Notes on Intuitionistic Fuzzy Sets
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Derivative-free Newton's method for solving intuitionistic fuzzy nonlinear equations with an application

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Abstract: In this paper, we present a derivative-free Newton's method that avoids computing the derivative by generating an approximation of the derivative for the intuitionistic fuzzy nonlinear equation. We first consider transforming the intuitionistic fuzzy quantities into their equivalent membership and non-membership parametric forms and insert the approximation from the forward difference method applied to $F'(x_k) = 0$ in Newton's method to avoid computing the Jacobian matrix. Numerical experiments were carried out, which shows that the approach is a good option for computing Jacobian and is an efficient one.
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1 Introduction

Many real-life problems can be transformed into a system of nonlinear equations and require the numerical solution of the form

$$F(x) = 0 \tag{1}$$

where $F: R^n \rightarrow R^n$ and that it is required to find $x^* \in R^n$ such that $F(x^*) = 0$. When the coefficients of (1) are written in crisp numbers, it may be convenient to represent some or all of them with fuzzy numbers, that is, a fuzzy set containing a degree of membership to a given crisp number, or with intuitionistic fuzzy numbers, that is an extension of fuzzy numbers that includes a degree of non-membership. Zadeh [22–24] introduced and investigated the idea of fuzzy numbers and the arithmetic operations involving these numbers. In 1983, Atanassov [3–5] introduced the intuitionistic fuzzy sets as an extension of Zadeh's fuzzy sets that contain the degree of non-membership explicitly together with the degree of membership to the set. The idea of fuzzy numbers and the arithmetic operations involving these numbers also applies to intuitionistic fuzzy sets. Among the widely used application of fuzzy number arithmetic is the nonlinear equation whose parametric forms are fully or partially represented by fuzzy numbers [9, 10, 12, 13, 15–17, 19–21]. The numerical solution to the intuitionistic fuzzy nonlinear equation with intuitionistic fuzzy coefficient involving fuzzy variable is one when the Jacobian is a nonsingular near-exact root (x^*). In particular, Amma et al. [2] considered the numerical solution of intuitionistic fuzzy differential equations by Euler and Taylor methods. Biswas et al. [6] solved fuzzy differential equations with a linear differential operator using the Adomian decomposition method, and Ettoussi et al. [11] worked on the solution of intuitionistic fuzzy differential equations by successive approximation method. However, in the literature, little has been done on intuitionistic fuzzy nonlinear equations among which is that of Keyanpour and Akbarian [14] that considered the mid-point of Newton's method for solving the intuitionistic fuzzy nonlinear equation.

Nevertheless, in an attempt to overcome some of the disadvantages of Newton's method, it has been suggested that the Jacobian matrix be evaluated either once for all iterations or once for every few iterations, instead of at every iteration as is strictly required [8]. In this paper, we consider the forward difference approach applied to Newton's method that does not compute the Jacobian for solving systems of intuitionistic fuzzy nonlinear equations because in [16] it was shown that the forward difference method is more efficient in numerical computation. Newton's method is an iterative method whose iterative function is generated either by computing the Jacobian or by a derivative approximation. We introduce the finite difference approach to the method that avoids computing the derivative of the function f. This is made possible by inserting the approximate $g(x)$ of $F'(x)$ in Newton's method. The anticipation has been to reduce the computational burden of computing the Jacobian matrix at each iteration.

This paper is arranged as follows: in the next section, we present some basic definitions and a brief overview of intuitionistic fuzzy nonlinear equations. We present the description of our method in Section 3. The next Section 4 contains the alternative approach for solving an intuitionistic fuzzy nonlinear equation. And finally, we report our numerical results in Section 5, and the conclusion is given in Section 6.

2 Preliminaries

We present the following definitions of fuzzy numbers and intuitionistic fuzzy numbers.

Definition 1. [24] A fuzzy set A of the real line R with a membership function $\mu_A: R \to [0,1]$ is called a fuzzy number if:

a) A is normal, i.e., there exists an element x_0 such that $\mu_A(x_0) = 1$;
b) A is fuzzy convex for the membership function $\mu_A(x)$, i.e., $\forall\, x_1, x_2 \in R, \forall \lambda \in [0,1]$, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\big(\mu_A(x_1), \mu_A(x_2)\big)$;
c) μ_A is upper semi continuous;
d) $\mathrm{supp}(A)$ is bounded.

Definition 2. [5] An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in R\}$, of the real line is called an intuitionistic fuzzy number (IFN) if:

a) A is IF-normal, i.e., there exist at least two points $x_0, x_1 \in X$ such that $\mu_A(x_0) = 1$ and $\nu_A(x_1) = 1$;
b) A is IF-convex, i.e., its membership function μ is fuzzy convex and its non-membership function ν is fuzzy concave;
c) μ_A is upper semicontinuous and ν_A is lower semicontinuous;
d) $\mathrm{supp}(A) = \{x \in X \mid \nu_A(x) < 1\}$ is bounded.

Definition 3. [5] An intuitionistic fuzzy set (IFS) A in E is defined as an object of the following form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where the functions $\mu_A: E \to [0,1]$ and $\nu_A: E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Definition 4. [14] An intuitionistic fuzzy number (IFN) n in parametric form is a pair $n = \big([\underline{n}, \overline{n}], [\underline{\underline{n}}, \overline{\overline{n}}]\big)$ of function $\underline{n}, \overline{n}, \underline{\underline{n}}, \overline{\overline{n}}$, that satisfies the following requirements:

1. $\underline{n}(\alpha)$ is a bounded monotonic increasing left continuous function;
2. $\overline{n}(\alpha)$ is a bounded monotonic decreasing left continuous function;
3. $\underline{\underline{n}}(\alpha)$ is a bounded monotonic increasing left continuous function;
4. $\overline{\overline{n}}(\alpha)$ is a bounded monotonic decreasing left continuous function;
5. $\underline{n}(\alpha) \leq \overline{n}(\alpha)$, $\underline{\underline{n}}(\alpha) \leq \overline{\overline{n}}(\alpha)$, $0 \leq \alpha \leq 1$.

Definition 5. [11] A triangular intuitionistic fuzzy number (TIFN) $\langle u, v\rangle$ is an intuitionistic fuzzy set in R with the following membership function u and non-membership function v:

$$u(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \frac{a_2-x}{a_2-a_1^1}, & a_1' \leq x \leq a_2 \\ \frac{x-a_2}{a_3^1-a_2}, & a_2 \leq x \leq a_3' \\ 1, & \text{otherwise} \end{cases},$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$ and $\forall x \in R, u(x), v(x) \leq 0.5$ for $u(x) = v(x)$.
This TIFN is denoted by $\langle u, v\rangle = \langle a_1, a_2, a_3; a_1', a_2, a_3'\rangle$ where the parametric form is given as

$$\underline{u}(\alpha) = a_1 + \alpha(a_2 - a_1), \qquad \overline{u}(\alpha) = a_3 - \alpha(a_3 - a_2),$$
$$\underline{\underline{v}}(\alpha) = a_1' + \alpha(a_2 - a_1'), \qquad \overline{\overline{v}}(\alpha) = a_3' - \alpha(a_3' - a_2).$$

3 Newton's method

The Newton's method [1, 8] is an iterative scheme that generates a sequence of approximation to the minimum and is given as

$$x_{n+1} = x_n - [J(x_n)]^{-1} F(x_n). \tag{3.0}$$

The direction or the correction factor, is given by

$$d_k = -(J_k)^{-1} F_k \tag{3.1}$$

The Jacobian matrix $J(x)$ in (3.1) will be approximated using $J(x_k, F(x_k))$ and the new d_k is represented as

$$d_k = -(D_k)^{-1} F_k \tag{3.2}$$

where the matrix D_k is component wise computed by two possible choices of forward difference or central difference and is defined as [7]

$$D_k^{ij} = \frac{F_i\left(x_k + h_k^j e_j\right) - F(x_k)}{h_k^j} \tag{3.3}$$

and

$$D_k^{ij} = \frac{F_i\left(x_k + h_k^j e_j\right) - F_i\left(x_k - h_k^j e_j\right)}{h_k^j} \tag{3.4}$$

with e_j the j-th unit column vector.

In this paper, we consider the derivative-free approach by adopting the forward difference method where D_k is derived using (3.3) because Omesa et al. [16] showed that the forward difference method is more efficient in numerical computations. The derivative-free Newton's method is presented via the following Algorithm.

Algorithm 1. Derivative-free Newton's method	
Given x_0, Solve $J(x_k, F(x_k))$ and $F(x_k)$	
Compute $x_{k+1} = x_k - [J(x_k, F(x_k))]^{-1} F(x_k)$	(3.5)
where $k = 0,1,2, \ldots$	

4 Iterative approach for solving intuitionistic fuzzy nonlinear equations

In this section, we intend to obtain a solution for the system of nonlinear equation $F(x) = 0$. The parametric form for the membership and non-membership is represented for all $r \in [0,1]$ as:

$$\mu_A(x) = \begin{cases} \underline{F}\left(\underline{x}, \overline{x}, r\right) = 0, \\ \overline{F}\left(\overline{x}, \underline{x}, r\right) = 0, \end{cases} \qquad \nu_A(x) = \begin{cases} \underline{\underline{F}}\left(\underline{\underline{x}}, \bar{\bar{x}}, r\right) = 0, \\ \bar{\bar{F}}\left(\bar{\bar{x}}, \underline{\underline{x}}, r\right) = 0. \end{cases} \tag{4.1}$$

Assume that $\alpha = (\underline{\alpha}, \overline{\alpha})$ is the solution to the nonlinear system (4.1), that is, for all $r \in [0,1]$,

$$\mu_A(x) = \begin{cases} \underline{F}\big(\underline{\alpha},\overline{\alpha};r\big) = 0, \\ \overline{F}\big(\overline{\alpha},\underline{\alpha};r\big) = 0, \end{cases} \qquad \nu_A(x) = \begin{cases} \underline{\underline{F}}\big(\underline{\underline{\alpha}},\overline{\overline{\alpha}};r\big) = 0, \\ \overline{\overline{F}}\big(\overline{\overline{\alpha}},\underline{\underline{\alpha}}.r\big) = 0. \end{cases} \tag{4.2}$$

Now, if $x_0 = (\underline{x}_0, \overline{x}_0, \underline{\underline{x}}_0, \overline{\overline{x}}_0)$ is an approximate solution for this nonlinear system, then for all $r \in [0,1]$ there are $h(r), k(r), p(x)\ and\ q(x)$ such that

$$\underline{\alpha}(r) = \underline{x}_0(r) + h(r), \qquad \underline{\underline{\alpha}}(r) = \underline{\underline{x}}_0(r) + p(r),$$

$$\overline{\alpha}(r) = \overline{x}_0(r) + k(r), \qquad \overline{\overline{\alpha}}(r) = \overline{\overline{x}}_0(r) + q(r).$$

Now by applying the Taylor series of $\underline{F}, \overline{F}$ about $(\underline{x}_0, \overline{x}_0)$ to the membership function and $\underline{\underline{F}}, \overline{\overline{F}}$ about $(\underline{\underline{x}}_0, \overline{\overline{x}}_0)$ to the non-membership function, then for all $r \in [0,1]$,

$$\underline{F}\big(\underline{\alpha},\overline{\alpha};r\big) = \underline{F}\big(\underline{x}_0,\overline{x}_0,r\big) + h\,\underline{F}_{\underline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) + g\,\underline{F}_{\overline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) + 0(h^2 + hk + h^2) = 0$$

$$\overline{F}\big(\underline{\alpha},\overline{\alpha};r\big) = \overline{F}\big(\underline{x}_0,\overline{x}_0,r\big) + h\,\overline{F}_{\underline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) + g\overline{F}_{\overline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) + 0(h^2 + hk + h^2) = 0$$

$$\underline{\underline{F}}\big(\underline{\underline{\alpha}},\overline{\overline{\alpha}};r\big) = \underline{\underline{F}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) + h\underline{\underline{F}}_{\underline{\underline{x}}}(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\ ) + g\underline{\underline{F}}_{\overline{\overline{x}}}(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\ ) + 0(h^2 + hk + h^2) = 0$$

$$\overline{\overline{F}}\big(\underline{\underline{\alpha}},\overline{\overline{\alpha}};r\big) = \overline{\overline{F}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) + h\overline{\overline{F}}_{\underline{\underline{x}}}(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\ ) + g\overline{\overline{F}}_{\overline{\overline{x}}}(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\ ) + 0(h^2 + hk + h^2) = 0$$

and if $\underline{x}_0, \overline{x}_0, \underline{\underline{x}}_0$ and $\overline{\overline{x}}_0$ are near to $\underline{\alpha}, \overline{\alpha}, \underline{\underline{\alpha}}$ and $\overline{\overline{\alpha}}$, respectively, then $h(r), k(r), p(r)$ and $q(r)$ are small enough.

Let us assume that all needed partial derivatives exist and are bounded. Therefore, for enough small $h(r), k(r), p(r)$ and $q(r)$, where for all $r \in [0,1]$, we have,

$$\underline{F}\big(\underline{x}_0,\overline{x}_0,r\big) + h\underline{F}_{\underline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) + g\underline{F}_{\overline{x}}\big(\underline{x}_0,\overline{x}_0;r\big) = 0,$$

$$\overline{F}\big(\underline{x}_0,\overline{x}_0,r\big) + h\overline{F}_{\underline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) + g\overline{F}_{\overline{x}}\big(\underline{x}_0,\overline{x}_0;r\big) = 0,$$

$$\underline{\underline{F}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) + h\underline{\underline{F}}_{\underline{\underline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) + g\underline{\underline{F}}_{\overline{\overline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0;r\big) = 0,$$

$$\overline{\overline{F}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) + h\overline{\overline{F}}_{\underline{\underline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) + g\overline{\overline{F}}_{\overline{\overline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0;r\big) = 0,$$

and hence $h(r), k(r), p(r)$ and $q(r)$ are unknown quantities that can be obtained by solving the following equations, for all $r \in [0,1]$,

$$J\big(\underline{x}_0,\overline{x}_0,\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big)\begin{pmatrix} h(r) \\ g(r) \\ p(r) \\ q(r) \end{pmatrix} = \begin{pmatrix} -\underline{F}\big(\underline{x}_0,\overline{x}_0,r\big) \\ -\overline{F}\big(\underline{x}_0,\overline{x}_0,r\big) \\ -\underline{\underline{F}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) \\ -\overline{\overline{F}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) \end{pmatrix} \tag{4.3}$$

where

$$J(\underline{x}_0,\overline{x}_0,\underline{\underline{x}}_0,\overline{\overline{x}}_0,r) = \begin{bmatrix} \underline{F}_{\underline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) & \underline{F}_{\overline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) \\ \overline{F}_{\underline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) & \overline{F}_{\overline{x}}\big(\underline{x}_0,\overline{x}_0,r\big) \\ \underline{\underline{F}}_{\underline{\underline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) & \underline{\underline{F}}_{\overline{\overline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) \\ \overline{\overline{F}}_{\underline{\underline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) & \overline{\overline{F}}_{\overline{\overline{x}}}\big(\underline{\underline{x}}_0,\overline{\overline{x}}_0,r\big) \end{bmatrix}$$

is the Jacobian matrix of the function $F = (\underline{F}, \overline{F}, \underline{\underline{F}}, \overline{\overline{F}})$ evaluated in $x_0 = \big(\underline{x}_0, \overline{x}_0, \underline{\underline{x}}_0, \overline{\overline{x}}_0\big)$.

Hence, the next approximations for $\underline{x}(r), \overline{x}(r), \underline{\underline{x}}(r)$ and $\overline{\overline{x}}(r)$ are as follows:

$$\underline{x}_1(r) = \underline{x}_0(r) + h(r),$$
$$\overline{x}_1(r) = \overline{x}_0(r) + k(r),$$
$$\underline{\underline{x}}_1(r) = \underline{\underline{x}}_1(r) + p(r),$$
$$\overline{\overline{x}}_1(r) = \overline{\overline{x}}_1(r) + q(r),$$

for all $r \in [0,1]$.

We can obtain approximated solution, $r \in [0,1]$, by using the recursive scheme

$$\begin{aligned} \underline{x}_{n+1}(r) &= \underline{x}_n(r) + h_n(r), \\ \overline{x}_{n+1}(r) &= \overline{x}_n(r) + k_n(r), \\ \underline{\underline{x}}_{n+1}(r) &= \underline{\underline{x}}_n(r) + p_n(r), \\ \overline{\overline{x}}_{n+1}(r) &= \overline{\overline{x}}_n(r) + q_n(r), \end{aligned} \tag{4.4}$$

when $n = 1, 2, \ldots$. Analogous to (4.3)

$$J(\underline{x}_n, \overline{x}_n, \underline{\underline{x}}_n, \overline{\overline{x}}_n, r) \begin{pmatrix} h(r) \\ g(r) \\ p(r) \\ q(r) \end{pmatrix} = \begin{pmatrix} -\underline{F}\left(\underline{x}_0, \overline{x}_0, r\right) \\ -\overline{F}\left(\underline{x}_0, \overline{x}_0, r\right) \\ -\underline{\underline{F}}\left(\underline{\underline{x}}_0, \overline{\overline{x}}_0, r\right) \\ -\overline{\overline{F}}\left(\underline{\underline{x}}_0, \overline{\overline{x}}_0, r\right) \end{pmatrix}$$

Now, if $J(\underline{x}_n, \overline{x}_n, \underline{\underline{x}}_n, \overline{\overline{x}}_n, r)$ is nonsingular, then from (4.2) we obtain the recursive scheme of Newton's method as follows,

$$\begin{bmatrix} \underline{x}_{n+1}(r) \\ \overline{x}_{n+1}(r) \\ \underline{\underline{x}}_{n+1}(r) \\ \overline{\overline{x}}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \\ \underline{\underline{x}}_n(r) \\ \overline{\overline{x}}_n(r) \end{bmatrix} - J(\underline{x}_n, \overline{x}_n, \underline{\underline{x}}_n, \overline{\overline{x}}_n, r)^{-1} \begin{bmatrix} \underline{F}\left(\underline{x}_n, \overline{x}_n, r\right) \\ \overline{F}\left(\underline{x}_n, \overline{x}_n, r\right) \\ \underline{\underline{F}}\left(\underline{\underline{x}}_n, \overline{\overline{x}}_n, r\right) \\ \overline{\overline{F}}\left(\underline{\underline{x}}_n, \overline{\overline{x}}_n, r\right) \end{bmatrix}.$$

Finally, we present the algorithm for our proposed approach (derivative-free method) as follows:

Algorithm 2. Derivative-free Newton's method	
Step 1.	Transform the intuitionistic fuzzy nonlinear equations into parametric form.
Step 2.	Determine the initial guess x_0 by solving the parametric equations for $r = 0$ and $r = 1$ and for $k = 0,1,2$ ...
Step 3.	Solve $J(x_k, F(x_k))^{-1}$ and $F(x_k)$.
Step 4.	Compute $[J(x_k, F(x_k))]^{-1}$ via (3.3) above.
Step 5.	Compute $x_{k+1} = x_k - [J(x_k, F(x_k))]^{-1} F(x_k)$.
Step 6.	Repeat Step 3 to Step 5 and continue with the next k until $\epsilon \leq 10^{-4}$ are satisfied.

5 Numerical result

In this section, we consider two problems to illustrate the performances of the forward difference method applied to Newton's iterative method without computing the Jacobian for solving intuitionistic fuzzy nonlinear equations. The computations are performed in MATLAB (R2015a) using the double precision computer. The benchmark problems are from [18].

Example 1. Consider the engineering problem that concerns the motion of an object under an intuitionistic fuzzy environment, resulting in two cases.

Case 1. Vertical motion of objects. Consider the vertical motion of the ball by neglecting air resistance and assuming constant acceleration of 32 ft/sec^2. The positive direction of the object y is upward from the earth. Then, the resulting intuitionistic fuzzy nonlinear equation is written as

$$A_1 * x(r)^2 + V_1 * x(r) = Y_1 \tag{5.1}$$

where the free parameter, $A_1 = (0.8,1.0,1.2; 0.25,1.0,1.2)$, velocity of the moving object under constant force of gravity $V_1 = (1.3,1.75,2.2; 1.2,1.7,2.1)$, and initial position of the force $Y_1 = (0.1,0.15,0.20; 0.1,0.13,0.18)$. Find the time $x(t)$ of the ball to hit the ground, which depends on fuzzy parameter $t = [0,1]$. Equation (5.1), becomes

$$(0.8,1.0,1.2; 0.25,1.0,1.2)x(t)^2 + (1.3,1.75,2.2; 1.2,1.7,2.1)x(t) = (0.1,0.15,0.20; 0.1,0.13,0.18) \tag{5.2}$$

Without any loss of generality, let x be positive, then the parametric form of this equation for membership and non-membership functions is as follows:

$$\begin{aligned} (0.8 + 0.2t)\underline{x}^2(t) + (1.3 + 0.45t)\underline{x}(t) &= (0.1 + 0.05t) \\ (1.2 - 0.2t)\overline{x}^2(r) + (2.2 - 0.45r)\overline{x}\,(r) &= (0.2 - 0.05t) \\ (0.25 + 0.2t)\underline{\underline{x}}^2(r) + (1.2 + 0.50r)\underline{\underline{x}}(r) &= (0.1 + 0.05t) \\ (1.2 - 0.2t)\overline{\overline{x}}^2(r) + (2.1 - 0.40r)\overline{\overline{x}}\,(r) &= (0.18 - 0.07t) \end{aligned} \tag{5.3}$$

For $r = 0$, we obtain the approximate Jacobian (3.3) as

$$J(x_k, F(x_k)) = \begin{bmatrix} 1.940000019967556 & 0 & 0 & 0 \\ 0 & 3.400000020861626 & 0 & 0 \\ 0 & 0 & 1.400000005960465 & 0 \\ 0 & 0 & 0 & 3.300000011920929 \end{bmatrix}$$

Then the inverse is given as

$$= \begin{bmatrix} \frac{1}{1.940000019967556} & 0 & 0 & 0 \\ 0 & \frac{1}{3.400000020861626} & 0 & 0 \\ 0 & 0 & \frac{1}{1.400000005960465} & 0 \\ 0 & 0 & 0 & \frac{1}{3.300000011920929} \end{bmatrix}$$

obtaining the initial guess requires assuming $r = 0$ and $r = 1$ in the above system (5.3), therefore $r = 0$ gives

$$\begin{aligned} 0.8\underline{x}^2(0) + 1.3\underline{x}(0) &= 0.1 \\ 1.2\overline{x}^2(0) + 2.2\overline{x}\,(0) &= 0.2 \\ 0.25\underline{\underline{x}}^2(0) + 1.2\underline{\underline{x}}(0) &= 0.1 \\ 1.2\overline{\overline{x}}^2(0) + 2.1\overline{\overline{x}}\,(0) &= 0.18 \end{aligned}$$

and $r = 1$ gives

$$1\underline{x}^2(1) + 1.75\underline{x}(1) = 0.15$$

$$1\overline{x}^2(1) + 1.75\overline{x}\,(1) = 0.15$$

$$0.45\underline{\underline{x}}^2(1) + 1.7\underline{\underline{x}}(1) = 0.15$$

$$0.6\overline{\overline{x}}^2(1) + 1.7\overline{\overline{x}}\,(1) = 0.11$$

When $t = 0$, we have $\underline{x}(0) = 0.0736$, $\overline{x}(0) = 0.0868$, $\underline{\underline{x}}(t) = 0.0863$, $\overline{\overline{x}}\,(t) = 0.0733$ and when $t = 1$, we have $\underline{x}(1) = \overline{x}(1) = \underline{\underline{x}}(1) = \overline{\overline{x}}\,(1) = 0.0819$ and we consider $x_0 = (0.4,0.5,0.4,0.5)$, as our initial guess. Via Algorithm 2 with $x_0 = (0.4,0.5,0.4,0.5)$ and approximate Jacobian $J(x_k, F(x_k))$ the number of iterations is 4 for the forward difference with maximum error less than 10^{-5}. We present the details of our solution for all $r \in [0,1]$ in Table 1 and Figure 1.

t	$\underline{x}(t)$	$\overline{x}\,(t)$	$\underline{\underline{x}}(t)$	$\overline{\overline{x}}\,(t)$
0.0	0.073594532594548	0.086809304749426	0.081934733466943	0.081896655277461
0.1	0.074671646080491	0.086409758542523	0.082528827952354	0.081190860209388
0.2	0.075679388968530	0.085993290501209	0.083075806073071	0.080456876932336
0.3	0.076624279545864	0.085558800357428	0.083581048609557	0.079692977756124
0.4	0.077512041333624	0.085105090157568	0.084049148498915	0.078897290441812
0.5	0.078347721040025	0.084630853158743	0.084484049697644	0.078067782742638
0.6	0.079135786045999	0.084134661165962	0.084889157719899	0.077202244859066
0.7	0.079880205463002	0.083614950056001	0.085267428359982	0.076298269572200
0.8	0.080584517912303	0.083070003174197	0.085621439455279	0.075353229611903
0.9	0.081251888508856	0.082497932238417	0.085953449350112	0.074364251829368
1.0	0.081885156997700	0.081896655279563	0.086265444846202	0.073328187649919

Table 1. Analytical solution of Problem 1 for all $t \in [0,1]$

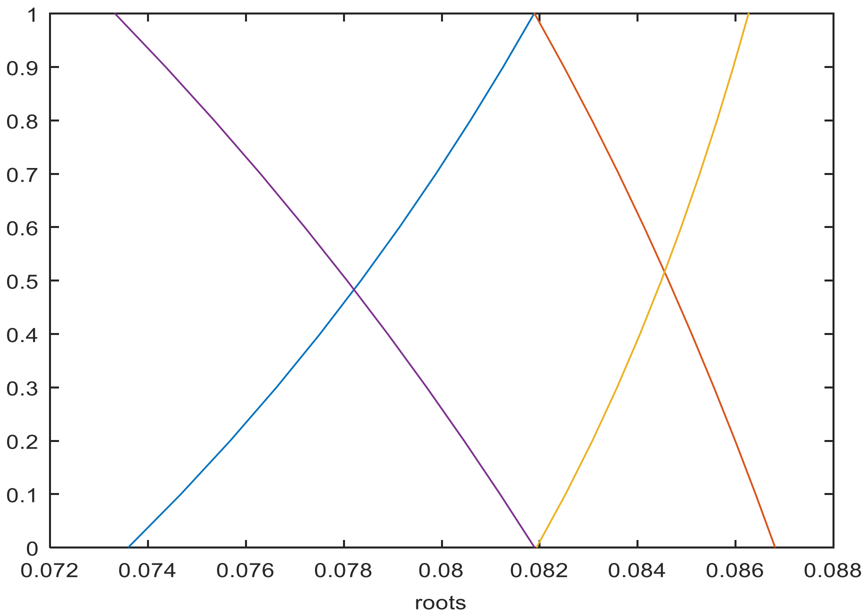

Figure 1. Analytical and numerical approximate solution of intuitionistic fuzzy nonlinear equation in Example 1

Case 2. Downward motion of object. Consider the downward motion of an object represented by the following intuitionistic fuzzy nonlinear equation

$$(3,4,5;2,3,4)x^2 + (1,2,3;1\ 2\ 2)x = (1,2,3;1\ 2\ 2)$$

Without any loss of generality, let x be positive, then the parametric form of this equation for the membership and non-membership functions is as follows:

$$(3+r)\underline{x}^2(r) + (1+r)\underline{x}(r) = (1+r)$$

$$(5-r)\overline{x}^2(r) + (3-r)\overline{x}\,(r) = (3-r)$$

$$(2+r)\underline{\underline{x}}^2(r) + (1+r)\underline{\underline{x}}(r) = (1+r)$$

$$(4-r)\overline{\overline{x}}^2(r) + (2-r)\overline{\overline{x}}\,(r) = (2-r)$$

For $r = 0$, we obtain the approximate Jacobian (3.3) as

$$J(x_k, F(x_k)) = \begin{bmatrix} 3.400000043213368 & 0 & 0 & 0 \\ 0 & 8.000000059604645 & 0 & 0 \\ 0 & 0 & 2.600000031292439 & 0 \\ 0 & 0 & 0 & 6.000000059604645 \end{bmatrix}$$

Then the inverse is given as

$$= \begin{bmatrix} \frac{1}{3.400000043213368} & 0 & 0 & 0 \\ 0 & \frac{1}{8.000000059604645} & 0 & 0 \\ 0 & 0 & \frac{1}{2.600000031292439} & 0 \\ 0 & 0 & 0 & \frac{1}{6.000000059604645} \end{bmatrix}$$

To obtain initial guess, we let $r = 0$ and $r = 1$ in the above system, therefore $r = 0$ gives

$$3\underline{x}^2(0) + \underline{x}(0) = 1$$

$$5\overline{x}^2(0) + 3\overline{x}\,(0) = 3$$

$$2\underline{\underline{x}}^2(0) + \underline{\underline{x}}(0) = 1$$

$$4\overline{\overline{x}}^2(0) + 2\overline{\overline{x}}\,(0) = 2$$

and $r = 1$ gives

$$4\underline{x}^2(1) + 2\underline{x}(1) = 2$$

$$4\overline{x}^2(1) + 2\overline{x}\,(1) = 2$$

$$3\underline{\underline{x}}^2(1) + 2\underline{\underline{x}}(1) = 2$$

$$3\overline{\overline{x}}^2(1) + \overline{\overline{x}}\,(1) = 1.$$

When $t = 0$, we have $\underline{x}(0) = 0.4343$, $\overline{x}(0) = 0.5306$, $\underline{\underline{x}}(t) = 0.5485$, $\overline{\overline{x}}\,(t) = 0.4343$. And when $t = 1$, we have $\underline{x}(1) = \overline{x}(1) = \underline{\underline{x}}(1) = \overline{\overline{x}}\,(1) = 0.5000$ and we consider $x_0 = (0.4,0.5,0.4,0.5)$, as our initial guess. Via Algorithm 2 with $x_0 = (0.4,0.5,0.4,0.5)$ and approximate Jacobian $J(x_k, F(x_k))$, the number of iterations is 2 for the forward difference with maximum error less than 10^{-5}. We present the details of our solution for all $r \in [0,1]$ in Table 2 and Figure 2.

t	$\underline{x}(t)$	$\overline{x}(t)$	$\underline{\underline{x}}(t)$	$\overline{\overline{x}}(t)$
0.0	0.434258313829634	0.530662352901064	0.499971064047340	0.500000000000000
0.1	0.444124152561498	0.528341552083287	0.507771812473090	0.495677089401974
0.2	0.452934408420214	0.525892369581886	0.514567101184687	0.491016667681032
0.3	0.460857339183747	0.523303729728293	0.520561285178602	0.485976360911993
0.4	0.468025767652303	0.520563182688385	0.525889994345204	0.480506146707827
0.5	0.474546360679261	0.517656725736360	0.530659516263694	0.474546482953888
0.6	0.480505952499601	0.514568548894616	0.534954407760406	0.468025837629962
0.7	0.485976074212196	0.511280727918697	0.538842831702757	0.460857374728988
0.8	0.491016267869303	0.507772851202120	0.542380381119442	0.452934425250540
0.9	0.495676556504057	0.504021563036460	0.545612872153780	0.444124165633310
1.0	0.499999315209352	0.500000000000000	0.548578418582904	0.434258575314830

Table 2. Analytical solution of Problem 2 for all $t \in [0,1]$

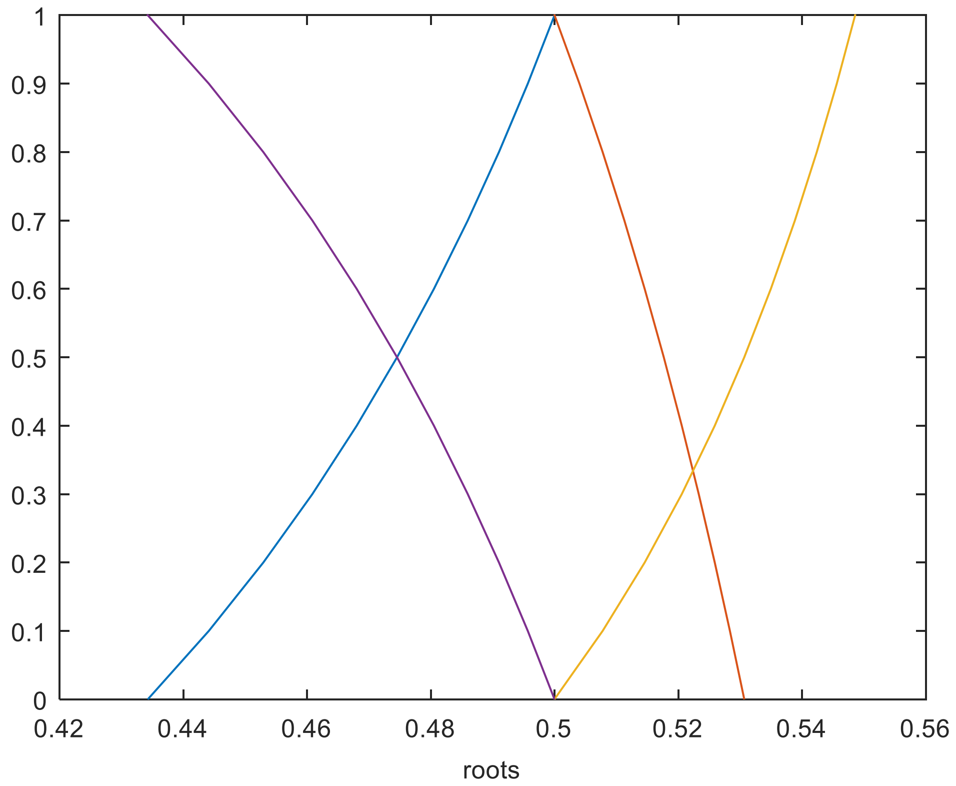

Figure 2. Analytical and numerical approximate solution
of intuitionistic fuzzy nonlinear equation in Case 2.

6 Conclusion

A derivative-free Newton's method for solving intuitionistic fuzzy nonlinear equations was presented. We were mainly interested in reducing the computational cost of the Jacobian matrix by computing the approximation to the Jacobian matrix throughout the iteration process. This was achieved by transforming the intuitionistic fuzzy nonlinear equation into parametric form and then solved via Newton's method without computing the Jacobian. The numerical result shows that the derivative-free approach is efficient when the Jacobian is non-singular. However, further research is required to examine the performance of the derivative-free Newton's method where the Jacobian is singular or near a singularity.

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