intuitionistic fuzzy differential and integral equations.

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