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On a special type of intuitionistic fuzzy implications

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To Vladik for his 65th birthday!

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Abstract: The concept of a tautologically asymmetric intuitionistic fuzzy implication is introduced and the list of all implications with this property are given. Open problems are formulated.

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1 Introduction

In [2, 4–7], 189 different intuitionistic fuzzy implications are introduced and some of their properties are studied. Here, we define a special property and study which of these 189 implications satisfy it. In the conclusion section we discuss a possible application of the determined subset of implications.

Initially, we remind that in intuitionistic fuzzy logic (see [1, 2]), each proposition, variable or formula is evaluated with two degrees: "truth degree" or "degree of validity" $\mu(p)$ and "falsity degree" or "degree of non-validity" $\nu(p)$. Thus, to each one of these objects, e.g., p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0, 1] \text{ and } \mu(p) + \nu(p) \leq 1.$$

Let an evaluation function V be defined over a set of propositions $\mathcal{S}$, in such a way that for $p \in \mathcal{S}$:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence the function $V : \mathcal{S} \to [0, 1] \times [0, 1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$.

For the needs of the discussion below, we define the notions of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [1, 2]) and tautology.

Formula A is an IFT if and only if (iff) for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a \geq b,$$

while it is a (classical) tautology if and only if for every evaluation function V, if $V(A) = \langle a, b \rangle$, then,

$$a = 1,\ b = 0.$$

Below, when it is clear, we will omit notation "$V(A)$", using directly "A" instead of the intuitionistic fuzzy evaluation of A.

In [3], we called the object $\langle \mu(p), \nu(p) \rangle$ an Intuitionistic Fuzzy Pair (IFP).

For brevity, below, instead of the IFP $\langle \mu(A), \nu(A) \rangle$ we will use the IFP $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$.

It is also suitable, if $\langle a, b \rangle$ and $\langle c, d \rangle$ are IFPs, to define the relations

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ iff } a \leq c \text{ and } b \geq d$$

and

$$\langle a, b \rangle \geq \langle c, d \rangle \text{ iff } a \geq c \text{ and } b \leq d.$$

2 Main results

Initially, by analogy with definitions from, e.g., [8, 9], we define for two IFPs $x = \langle a, b \rangle$ and $y = \langle c, d \rangle$ a binary operation $\circ$ such that:

1. $\circ$ is *symmetric* iff $x \circ y = y \circ x$,

2. $\circ$ is *antisymmetric* iff if $x \circ y = y \circ x$, then $x = y$,

3. $\circ$ is *asymmetric* iff $x \circ y \neq y \circ x$, for $x \neq y$.

For example, operations

$$V(x\&y) = \langle \min(a, c), \max(b, d)\rangle$$

and

$$V(x \vee y) = \langle \max(a, c), \min(b, d)\rangle$$

are symmetric.

An implication $\rightarrow$ is called *tautologically asymmetric* if for every two different IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$ (i.e., $a \neq c$ or $b \neq d$):

$$x \rightarrow y \text{ is a tautology iff } y \rightarrow x \text{ is not a tautology.}$$

In the present paper, we study which intuitionistic fuzzy implications are tautologically asymmetric.

Theorem 1. *For* $i \in \{1, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 17, 18, 19, 21, 24, 25, 26, 28, 29, 46, \ldots, 56, 58, 60, 61, 64, 66, 67, 69, 71, 72, 73, 75, 78, 80, 81, 91, \ldots, 96, 98, 99, 100, 102, 106, 108, \ldots, 113, 119, \ldots, 128, 134, \ldots, 152, 154, \ldots, 166, 169, 175, 179, 184, 186, 187, 188, 189\}$, $\rightarrow_i$ *is tautologically asymmetric (see Table 1).*

Table 1

$\rightarrow_1$	$\langle \max(b, \min(a, c)), \min(a, d)\rangle$
$\rightarrow_4$	$\langle \max(b, c), \min(a, d)\rangle$
$\rightarrow_5$	$\langle \min(1, b + c), \max(0, a + d - 1)\rangle$
$\rightarrow_6$	$\langle b + ac, ad\rangle$
$\rightarrow_7$	$\langle \min(\max(b, c), \max(a, b), \max(c, d)),$ $\max(\min(a, d), \min(a, b), \min(c, d))\rangle$
$\rightarrow_9$	$\langle b + a^2c, ab + a^2d\rangle$
$\rightarrow_{10}$	$\langle c\overline{\text{sg}}(1 - a) + \text{sg}(1 - a)(\overline{\text{sg}}(1 - c) + b\text{sg}(1 - c)),$ $d\overline{\text{sg}}(1 - a) + a\text{sg}(1 - a)\text{sg}(1 - c)\rangle$
$\rightarrow_{12}$	$\langle \max(b, c), 1 - \max(b, c)\rangle$
$\rightarrow_{13}$	$\langle b + c - bc, ad\rangle$
$\rightarrow_{14}$	$\langle 1 - (1 - c)\text{sg}(a - c) - d\overline{\text{sg}}(a - c)\text{sg}(d - b), d\text{sg}(d - b)\rangle$
$\rightarrow_{15}$	$\langle 1 - (1 - \min(b, c))\text{sg}(\text{sg}(a - c) + \text{sg}(d - b))$ $- \min(b, c)\text{sg}(a - c)\text{sg}(d - b),\ 1 - (1 - \max(a, d))\text{sg}(\overline{\text{sg}}(a - c)$ $+ \overline{\text{sg}}(d - b)) - \max(a, d)\overline{\text{sg}}(a - c)\overline{\text{sg}}(d - b)\rangle$
$\rightarrow_{17}$	$\langle \max(b, c), \min(ab + a^2, d)\rangle$
$\rightarrow_{18}$	$\langle \max(b, c), \min(1 - b, d)\rangle$

$\to_{19}$	$\langle \max(1 - \mathrm{sg}(\mathrm{sg}(a) + \mathrm{sg}(1-b)), c), \min(\mathrm{sg}(1-b), d)\rangle$
$\to_{21}$	$\langle \max(b, c(c+d)), \min(a(a+b), d(c^2+d+cd))\rangle$
$\to_{24}$	$\langle \overline{\mathrm{sg}}(a-c)\overline{\mathrm{sg}}(d-b), \mathrm{sg}(a-c)\mathrm{sg}(d-b)\rangle$
$\to_{25}$	$\langle \max(b, \overline{\mathrm{sg}}(a)\overline{\mathrm{sg}}(1-b)), c\overline{\mathrm{sg}}(d)\overline{\mathrm{sg}}(1-c)), \min(a, d)\rangle$
$\to_{26}$	$\langle \max(\overline{\mathrm{sg}}(1-b), c), \min(\mathrm{sg}(a), d)\rangle$
$\to_{28}$	$\langle \max(\overline{\mathrm{sg}}(1-b), c), \min(a, d)\rangle$
$\to_{29}$	$\langle \max(\overline{\mathrm{sg}}(1-b), \overline{\mathrm{sg}}(1-c)), \min(a, \overline{\mathrm{sg}}(1-d))\rangle$
$\to_{46}$	$\langle \max(b, \min(1-b, c)), 1 - \max(b, c)\rangle$
$\to_{47}$	$\langle \overline{\mathrm{sg}}(1-b-c), (1-c)\mathrm{sg}(1-b-c)\rangle$
$\to_{48}$	$\langle 1 - (1-c)\mathrm{sg}(1-b-c), (1-c)\mathrm{sg}(1-b-c)\rangle$
$\to_{49}$	$\langle \min(1, b+c), \max(0, 1-b-c)\rangle$
$\to_{50}$	$\langle b + c - bc, 1 - b - c + bc\rangle$
$\to_{51}$	$\langle \min(\max(b, c), \max(1-b, b), \max(c, 1-c)),$ $\max(1 - \max(b, c), \min(1-b, b), \min(c, 1-c))\rangle$
$\to_{52}$	$\langle 1 - (1 - \min(b, c))\mathrm{sg}(1-b-c), 1 - \min(b, c)\mathrm{sg}(1-b-c)\rangle$
$\to_{53}$	$\langle b + (1-b)^2 c, (1-b)b + (1-b)^2(1-c)\rangle$
$\to_{54}$	$\langle c\overline{\mathrm{sg}}(b) + \mathrm{sg}(b)(\overline{\mathrm{sg}}(1-c) + b\mathrm{sg}(1-c)),$ $(1-c)\overline{\mathrm{sg}}(b) + (1-b)\mathrm{sg}(b)\mathrm{sg}(1-c)\rangle$
$\to_{55}$	$\langle 1 - \mathrm{sg}(1-b-c), 1 - \overline{\mathrm{sg}}(1-b-c)\rangle$
$\to_{56}$	$\langle \max(\overline{\mathrm{sg}}(1-b), c), \min(\mathrm{sg}(1-b), 1-c)\rangle$
$\to_{58}$	$\langle \max(\overline{\mathrm{sg}}(1-b), \overline{\mathrm{sg}}(1-c)), 1 - \max(b, c)\rangle$
$\to_{60}$	$\langle \max(\overline{\mathrm{sg}}(1-b), \overline{\mathrm{sg}}(1-c)), \min(1-b, \overline{\mathrm{sg}}(c))\rangle$
$\to_{61}$	$\langle \max(c, \min(b, d)), \min(a, d)\rangle$
$\to_{64}$	$\langle c + bd, ad\rangle$
$\to_{66}$	$\langle c + d^2 b, bd + d^2 a\rangle$
$\to_{67}$	$\langle b\overline{\mathrm{sg}}(1-d) + \mathrm{sg}(1-d)(\overline{\mathrm{sg}}(1-b) + c\mathrm{sg}(1-b)),$ $a\overline{\mathrm{sg}}(1-d) + d\mathrm{sg}(1-d)\mathrm{sg}(1-b)\rangle$
$\to_{69}$	$\langle 1 - (1-b)\mathrm{sg}(d-b) - a\overline{\mathrm{sg}}(d-b)\mathrm{sg}(a-c), a\mathrm{sg}(a-c)\rangle$
$\to_{71}$	$\langle \max(b, c), \min(cd + d^2, a)\rangle$
$\to_{72}$	$\langle \max(b, c), \min(1-c, a)\rangle$
$\to_{73}$	$\langle \max(1 - \max(\mathrm{sg}(d), \mathrm{sg}(1-c)), b), \min(\mathrm{sg}(1-c), a)\rangle$
$\to_{75}$	$\langle \max(c, b(a+b)), \min(d(c+d), a(b^2+a) + ab)\rangle$
$\to_{78}$	$\langle \max(\overline{\mathrm{sg}}(1-c), b), \min(\mathrm{sg}(d), a)\rangle$
$\to_{80}$	$\langle \max(\overline{\mathrm{sg}}(1-c), b), \min(d, a)\rangle$
$\to_{81}$	$\langle \max(\overline{\mathrm{sg}}(1-b), \overline{\mathrm{sg}}(1-c)), \min(d, \overline{\mathrm{sg}}(1-a))\rangle$
$\to_{91}$	$\langle \max(c, \min(1-c, b)), 1 - \max(b, c)\rangle$
$\to_{92}$	$\langle \overline{\mathrm{sg}}(1-b-c), \min(1-b, \mathrm{sg}(1-b-c))\rangle$
$\to_{93}$	$\langle (1 - \min(1-b, \mathrm{sg}(1-b-c)), \min(1-b, \mathrm{sg}(1-b-c))\rangle$
$\to_{94}$	$\langle c + (1-c)^2 b, (1-c)c + (1-c)^2(1-b)\rangle$

$\rightarrow_{95}$	$\langle \min(b, \overline{\mathrm{sg}}(c)) + \mathrm{sg}(c)(\overline{\mathrm{sg}}(1-b) + \min(c, \mathrm{sg}(1-b))),$ $\min(1-b, \overline{\mathrm{sg}}(c)) + \min(1-c, \mathrm{sg}(c), \mathrm{sg}(1-b))\rangle$
$\rightarrow_{96}$	$\langle \max(\overline{\mathrm{sg}}(1-c), b), \min(\mathrm{sg}(1-b), 1-c)\rangle$
$\rightarrow_{98}$	$\langle \max(\overline{\mathrm{sg}}(1-c), b), 1-\max(b, c)\rangle$
$\rightarrow_{99}$	$\langle \max(\overline{\mathrm{sg}}(1-c), \overline{\mathrm{sg}}(1-b)), \min(1-c, \overline{\mathrm{sg}}(b))\rangle$
$\rightarrow_{100}$	$\langle \max(b\mathrm{sg}(a), c), \min(a\mathrm{sg}(b), d)\rangle$
$\rightarrow_{102}$	$\langle \max(b, c\mathrm{sg}(d)), \min(a, \mathrm{sg}(c)d)\rangle$
$\rightarrow_{106}$	$\langle \max(\min(b, \mathrm{sg}(1-b)), c), \min(1-b, \mathrm{sg}(b), 1-c)\rangle$
$\rightarrow_{108}$	$\langle \max(b, \min(c, \mathrm{sg}(1-c))), \min(1-b, 1-c, \mathrm{sg}(c))\rangle$
$\rightarrow_{109}$	$\langle b + \min(\overline{\mathrm{sg}}(1-a), c), ab + \min(\overline{\mathrm{sg}}(1-a), d))\rangle$
$\rightarrow_{110}$	$\langle \max(b, c), \min(ab + \overline{\mathrm{sg}}(1-a), d)\rangle$
$\rightarrow_{111}$	$\langle \max(b, cd + \overline{\mathrm{sg}}(1-c)), \min(ab + \overline{\mathrm{sg}}(1-a),$ $d(cd + \overline{\mathrm{sg}}(1-c)) + \overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{112}$	$\langle b + c - bc, ab + \overline{\mathrm{sg}}(1-a)d\rangle$
$\rightarrow_{113}$	$\langle b + cd - b(cd + \overline{\mathrm{sg}}(1-c)),$ $(ab + \overline{\mathrm{sg}}(1-a))(d(cd + \overline{\mathrm{sg}}(1-c)) + \overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{119}$	$\langle b + \min(\overline{\mathrm{sg}}(b), c), (1-b)b + \min(\overline{\mathrm{sg}}(b), 1-c)\rangle$
$\rightarrow_{120}$	$\langle \max(b, c), \min((1-b)b + \overline{\mathrm{sg}}(b), 1-c)\rangle$
$\rightarrow_{121}$	$\langle \max(b, c(1-c) + \overline{\mathrm{sg}}(1-c)),$ $\min((1-b)b + \overline{\mathrm{sg}}(b), (1-c)(c(1-c) + \overline{\mathrm{sg}}(1-c))) + \overline{\mathrm{sg}}(c)\rangle$
$\rightarrow_{122}$	$\langle b + c - bc, ((1-c)b + \overline{\mathrm{sg}}(b))(1-c)\rangle$
$\rightarrow_{123}$	$\langle b + c(1-c) - (b(c(1-c) + \overline{\mathrm{sg}}(1-c))),$ $((1-b)b + \overline{\mathrm{sg}}(b))(((1-c)(c(1-c) + \overline{\mathrm{sg}}(1-c))) + \overline{\mathrm{sg}}(c))\rangle$
$\rightarrow_{124}$	$\langle c + \min(\overline{\mathrm{sg}}(1-d), b), cd + \min(\overline{\mathrm{sg}}(1-d), a)\rangle$
$\rightarrow_{125}$	$\langle \max(b, c), \min(cd + \overline{\mathrm{sg}}(1-d), a)\rangle$
$\rightarrow_{126}$	$\langle \max(c, ab + \overline{\mathrm{sg}}(1-b)),$ $\min(cd + \overline{\mathrm{sg}}(1-d), a(ab + \overline{\mathrm{sg}}(1-b)) + \overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{127}$	$\langle b + c - bc, (cd + \overline{\mathrm{sg}}(1-d))a\rangle$
$\rightarrow_{128}$	$\langle c + ab - c(ab + \overline{\mathrm{sg}}(1-b)),$ $(cd + \overline{\mathrm{sg}}(1-d))(a(ab + \overline{\mathrm{sg}}(1-b)) + \overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{134}$	$\langle c + \min(\overline{\mathrm{sg}}(c), b), (1-c)c + \min(\overline{\mathrm{sg}}(c), (1-b))\rangle$
$\rightarrow_{135}$	$\langle \max(b, c), \min((1-c)c + \overline{\mathrm{sg}}(c), 1-b)\rangle$
$\rightarrow_{136}$	$\langle \max(c, b(1-b) + \overline{\mathrm{sg}}(1-b)),$ $\min((1-c)c + \overline{\mathrm{sg}}(c), (1-b)(b(1-b) + \overline{\mathrm{sg}}(1-b)) + \overline{\mathrm{sg}}(b))\rangle$
$\rightarrow_{137}$	$\langle b + c - bc, ((1-c)c + \overline{\mathrm{sg}}(c))(1-b)\rangle$
$\rightarrow_{138}$	$\langle c + b(1-b) - c(b(1-b) + \overline{\mathrm{sg}}(1-b)),$ $((1-c)c + \overline{\mathrm{sg}}(c))((1-b)(b(1-b) + \overline{\mathrm{sg}}(1-b)) + \overline{\mathrm{sg}}(b))\rangle$
$\rightarrow_{139}$	$\langle \frac{b+c}{2}, \frac{a+d}{2}\rangle$
$\rightarrow_{140}$	$\langle \frac{b+c+\min(b,c)}{3}, \frac{a+d+\max(a,d)}{3}\rangle$
$\rightarrow_{141}$	$\langle \frac{b+c+\max(b,c)}{3}, \frac{a+d+\min(a,d)}{3}\rangle$

$\rightarrow_{142}$	$\langle\frac{3-a-d-\max(a,d)}{3}, \frac{a+d+\max(a,d)}{3}\rangle$
$\rightarrow_{143}$	$\langle\frac{1-a+c+\min(1-a,c)}{3}, \frac{2+a-c-\min(1-a,c)}{3}\rangle$
$\rightarrow_{144}$	$\langle\frac{1+b-d+\min(b,1-d)}{3}, \frac{2-b+d-\min(b,1-d)}{3}\rangle$
$\rightarrow_{145}$	$\langle\frac{b+c+\min(b,c)}{3}, \frac{3-b-c-\min(b,c)}{3}\rangle$
$\rightarrow_{146}$	$\langle\frac{3-a-d-\min(a,d)}{3}, \frac{a+d+\min(a,d)}{3}\rangle$
$\rightarrow_{147}$	$\langle\frac{1-a+c+\max(1-a,c)}{3}, \frac{2+a-c-\max(1-a,c)}{3}\rangle$
$\rightarrow_{148}$	$\langle\frac{1+b-d+\max(b,1-d)}{3}, \frac{2-b+d-\max(b,1-d)}{3}\rangle$
$\rightarrow_{149}$	$\langle\frac{b+c+\max(b,c)}{3}, \frac{3-b-c-\max(b,c)}{3}$
$\rightarrow_{150,\lambda}$	$\langle\frac{b+c+\lambda-1}{2\lambda}, \frac{a+d+\lambda-1}{2\lambda}$, where $\lambda \geq 1$
$\rightarrow_{151,\gamma}$	$\langle\frac{b+c+\gamma}{2\gamma+1}, \frac{a+d+\gamma-1}{2\gamma+1}$, where $\gamma \geq 1$
$\rightarrow_{152,\alpha,\beta}$	$\langle\frac{b+c+\alpha-1}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta}$ where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{154,\lambda}$	$\langle\frac{-a+c+\lambda}{2\lambda}, \frac{a-c+\lambda}{2\lambda}\rangle$, where $\lambda \geq 1$
$\rightarrow_{155,\lambda}$	$\langle\frac{1-a-d+\lambda}{2\lambda}, \frac{a+d+\lambda-1}{2\lambda}\rangle$, where $\lambda \geq 1$
$\rightarrow_{156,\lambda}$	$\langle\frac{b+c+\lambda-1}{2\lambda}, \frac{1-b-c+\lambda}{2\lambda}\rangle$, where $\lambda \geq 1$
$\rightarrow_{157,\lambda}$	$\langle\frac{b-d+\lambda}{2\lambda}, \frac{-b+d+\lambda}{2\lambda}\rangle$, where $\lambda \geq 1$
$\rightarrow_{158,\gamma}$	$\langle\frac{1-a+c+\gamma}{2\gamma+1}, \frac{a-c+\gamma}{2\gamma+1}\rangle$, where $\gamma \geq 1$
$\rightarrow_{159,\gamma}$	$\langle\frac{2-a-d+\gamma}{2\gamma+1}, \frac{a+d+\gamma-1}{2\gamma+1}\rangle$, where $\gamma \geq 1$
$\rightarrow_{160,\gamma}$	$\langle\frac{b-d+\gamma+1}{2\gamma+1}, \frac{-b+d+\gamma}{2\gamma+1}\rangle$, where $\gamma \geq 1$
$\rightarrow_{161,\gamma}$	$\langle\frac{b+c+\gamma}{2\gamma+1}, \frac{1-b-c+\gamma}{2\gamma+1}\rangle$, where $\gamma \geq 1$
$\rightarrow_{162,\alpha,\beta}$	$\langle\frac{-a+c+\alpha}{\alpha+\beta}, \frac{a-c+\beta}{\alpha+\beta}\rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{163,\alpha,\beta}$	$\langle\frac{1-a-d+\alpha}{\alpha+\beta}, \frac{a+d+\beta-1}{\alpha+\beta}\rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{164,\alpha,\beta}$	$\langle\frac{b-d+\alpha}{\alpha+\beta}, \frac{-b+d+\beta}{\alpha+\beta}\rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{165,\alpha,\beta}$	$\langle\frac{b+c+\alpha-1}{\alpha+\beta}, \frac{1-b-c+\beta}{\alpha+\beta}\rangle$, where $\alpha \geq 1, \beta \in [1, \alpha]$
$\rightarrow_{166}$	$\langle\max(b, \min(a, c)), \min(a, \max(b, d))\rangle$
$\rightarrow_{169}$	$\langle\max(b, \min(1-b, c)), 1 - \max(b, \min(1-b, c))\rangle$
$\rightarrow_{175}$	$\langle\overline{\text{sg}}(d-b), \text{sg}(d-b)\rangle$
$\rightarrow_{179}$	$\langle\overline{\text{sg}}(1-b-c) + \text{sg}(1-b-c)\max(b, c),$ $\text{sg}(1-b-c)(1-\max(b, c))\rangle$
$\rightarrow_{184}$	$\langle 1 - \text{sg}(1-b).d, d.\text{sg}(1-b)\rangle$
$\rightarrow_{186}$	$\langle\overline{\text{sg}}(d-b) + \text{sg}(d-b)\max(b, c), \text{sg}(d-b)\min(a, d)\rangle$
$\rightarrow_{187}$	$\langle\max(b, c), ad\rangle$
$\rightarrow_{188}$	$\langle\min(b, c), ad\rangle$
$\rightarrow_{189}$	$\langle bc, ad\rangle$

Proof of the Theorem. Let $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$ be two different given IFPs, i.e.,

$$a \neq c \text{ or } b \neq d. \tag{1}$$

First, we prove the validity of the Theorem for implication $\rightarrow_1$.

Let

$$x \to_1 y = \langle \max(b, \min(a, c)), \min(a, d) \rangle$$

be a tautology, i.e.,

$$\max(b, \min(a, c)) = 1, \tag{2}$$

$$\min(a, d) = 0. \tag{3}$$

Let us assume that

$$y \to_1 x = \langle \max(d, \min(a, c)), \min(b, c) \rangle$$

be also a tautology, i.e.,

$$\max(d, \min(a, c)) = 1, \tag{4}$$

$$\min(b, c) = 0. \tag{5}$$

From (2) it follows that $b = 1$ or $\min(a, c) = 1$. If $b = 1$, then $a = 0$ and in (4) d must be equal to 1, i.e., $c = 0$. Therefore, the IFPs $\langle a, b \rangle$ and $\langle c, d \rangle$ coincide, which is a contradiction with the condition of the Theorem. Let $\min(a, c) = 1$, then $a = c = 1$ and hence $b = d = 0$, i.e., we obtain again a contradiction.

In a similar way, all the other cases when there is a tautology are checked.

Now, let us check the behaviour of the intuitionistic fuzzy implication

$$x \to_2 y = \langle \overline{\text{sg}}(a - c), d\ \text{sg}(a - c) \rangle$$

(see, [2]). Let $\langle \overline{\text{sg}}(a - c), d\ \text{sg}(a - c) \rangle$ be a tautology, i.e.,

$$\overline{\text{sg}}(a - c) = 1,$$

$$d\ \text{sg}(a - c) = 0.$$

Therefore, $a \leq c$. We show that there are cases, in which

$$y \to_2 x = \langle \overline{\text{sg}}(c - a), b\ \text{sg}(c - a) \rangle$$

is also a tautology. Indeed, if $a = c$,

$$\overline{\text{sg}}(c - a) = 1$$

and for all values of b, different from d,

$$b\ \text{sg}(c - a) = 0,$$

i.e., $y \to_2 x$ is a tautology and $\langle a, b \rangle \neq \langle c, d \rangle$ for $b \neq d$.

In a similar way, all the other cases when there is no tautology are checked. □

3 Conclusion

By analogy with the above definitions, we can define the concept of an tautologically symmetric and antisymmetric operations.

The implications discussed in the present paper will find applications in procedures related to intercriteria analysis and decision making processes. They can also be used in intuitionistic fuzzy databases, intuitionistic fuzzy expert systems, and intuitionistic fuzzy Prolog.

We finish with the following two open problems.

Open Problem 1. Which intuitionistic fuzzy implications are tautologically antisymmetric?

More general,

Open Problem 2. Which intuitionistic fuzzy operations are tautologically symmetric and which are antisymmetric?

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