Notes on Intuitionistic Fuzzy Sets
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A brief note on intuitionistic fuzzy operators

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Abstract: The research on intuitionistic fuzzy sets gives an extra dimension in mathematics. It has been observed that some operators including the modal operators have very interesting properties in intuitionistic fuzzy sets. More important properties are investigated in this paper and new findings are obtained and proved.
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1 Introduction

The notion of modal operators were first introduced by Atanassov [1] in 1986. Modal operators ($\Box, \Diamond$) defined over the set of all intuitionistic fuzzy sets that convert every intuitionistic fuzzy set into a fuzzy set. Atanassov [2] also introduced the operators ($\boxplus, \boxtimes$) in intuitionistic fuzzy set. More relations on these operators are regorously studied in [3] , [4] , [5] , [6] , [7] and [8]. The second extension of the operators $\boxplus$ and $\boxtimes$ are introduced by K. Dencheva [9]. The third extensions of the above operators are derived by Atanassov [4]. The natural extension of both the latest operators denoted by $\boxdot_{\alpha,\beta,\gamma,\delta}$ is also defined by Atanassov [5]. In this paper, we try to investigate various properties of this intuitionistic fuzzy operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ and establish some comperison with the help of four basic operations $\cup, \cap, \oplus$ and $\otimes$.

2 Preliminaries

Let a set X be fixed. An intuitionistic fuzzy set A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where the functions $\mu_A, \nu_A : x \to [0, 1]$ define, respectively, the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X$,$0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Furthermore, we have $\pi_A(x)$= 1-$\mu_A(x)$-$\nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. Then, $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0, 1]$ that is $\pi_A : x \to [0, 1]$ and $0 \leq \pi_A(x) \leq 1$ for every $x \in X$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Obviously, for every ordinary fuzzy set $\pi_A(x) = 0$ for each $x \in X$ and these sets have the form: $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$.

Definition 2.1 [1] Let A, B be two IFSs in X. The basic operations are defined as follows:

1. $A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$,
2. $A = B \Longleftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x), \forall x \in X$,
3. $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$,
4. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,
5. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,
6. $A \oplus B = \{\langle x, (\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)), \nu_A(x)\nu_B(x)\rangle : x \in X\}$,
7. $A \otimes B = \{\langle x, \mu_A(x)\mu_B(x), (\nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x)),\rangle : x \in X\}$,
8. $A - B = \{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\nu_A(x), \mu_B(x))\rangle : x \in X\}$,
9. $A \triangle B = \{\langle x, \max[\min(\mu_A(x), \nu_B(x)), \min(\mu_B(x), \nu_A(x))],$
   $\min[\max(\nu_A(x), \mu_B(x)), \max(\nu_B(x), \mu_A(x))]\rangle : x \in X\}$,
10. $A \times B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle : x \in X\}$.

Definition 2.2 [2] Let X be a nonempty set. If A is an IFS drawn from X, then the modal operators which are also termed as necessity and possibility operators can be defined as:

1. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$,
2. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$.

For a proper IFS, $\Box A \subset A \subset \Diamond A$ and $\Box A \neq A \neq \Diamond A$.

Definition 2.3 [2] Let X be a nonempty set. If A is an IFS drawn from X, then,

1. $\boxplus A = \{\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\rangle : x \in X\}$,
2. $\boxtimes A = \{\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\rangle : x \in X\}$.

For a proper IFS, $\boxplus A \subset A \subset \boxtimes A$ and $\boxplus A \neq A \neq \boxtimes A$.

Definition 2.4 [2] Let $\alpha \in [0, 1]$ and let A be an IFS. Then the first extension of the operators $\boxplus$ and $\boxtimes$ can be defined as:

1. $\boxplus_\alpha A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha\rangle : x \in X\}$,
2. $\boxtimes_\alpha A = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x)\rangle : x \in X\}$.

Definition 2.5 [9] Let $\alpha, \beta \in [0, 1]$ and let A be an IFS. Then the second extension of the operators $\boxplus$ and $\boxtimes$ can be defined as:

1. $\boxplus_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta\rangle : x \in X\}$,
2. $\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x)\rangle : x \in X\}$, where $\alpha, \beta, \alpha + \beta \in [0, 1]$.

Definition 2.6 [4] Let $\alpha, \beta, \gamma \in [0, 1]$ and let A be an IFS. Then the third extensions of the operators $\boxplus$ and $\boxtimes$ can be defined as:

1. $\boxplus_{\alpha,\beta,\gamma} A = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma\rangle : x \in X\}$,
2. $\boxtimes_{\alpha,\beta,\gamma} A = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x)\rangle : x \in X\}$, where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha + \beta) + \gamma \leq 1$.

Definition 2.7 [5] Let $\alpha, \beta, \gamma, \delta \in [0, 1]$ and let A be an IFS. Then the natural extension of the operators $\boxplus_{\alpha,\beta,\gamma}$ and $\boxtimes_{\alpha,\beta,\gamma}$ can be defined as:

$$\boxdot_{\alpha,\beta,\gamma,\delta} A = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta\rangle : x \in X\},$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\max(\alpha + \beta) + \gamma + \delta \leq 1$.

3 Main results

Throughout this paper, intuitionistic fuzzy set and fuzzy set are denoted by IFS and FS respectively.

Theorem 3.1 *For every IFS A and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$ for which* $\max(\alpha + \beta) + \gamma + \delta \leq 1$.

1. $\boxdot_{0.5,0.5,0,0.5} \Box A = \Box \boxplus A$,
2. $\boxdot_{0.5,0.5,0.5,0} \Diamond A = \Diamond \boxtimes A$,
3. $\boxdot_{\alpha,\alpha,0,1-\alpha} \Box A = \Box \boxplus_\alpha A$,
4. $\boxdot_{\alpha,\alpha,1-\alpha,0} \Diamond A = \Diamond \boxtimes_\alpha A$,
5. $\boxdot_{\alpha,\alpha,0,1-\alpha} \Box A = \Box \boxplus_\alpha A$,
6. $\boxdot_{\alpha,\alpha,1-\alpha,0} \Diamond A = \Diamond \boxtimes_\alpha A$.

Proof. Let us prove the statements in items 1. and 3.

1. Let us have $A = \langle \mu_A(x), \nu_A(x) \rangle$ be an IFS.
   Then $\Box A = \langle \mu_A(x), 1 - \mu_A(x) \rangle$
   Now, L.H.S= $\boxdot_{0.5,0.5,0,0.5} \Box A = \langle \frac{\mu_A(x)}{2}, \frac{(1-\mu_A(x))}{2} + \frac{1}{2} \rangle = \langle \frac{\mu_A(x)}{2}, 1 - \frac{\mu_A(x)}{2} \rangle$
   Again, R.H.S $= \Box \boxplus A = \Box \langle \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2} \rangle = \langle \frac{\mu_A(x)}{2}, 1 - \frac{\mu_A(x)}{2} \rangle$.
   Hence the proof.

3. L.H.S $= \boxdot_{\alpha,\alpha,0,1-\alpha} \Box A = \langle \alpha\mu_A(x), 1 - \alpha\mu_A(x) \rangle$
   Again, R.H.S $= \Box \boxplus_\alpha A = \Box \langle \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha \rangle = \langle \alpha\mu_A(x), 1 - \alpha\mu_A(x) \rangle$.
   Hence the proof.

Similarly, the other statements can be proved. □

Theorem 3.2 *For every IFS A and for every $\alpha, \beta \in [0, 1]$ where $\alpha + \beta = 1$.*

1. $\boxdot_{\alpha,\alpha,0,\beta} \Box A = \Box \boxplus_{\alpha,\beta} A$,
2. $\boxdot_{\alpha,\alpha,\beta,0} \Diamond A = \Diamond \boxtimes_{\alpha,\beta} A$.

Proof. Let us prove the statement in item 1.
Let $A = \langle \mu_A(x), \nu_A(x) \rangle$ be an IFS. Then $\Box A = \langle \mu_A(x), 1 - \mu_A(x) \rangle$. Now,

$$\begin{aligned} L.H.S. &= \boxdot_{\alpha,\alpha,0,\beta} \Box A \\ &= \langle \alpha\mu_A(x), \alpha(1 - \mu_A(x)) + \beta \rangle \\ &= \langle \alpha\mu_A(x), \alpha(1 - \mu_A(x)) + 1 - \alpha \rangle \\ &= \langle \alpha\mu_A(x), 1 - \alpha\mu_A(x) \rangle. \end{aligned}$$

Again,

$$\begin{aligned} R.H.S. &= \Box \boxplus_{\alpha,\beta} A = \Box \langle \alpha\mu_A(x), \alpha\nu_A(x) + \beta \rangle \\ &= \langle \alpha\mu_A(x), 1 - \alpha\mu_A(x) \rangle. \end{aligned}$$

Hence the proof.
Similarly, the other statement can be proved. □

Remark 3.3 *For every IFS A and for every $\alpha, \beta \in [0, 1]$ where $\alpha + \beta < 1$, we have:*

1. $\boxdot_{\alpha,\alpha,0,\beta} \Box A \neq \Box \boxplus_{\alpha,\beta} A$,

2. $\boxdot_{\alpha,\alpha,\beta,0} \Diamond A \neq \Diamond \boxtimes_{\alpha,\beta} A$.

Counterexample. *Let $A = \langle 0.7, 0.2, 0.1 \rangle$ and $\alpha = 0.5, \beta = 0.3$. Then we have,*

$$\boxdot_{0.5,0.5,0,0.3} \Box \langle 0.7, 0.2 \rangle = \langle 0.35, 0.45 \rangle$$

and

$$\Box \boxplus_{0.5,0.3} \langle 0.7, 0.2 \rangle = \langle 0.35, 0.65 \rangle.$$

Theorem 3.4 *For every two IFSs A and B and for every $\alpha, \beta, \gamma, \delta \in [0,1]$ with* $\max(\alpha+\beta)+\gamma+\delta \leq 1$,

1. $\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \cup B) = \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \cup \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B)$,

2. $\boxdot_{\alpha,\beta,\gamma,\delta}\Diamond(A \cup B) = \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond A) \cup \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond B)$,

3. $\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \cap B) = \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \cap \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B)$,

4. $\boxdot_{\alpha,\beta,\gamma,\delta}\Diamond(A \cap B) = \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond A) \cap \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond B)$.

Proof. Let us prove the statement in item 1.

Let $A = \langle x, \mu_A, \nu_A \rangle$ and $B = \langle x, \mu_B, \nu_B \rangle$ be two IFSs. Then for every $\alpha, \beta, \gamma, \delta \in [0,1]$ with $\max(\alpha+\beta)+\gamma+\delta \leq 1$, we have

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \cup B) &= \boxdot_{\alpha,\beta,\gamma,\delta}\langle \max(\mu_A,\mu_B), 1-\max(\mu_A,\mu_B)\rangle \\ &= \langle \alpha\max(\mu_A,\mu_B)+\gamma, \beta[1-\max(\mu_A,\mu_B)]+\delta\rangle.\end{aligned}$$

Again,

$$\begin{aligned}\boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \cup \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B) &= \boxdot_{\alpha,\beta,\gamma,\delta}\langle \mu_A, 1-\mu_A\rangle \cup \boxdot_{\alpha,\beta,\gamma,\delta}\langle \mu_B, 1-\mu_B\rangle \\ &= \langle(\alpha\mu_A+\gamma, \beta(1-\mu_A)+\delta)\rangle \cup \langle(\alpha\mu_B+\gamma, \beta(1-\mu_B)+\delta)\rangle \\ &= \langle \max(\alpha\mu_A+\gamma, \alpha\mu_B+\gamma), \min(\beta(1-\mu_A)+\delta, \beta(1-\mu_B)+\delta)\rangle \\ &= \langle \alpha\max(\mu_A,\mu_B)+\gamma, \beta\min(1-\mu_A, 1-\mu_B)+\delta\rangle \\ &= \langle \alpha\max(\mu_A,\mu_B)+\gamma, \beta[1-\max(\mu_A,\mu_B)]+\delta\rangle.\end{aligned}$$

Hence

$$\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \cup B) = \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \cup \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B).$$

Similarly we can prove the other parts of the Theorem. □

Theorem 3.5 *For every two IFSs A and B and for every α, β, γ, δ $\in$ $[0,1]$ with* $\max(\alpha+\beta)+\gamma+\delta \leq 1$,

1. $\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \oplus B) \subset \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \oplus \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B)$,

2. $\boxdot_{\alpha,\beta,\gamma,\delta}\Diamond(A \oplus B) \subset \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond A) \oplus \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond B)$,

3. $\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \otimes B) \supset \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \otimes \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B)$,

4. $\boxdot_{\alpha,\beta,\gamma,\delta}\Diamond(A \otimes B) \supset \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond A) \otimes \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond B)$,

5. $\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A - B) \subset \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) - \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B)$,

6. $\boxdot_{\alpha,\beta,\gamma,\delta}\Diamond(A - B) \subset \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond A) - \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond B)$,

7. $\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \triangle B) \subset \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \triangle \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B)$,

8. $\boxdot_{\alpha,\beta,\gamma,\delta}\Diamond(A \triangle B) \subset \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond A) \triangle \boxdot_{\alpha,\beta,\gamma,\delta}(\Diamond B)$.

Proof. 1. Let $A = \langle x, \mu_A, \nu_A \rangle$ and $B = \langle x, \mu_B, \nu_B \rangle$ be two IFSs.

$$\begin{aligned} L.H.S. &= \boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \oplus B) \\ &= \boxdot_{\alpha,\beta,\gamma,\delta}\langle \mu_A + \mu_B - \mu_A\mu_B, 1 - (\mu_A + \mu_B - \mu_A\mu_B)\rangle \\ &= \langle \alpha(\mu_A + \mu_B - \mu_A\mu_B) + \gamma, \beta[1 - (\mu_A + \mu_B - \mu_A\mu_B)] + \delta \rangle \end{aligned} \tag{1}$$

Respectively,

$$\begin{aligned} R.H.S. &= \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \oplus \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B) \\ &= \boxdot_{\alpha,\beta,\gamma,\delta}\langle \mu_A, 1 - \mu_A \rangle \oplus \boxdot_{\alpha,\beta,\gamma,\delta}\langle \mu_B, 1 - \mu_B \rangle \\ &= \langle \alpha\mu_A + \gamma, \beta(1 - \mu_A) + \delta \rangle \oplus \langle \alpha\mu_B + \gamma, \beta(1 - \mu_B) + \delta \rangle \\ &= \langle \alpha\mu_A + \gamma + \alpha\mu_B + \gamma - (\alpha\mu_A + \gamma)(\alpha\mu_B + \gamma), (\beta(1 - \mu_A) + \delta)(\beta(1 - \mu_B) + \delta)\rangle \\ &= \langle \alpha(\mu_A + \mu_B - \alpha\mu_A\mu_B - \gamma\mu_A - \gamma\mu_B) + 2\gamma - \gamma^2, \\ &\quad \beta^2(1 - \mu_A - \mu_B + \mu_A\mu_B) - \beta\delta(\mu_A + \mu_B) + 2\beta\delta + \delta^2\rangle. \end{aligned} \tag{2}$$

Now, from (1) and (2), we see that,
$(\mu_A + \mu_B - \mu_A\mu_B) \leq (\mu_A + \mu_B - \alpha\mu_A\mu_B)$, as $0 \leq \alpha \leq 1$
$\Rightarrow \alpha(\mu_A + \mu_B - \mu_A\mu_B) \leq \alpha(\mu_A + \mu_B - \alpha\mu_A\mu_B)$
$\Rightarrow \alpha(\mu_A + \mu_B - \mu_A\mu_B) \leq \alpha(\mu_A + \mu_B - \alpha\mu_A\mu_B) - \alpha\gamma\mu_A - \alpha\gamma\mu_B$
$\Rightarrow \alpha(\mu_A + \mu_B - \mu_A\mu_B) \leq \alpha(\mu_A + \mu_B - \alpha\mu_A\mu_B - \gamma\mu_A - \gamma\mu_B)$
$\Rightarrow \alpha(\mu_A + \mu_B - \mu_A\mu_B) + \gamma \leq \alpha(\mu_A + \mu_B - \alpha\mu_A\mu_B - \gamma\mu_A - \gamma\mu_B) + \gamma$
$\Rightarrow \alpha(\mu_A + \mu_B - \mu_A\mu_B) + \gamma \leq \alpha(\mu_A + \mu_B - \alpha\mu_A\mu_B - \gamma\mu_A - \gamma\mu_B) + \gamma + (\gamma - \gamma^2)$, as $\gamma - \gamma^2 > 0$
$\Rightarrow \alpha(\mu_A + \mu_B - \mu_A\mu_B) + \gamma \leq \alpha(\mu_A + \mu_B - \alpha\mu_A\mu_B - \gamma\mu_A - \gamma\mu_B) + 2\gamma - \gamma^2$.

Similarly it can be proved that

$$\beta[1 - (\mu_A + \mu_B - \mu_A\mu_B)] + \delta\rangle \geq \beta^2(1 - \mu_A - \mu_B + \mu_A\mu_B) - \beta\delta(\mu_A + \mu_B) + 2\beta\delta + \delta^2\rangle.$$

Hence,

$$\boxdot_{\alpha,\beta,\gamma,\delta}\Box(A \oplus B) \subset \boxdot_{\alpha,\beta,\gamma,\delta}(\Box A) \oplus \boxdot_{\alpha,\beta,\gamma,\delta}(\Box B).$$

The other statements can be proved in a similar manner. □

Theorem 3.6 em For every two IFSs A and B and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$ with $\max(\alpha+\beta)+\gamma + \delta \leq 1$,

1. $\boxdot_{0,0,0,0}A = \langle 0, 0\rangle = \boxplus_{0,0,0}A = \boxtimes_{0,0,0}A$,
2. $\boxdot_{0,0,0,1}A = \langle 0, 1\rangle = \boxplus_{0,0,1}A$,
3. $\boxdot_{0,0,1,0}A = \langle 1, 0\rangle = \boxtimes_{0,0,1}A$,
4. $\boxdot_{0,1,0,0}A = \langle 0, \nu_A\rangle = \boxplus_{0,1,0}A$,
5. $\boxdot_{1,0,0,0}A = \langle \mu_A, 0\rangle = \boxtimes_{1,0,0}A$,
6. $\boxdot_{1,1,0,0}A = \langle \mu_A, \nu_A\rangle = \boxplus_{1,1,0}A = \boxtimes_{1,1,0}A$.

Proof Let us prove the statement in item 5.

We have $\boxdot_{1,0,0,0}A = \langle \mu_A, 0\rangle$. Again $\boxtimes_{1,0,0}A = \langle \mu_A, 0\rangle$. Hence the proof.
Similarly, the other statements can be proved. □

4 Some observations

Let $A = \langle x, \mu_A, \nu_A \rangle$ be any IFS. Then we constuct the following tables.

Table 1. Composition table under union.

$\cup$	$\boxdot_{0,0,0,0}A$	$\boxdot_{0,0,0,1}A$	$\boxdot_{0,0,1,0}A$	$\boxdot_{0,1,0,0}A$	$\boxdot_{1,0,0,0}A$	$\boxdot_{1,1,0,0}A$
$\boxdot_{0,0,0,0}A$	$\langle 0,0 \rangle$	$\langle 0,0 \rangle$	$\langle 1,0 \rangle$	$\langle 0,0 \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,0 \rangle$
$\boxdot_{0,0,0,1}A$	$\langle 0,0 \rangle$	$\langle 0,1 \rangle$	$\langle 1,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$
$\boxdot_{0,0,1,0}A$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$
$\boxdot_{0,1,0,0}A$	$\langle 0,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle 1,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$
$\boxdot_{1,0,0,0}A$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,0 \rangle$	$\langle 1,0 \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,0 \rangle$
$\boxdot_{1,1,0,0}A$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$	$\langle 1,0 \rangle$	$\langle \mu_A,\nu_A \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$

Table 2. Composition table under intersection.

$\cap$	$\boxdot_{0,0,0,0}A$	$\boxdot_{0,0,0,1}A$	$\boxdot_{0,0,1,0}A$	$\boxdot_{0,1,0,0}A$	$\boxdot_{1,0,0,0}A$	$\boxdot_{1,1,0,0}A$
$\boxdot_{0,0,0,1}A$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$
$\boxdot_{0,0,1,0}A$	$\langle 0,0 \rangle$	$\langle 0,1 \rangle$	$\langle 1,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$
$\boxdot_{0,1,0,0}A$	$\langle 0,\nu_A \rangle$	$\langle 0,1 \rangle$	$\langle 0,\nu_A \rangle$	$\langle 0,\nu_A \rangle$	$\langle 0,\nu_A \rangle$	$\langle 0,\nu_A \rangle$
$\boxdot_{1,0,0,0}A$	$\langle 0,0 \rangle$	$\langle 0,1 \rangle$	$\langle \mu_A,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$
$\boxdot_{1,1,0,0}A$	$\langle 0,\nu_A \rangle$	$\langle 0,1 \rangle$	$\langle \mu_A,\nu_A \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A,\nu_A \rangle$	$\langle \mu_A,\nu_A \rangle$

Table 3. Composition table under addition.

$\oplus$	$\boxdot_{0,0,0,0}A$	$\boxdot_{0,0,0,1}A$	$\boxdot_{0,0,1,0}A$	$\boxdot_{0,1,0,0}A$	$\boxdot_{1,0,0,0}A$	$\boxdot_{1,1,0,0}A$
$\boxdot_{0,0,0,0}A$	$\langle 0,0 \rangle$	$\langle 0,0 \rangle$	$\langle 1,0 \rangle$	$\langle 0,0 \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,0 \rangle$
$\boxdot_{0,0,0,1}A$	$\langle 0,0 \rangle$	$\langle 0,1 \rangle$	$\langle 1,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$
$\boxdot_{0,0,1,0}A$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$	$\langle 1,0 \rangle$
$\boxdot_{0,1,0,0}A$	$\langle 0,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle 1,0 \rangle$	$\langle 0,\nu_A^2 \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A^2 \rangle$
$\boxdot_{1,0,0,0}A$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,0 \rangle$	$\langle 1,0 \rangle$	$\langle \mu_A,0 \rangle$	$\langle 2\mu_A-\mu_A^2,0 \rangle$	$\langle 2\mu_A-\mu_A^2,0 \rangle$
$\boxdot_{1,1,0,0}A$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$	$\langle 1,0 \rangle$	$\langle \mu_A,\nu_A^2 \rangle$	$\langle 2\mu_A-\mu_A^2,0 \rangle$	$\langle 2\mu_A-\mu_A^2,\nu_A^2 \rangle$

Table 4. Composition table under multiplication.

$\otimes$	$\boxdot_{0,0,0,0}A$	$\boxdot_{0,0,0,1}A$	$\boxdot_{0,0,1,0}A$	$\boxdot_{0,1,0,0}A$	$\boxdot_{1,0,0,0}A$	$\boxdot_{1,1,0,0}A$
$\boxdot_{0,0,0,0}A$	$\langle 0,0 \rangle$	$\langle 0,1 \rangle$	$\langle 0,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle 0,0 \rangle$	$\langle 0,\nu_A \rangle$
$\boxdot_{0,0,0,1}A$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$	$\langle 0,1 \rangle$
$\boxdot_{0,0,1,0}A$	$\langle 0,0 \rangle$	$\langle 0,1 \rangle$	$\langle 1,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A,0 \rangle$	$\langle \mu_A,\nu_A \rangle$
$\boxdot_{0,1,0,0}A$	$\langle 0,\nu_A \rangle$	$\langle 0,1 \rangle$	$\langle 0,\nu_A \rangle$	$\langle 0,2\nu_A-\nu_A^2 \rangle$	$\langle 0,\nu_A \rangle$	$\langle 0,2\nu_A-\nu_A^2 \rangle$
$\boxdot_{1,0,0,0}A$	$\langle 0,0 \rangle$	$\langle 0,1 \rangle$	$\langle \mu_A,0 \rangle$	$\langle 0,\nu_A \rangle$	$\langle \mu_A^2,0 \rangle$	$\langle \mu_A^2,\nu_A \rangle$
$\boxdot_{1,1,0,0}A$	$\langle 0,\nu_A \rangle$	$\langle 0,1 \rangle$	$\langle \mu_A,\nu_A \rangle$	$\langle 0,2\nu_A-\nu_A^2 \rangle$	$\langle \mu_A^2,\nu_A \rangle$	$\langle \mu_A^2,2\nu_A-\nu_A^2 \rangle$

We may quickly review some of the properties for the IFS A using the tables above. Some novel comparisons employing the fundamental operations $\cup, \cap, \oplus$, and $\otimes$ are derived by utilizing the properties of this operator:

1. $\boxdot_{0,0,1,0}A \cup \boxdot_{0,0,0,0}A = \boxdot_{0,0,1,0}A \cup \boxdot_{0,0,0,1}A = \boxdot_{0,0,1,0}A \cup \boxdot_{0,0,1,0}A = \boxdot_{0,0,1,0}A \cup \boxdot_{0,1,0,0}A$ $= \boxdot_{0,0,1,0}A \cup \boxdot_{1,0,0,0}A = \boxdot_{0,0,1,0}A \cup \boxdot_{1,1,0,0}A = \langle 1, 0\rangle$.

2. $\boxdot_{0,0,0,1}A \cap \boxdot_{0,0,0,0}A = \boxdot_{0,0,0,1}A \cap \boxdot_{0,0,0,1}A = \boxdot_{0,0,0,1}A \cap \boxdot_{0,0,1,0}A = \boxdot_{0,0,0,1}A \cap \boxdot_{0,1,0,0}A$ $= \boxdot_{0,0,0,1}A \cap \boxdot_{1,0,0,0}A = \boxdot_{0,0,0,1}A \cap \boxdot_{1,1,0,0}A = \langle 0, 1\rangle$.

3. $\boxdot_{0,0,1,0}A \oplus \boxdot_{0,0,0,0}A = \boxdot_{0,0,1,0}A \oplus \boxdot_{0,0,0,1}A = \boxdot_{0,0,1,0}A \oplus \boxdot_{0,0,1,0}A$ $= \boxdot_{0,0,1,0}A \oplus \boxdot_{0,1,0,0}A = \boxdot_{0,0,1,0}A \oplus \boxdot_{1,0,0,0}A = \boxdot_{0,0,1,0}A \oplus \boxdot_{1,1,0,0}A = \langle 1, 0\rangle$.

4. $\boxdot_{0,0,0,1}A \otimes \boxdot_{0,0,0,0}A = \boxdot_{0,0,0,1}A \otimes \boxdot_{0,0,0,1}A = \boxdot_{0,0,0,1}A \otimes \boxdot_{0,0,1,0}A$ $= \boxdot_{0,0,0,1}A \otimes \boxdot_{0,1,0,0}A = \boxdot_{0,0,0,1}A \otimes \boxdot_{1,0,0,0}A = \boxdot_{0,0,0,1}A \otimes \boxdot_{1,1,0,0}A = \langle 0, 1\rangle$.

5. $\boxdot_{0,0,0,1}A \cup \boxdot_{1,1,0,0}A = \boxdot_{0,1,0,0}A \cup \boxdot_{1,1,0,0}A = \boxdot_{1,1,0,0}A \cup \boxdot_{0,0,0,1}A = \boxdot_{1,1,0,0}A \cup \boxdot_{0,1,0,0}A$ $= \boxdot_{1,1,0,0}A \cup \boxdot_{1,1,0,0}A = \boxdot_{0,0,1,0}A \cap \boxdot_{1,1,0,0}A = \boxdot_{1,0,0,0}A \cap \boxdot_{1,1,0,0}A$ $= \boxdot_{1,1,0,0}A \cap \boxdot_{0,0,1,0}A = \boxdot_{1,1,0,0}A \cap \boxdot_{1,0,0,0}A = \boxdot_{1,1,0,0}A \cap \boxdot_{1,1,0,0}A$ $= \boxdot_{0,0,0,1}A \oplus \boxdot_{1,1,0,0}A = \boxdot_{1,1,0,0}A \oplus \boxdot_{0,0,0,1}A = \boxdot_{0,0,1,0}A \otimes \boxdot_{1,1,0,0}A$ $= \boxdot_{1,1,0,0}A \otimes \boxdot_{0,0,1,0}A = \langle \mu_A, \nu_A\rangle$.

5 Conclusion

Some new interesting properties of $\boxdot_{\alpha,\beta,\gamma,\delta}$ have been investigated in this paper. Using the features of this operator, some new equalities have been obtained and some comperisons using the basic operations $\cup, \cap, \boxplus$ and $\boxtimes$ are done.These equalities are very useful because they are shorter and more practical. These equalities could be made use in many application areas of intuitioistic fuzzy operators and will provide more practical solutions.

References

[1] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[2] Atanassov, K. T. (1997). Some operators on intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 3(4), 28–33.

[3] Atanassov, K. T. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Springer Physica-Verlag, Berlin.

[4] Atanassov, K. T. (2005). On one type of intuitionistic fuzzy modal operators. *Notes on Intuitionistic Fuzzy Sets*, 11(5), 24–28.

[5] Atanassov, K. T. (2006). The most general form of one type of intuitionistic fuzzy modal operators. *Notes on Intuitionistic Fuzzy Sets*, 12(2), 36–38.

[6] Bhattacharya, J. (2021). Some results on certain properties of intuitionistic fuzzy sets. *Journal of Fuzzy Extension and Applications*, 2(4), 377–387.

[7] Bhattacharya, J. (2022). Several significant equalities on intuitionistic fuzzy operators. *Notes on Intuitionistic Fuzzy Sets*, 28(2), 132–148.

[8] Bhattacharya, J. (2023). Some new results on intuitionistic fuzzy operators. *Notes on Intuitionistic Fuzzy Sets*, 28(3), 247–260.

[9] Dencheva, K. (2004). Extension of intuitionistic fuzzy modal operators $\boxplus$ and $\boxtimes$. *Proceedings of the Second Int. IEEE Conference on Intelligent Systems, Varna, June 22–24, 2004*, Vol. 3, 21–22.