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Extension of intuitionistic fuzzy modal operators diagram with new operators

Sinem Yılmaz and Arif Bal

Department Of Mathematics, University of Mersin
Mersin, Turkey
e-mails: sinemnyilmaz@gmail.com,
arif.bal.math@gmail.com

Abstract: Intuitionistic Fuzzy Modal Operator was defined by Atanassov in 1999, he introduced the generalization of these operators. After this study, some authors defined some modal operators which are called one type and two type modal operators on Intuitionistic Fuzzy Sets. In this paper, we defined new operators which are called $L^{\omega}_{\alpha,\beta}$ and $K^{\omega}_{\alpha,\beta}$ and examined some of their properties. $L^{\omega}_{\alpha,\beta}$ and $K^{\omega}_{\alpha,\beta}$ are One Type Modal Operators on Intuitionistic Fuzzy Sets. These operators are shown on the diagram.
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy modal operators, OTMO diagram.
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1 Introduction

The concept of fuzzy sets was introduced by Zadeh [13] as an extension of crisp sets by expanding the truth value set to the real unit interval $[0, 1]$. Let X be a set. The function $\mu : X \to [0, 1]$ is called a fuzzy set over $X(FS(X))$. For $x \in X$, $\mu(x)$ is the membership degree of x and the non-membership degree is $1 - \mu(x)$. Intuitionistic fuzzy sets have been introduced by Atanassov [1], as an extension of fuzzy sets. If X is a universal then a intuitionistic fuzzy set A, the membership and non-membership degree for each $x \in X$ respectively, $\mu_A(x)(\mu_A : X \to [0, 1])$ and $\nu_A(x)(\nu_A{:}X \to [0, 1])$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$. While the sum of membership degree and non-membership degree is 1 on FS, this sum is less than 1 on IFS.

Intuitionistic fuzzy sets have been introduced by Atanassov in 1986 [1], form an extension of fuzzy sets by enlarging the truth value set to the lattice $[0, 1] \times [0, 1]$ is defined as following.

Definition 1. *Let* $L = [0, 1]$ *then*

$$L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\}$$

is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".
For $(x_1, y_1), (x_2, y_2) \in L^*$, the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as following;
$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$
$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$

For each $J \subseteq L^*$
$\sup J = (sup\{x : (x, y \in [0, 1]), ((x, y) \in J)\}, inf\{y :(x, y \in [0, 1])((x, y) \in J)\})$ and
$\inf J = (inf\{x : (x, y \in [0, 1])((x, y) \in J)\}, sup\{y : (x, y \in [0, 1])((x, y) \in J)\})$.

Definition 2. *[1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0, 1])$ is called the "degree of membership of x in A", $\nu_A(x), (\nu_A : X \to [0, 1])$is called the "degree of non-membership of x in A" and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 3. *[1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 4. *[1]Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the above set is callede the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$$

The notion of Intuitionistic Fuzzy Operators was firstly introduced by Atanassov. The simplest one among them is presented as in the following definition.

Definition 5. *[3] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta \in [0, 1]$.*

1. $\boxplus A = \left\{\left\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\right\rangle : x \in X\right\}$
2. $\boxtimes A = \left\{\left\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\right\rangle : x \in X\right\}$

After this definition, in 2001, Atanassov, defined the extension of these operators as following:

Definition 6. *[4] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta \in [0, 1]$.*

1. $\boxplus_\alpha A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha\rangle : x \in X\}$
2. $\boxtimes_\alpha A = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x)\rangle : x \in X\}$

In these operators $\boxplus_\alpha$and $\boxtimes_\alpha$, if we choose $\alpha = \frac{1}{2}$, we get the operators $\boxplus$, $\boxtimes$, resp. Therefore, the operators $\boxplus_\alpha$and $\boxtimes_\alpha$ are the extensions of the operators $\boxplus$, $\boxtimes$, respectively. Some relationships between these operators were studied by several authors ([12], [7])

In 2004, the second extension of these operators was introduced by Dencheva in [12].

Definition 7. *[12] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta \in [0, 1]$.*

1. $\boxplus_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta\rangle : x \in X\}$ where $\alpha + \beta \in [0, 1]$.
2. $\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x)\rangle : x \in X\}$where $\alpha + \beta \in [0, 1]$.

In 2006, the third extension of the above operators was studied by Atanassov. He defined the following operators in [6]

Definition 8. *[6] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$.*

1. $\boxplus_{\alpha,\beta,\gamma}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma\rangle : x \in X\}$
   where $\alpha, \beta, \gamma \in [0, 1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.
2. $\boxtimes_{\alpha,\beta,\gamma}(A) = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x)\rangle : x \in X\}$
   where $\alpha, \beta, \gamma \in [0, 1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.

If we choose $\alpha = \beta$ and $\gamma = \beta$ in above operators then we can see easily that $\boxplus_{\alpha,\alpha,\gamma} = \boxplus_{\alpha,\beta}$and $\boxtimes_{\alpha,\alpha,\gamma} = \boxtimes_{\alpha,\beta}$.Therefore, we can say that $\boxplus_{\alpha,\beta,\gamma}$and $\boxtimes_{\alpha,\beta,\gamma}$are the extensions of the operators.

In 2007, after this diagram, Çuvalcıoğlu [8] defined a new operator and studied some of its properties. This operator is named $E_{\alpha,\beta}$ and defined as follows:

Definition 9. *[8]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta \in [0, 1]$. We define the following operator:*

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta)\rangle : x \in X\}$$

In 2007, Atanassov introduced the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ which is a natural extension of all these operators in [5] .

Definition 10. *[5] Let X be a set, $A \in IFS(X)$, $\alpha, \beta, \gamma, \delta \in [0,1]$ such that*

$$\max(\alpha, \beta) + \gamma + \delta \leqslant 1$$

then the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ defined by

$$\boxdot_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta\rangle : x \in X\}$$

In 2008, he defined this most general operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ as following:

Definition 11. *[6] Let X be a set, $A \in IFS(X)$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$ such that*

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leqslant 1$$

and

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0$$

then the operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ defined by

$$\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A) = \{\langle x, \alpha\mu_A(x) - \varepsilon\nu_A(x) + \gamma, \beta\nu_A(x) - \zeta\mu_A(x) + \delta\rangle : x \in X\}$$

In 2010, Çuvalcıoğlu [9] defined a new operator which is a generalization of $E_{\alpha,\beta}$.

Definition 12. *[9]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta, \omega \in [0,1]$ then*

$$Z^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta)\rangle : x \in X\}$$

Definition 13. *[10]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0,1]$ then*

$$Z^{\omega,\theta}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta)\rangle : x \in X\}$$

The operator $Z^{\omega,\theta}_{\alpha,\beta}$ is a generalization of $Z^{\omega}_{\alpha,\beta}$,and also, $E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$.

Definition 14. *[7] Let X be universal and $A \in IFS(X)$, $\alpha \in [0,1]$ then*

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle : x \in X\}$$

Definition 15. *[7] Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$ then*

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$$

Definition 16. *[7] Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0,1]$ then*

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle : x \in X\}$$

Definition 17. *[7] Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0,1]$ then*

1. $H_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$

2. $H^*_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x))\rangle : x \in X\}$

Definition 18. *[7] Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0,1]$ then*

1. $J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \beta\nu_A(x)\rangle : x \in X\}$

2. $J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta\nu_A(x)), \beta\nu_A(x)\rangle : x \in X\}$

Definition 19. *[7] Let X be universal and $A \in IFS(X)$ then*

1. $\Box(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$

2. $\Diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$

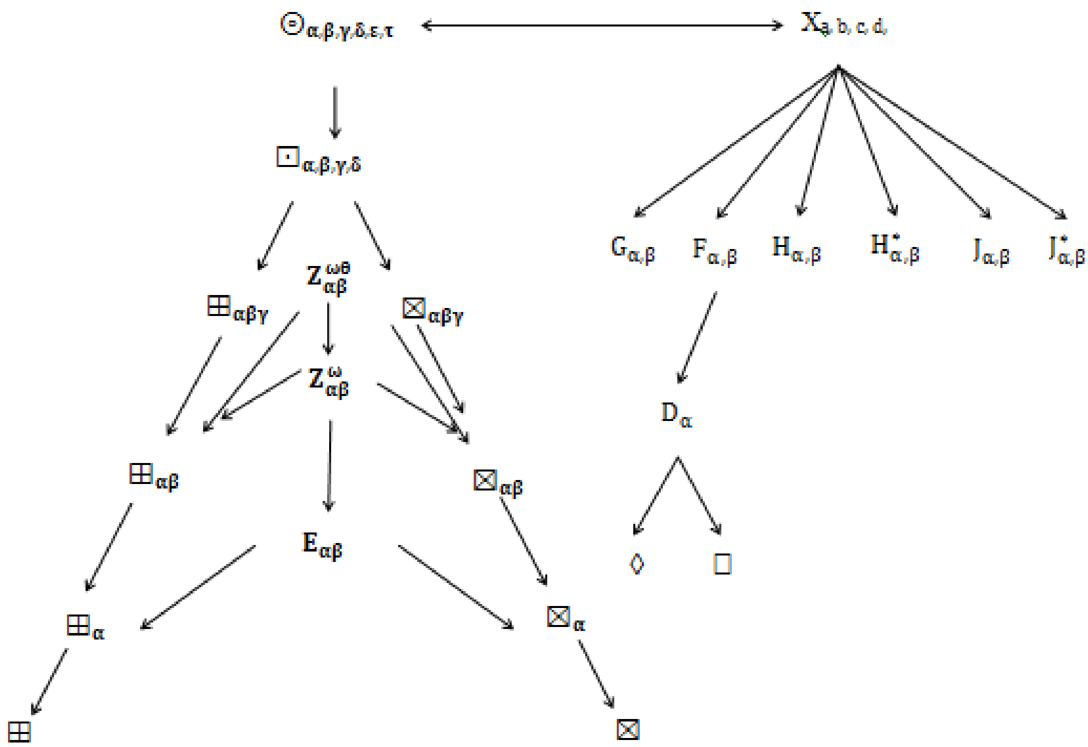

Figure 1:

Definition 20. *[11]Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$ then*

1. $\boxplus^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha(\beta\nu_A(x) + \omega - \omega\beta)\rangle : x \in X\}$

2. $\boxtimes^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega\alpha), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle : x \in X\}$

Definition 21. *[11]Let X be a set and $A \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0,1]$ then for $x \in X$*

$$E^{\omega,\theta}_{\alpha,\beta}(A) = \left\{ \left\langle \begin{array}{c} x, \beta((1-(1-\alpha)(1-\theta))\mu_A(x) + (1-\alpha)\theta\nu_A(x) + (1-\alpha)(1-\theta)\omega), \\ \alpha((1-\beta)\theta\mu_A(x) + (1-(1-\beta)(1-\theta))\nu_A(x) + (1-\beta)(1-\theta)\omega) \end{array} \right\rangle \right\}$$

Definition 22. *[11]Let X be a set, $A \in IFS(X)$ and $\alpha, \beta \in [0,1]$ then*

$$B_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle : x \in X\}$$

Definition 23. *[11]Let X be a set, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$ then*

$$\boxminus_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\beta)\nu_A(x)), \alpha((1-\alpha)\mu_A(x) + \nu_A(x))\rangle : x \in X\}$$

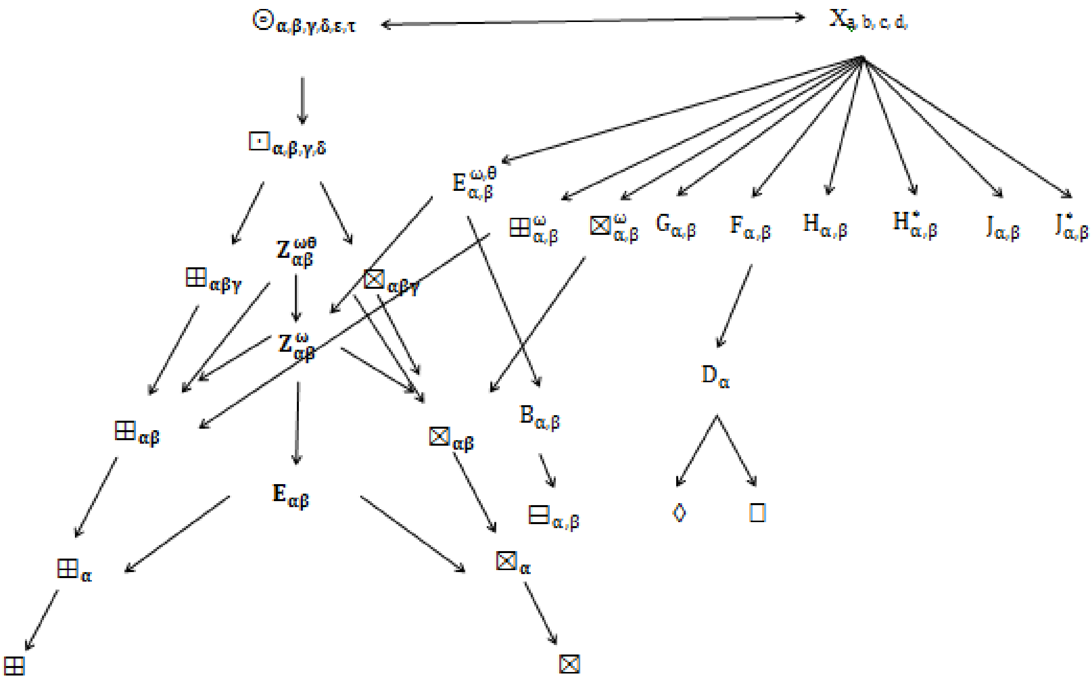

Figure 2:

2 $L^{\omega}_{\alpha,\beta}$ and $K^{\omega}_{\alpha,\beta}$

Definition 24. *Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X), \alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$*

1. $L^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\rangle\, x \in X\}$
2. $K^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha(1-\beta)\mu_A(x) + \alpha\beta(1-\omega), \alpha\nu_A(x) + \omega(1-\alpha)\rangle\, x \in X\}$

It is clear that:

$$\begin{aligned}
& \alpha\mu_A(x) + \omega(1-\alpha) + \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega) \\
= \;& \alpha(\mu_A(x) + \nu_A(x)) + \alpha\beta - \alpha\beta\nu_A(x) + \omega(1-\alpha-\alpha\beta) \\
\leq \;& \alpha + \alpha\beta + \omega(1-\alpha-\alpha\beta) \\
\leq \;& \alpha + \alpha\beta + 1 - \alpha - \alpha\beta = 1
\end{aligned}$$

Proposition 1. *Let X be a universal, $A \in IFS(X)$ and $\alpha \in [0, 1]$.*

1. $L^1_{\alpha,0}(A) = \boxtimes_\alpha(A)$
2. $K^1_{\alpha,0}(A) = \boxplus_\alpha(A)$

Proof. It is clear from definition. □

Proposition 2. *Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta \in [0, 1]$, $\alpha \neq 1$ and $\alpha + \beta \leq 1$.*

1. $L^{\frac{\beta}{1-\alpha}}_{\alpha,0}(A) = \boxtimes_{\alpha,\beta}(A)$
2. $K^{\frac{\beta}{1-\alpha}}_{\alpha,0}(A) = \boxplus_{\alpha,\beta}(A)$

Proof. It is clear from definition. □

As above, we get the following diagram.

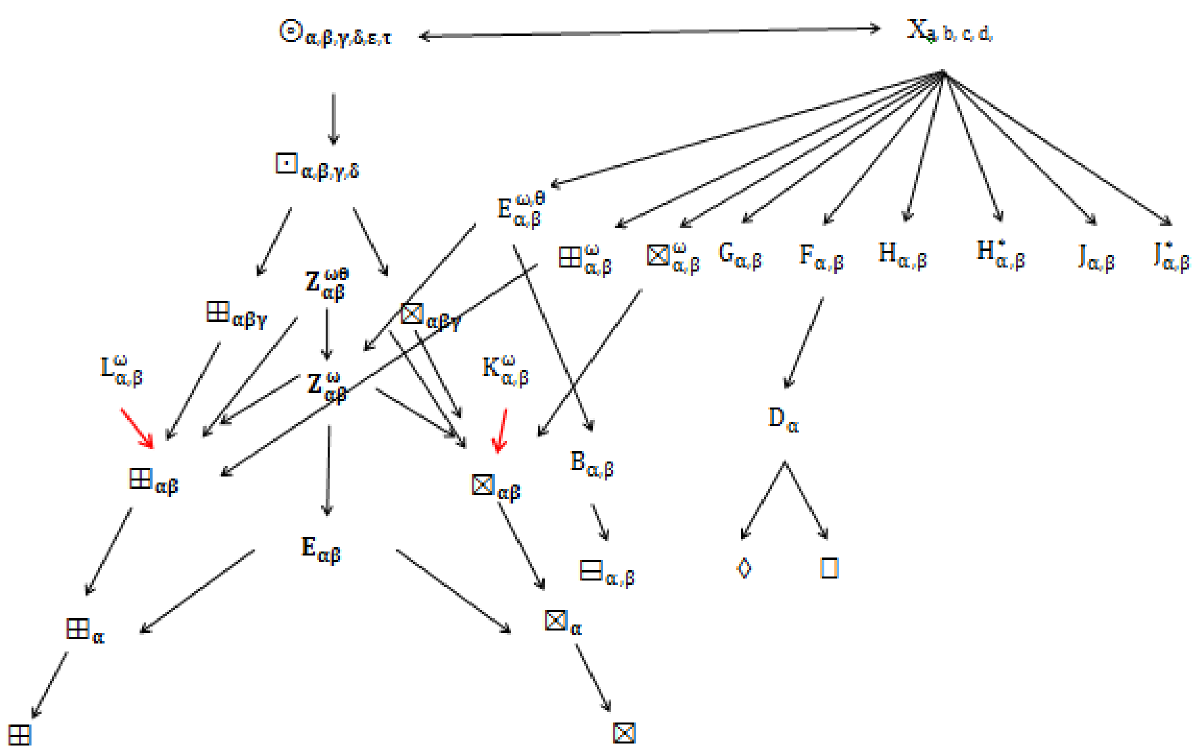

Figure 3:

Proposition 3. *Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0, 1]$ and $\alpha + \beta \leq 1$.*

1. $L^\omega_{\alpha,\beta}(A)^c = K^\omega_{\alpha,\beta}(A^c)$
2. $L^\omega_{\alpha,\beta}(A^c) = K^\omega_{\alpha,\beta}(A)^c$

Proof. It is clear from definition. □

Theorem 1. *Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$.*

1. If $\alpha \leq \beta$ then $L^{\omega}_{\beta,\alpha}(L^{\omega}_{\alpha,\beta}(A)) \sqsubseteq L^{\omega}_{\alpha,\beta}(L^{\omega}_{\beta,\alpha}(A))$
2. If $\alpha \leq \beta$ then $K^{\omega}_{\alpha,\beta}(K^{\omega}_{\beta,\alpha}(A)) \sqsubseteq K^{\omega}_{\beta,\alpha}(K^{\omega}_{\alpha,\beta}(A))$

Proof. (1) If we use $\alpha \leq \beta$ then we get,

$$\begin{aligned} \alpha \leq & \quad \beta \Rightarrow \alpha^2\beta(1-\omega) \leq \alpha\beta^2(1-\omega) \\ \Rightarrow & \quad \alpha^2\beta(1-\omega)(1-\beta) + \alpha\beta(1-\omega) \leq \alpha\beta^2(1-\omega)(1-\alpha) + \alpha\beta(1-\omega) \end{aligned}$$

and with this inequality we can say $L^{\omega}_{\beta,\alpha}(L^{\omega}_{\alpha,\beta}(A)) \sqsubseteq L^{\omega}_{\alpha,\beta}(L^{\omega}_{\beta,\alpha}(A))$.
(2) It can be seen in the same way. □

Theorem 2. *Let X be a universal, $A, B \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$. If $A \sqsubseteq B$ then*

1. $L^{\omega}_{\alpha,\beta}(A) \sqsubseteq L^{\omega}_{\alpha,\beta}(B)$
2. $K^{\omega}_{\alpha,\beta}(A) \sqsubseteq K^{\omega}_{\alpha,\beta}(B)$

Proof. If $A \sqsubseteq B$ the

$$\mu_A(x) \leq \mu_B(x) \Rightarrow \alpha\mu_A(x) + \omega(1-\alpha) \leq \alpha\mu_B(x) + \omega(1-\alpha)$$

and

$$\nu_A(x) \geq \nu_B(x) \Rightarrow \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega) \geq \alpha(1-\beta)\nu_B(x) + \alpha\beta(1-\omega)$$

Therefore; $L^{\omega}_{\alpha,\beta}(A) \sqsubseteq L^{\omega}_{\alpha,\beta}(B)$
Similarly, can show that $K^{\omega}_{\alpha,\beta}(A) \sqsubseteq K^{\omega}_{\alpha,\beta}(B)$. □

Theorem 3. *Let X be a universal, $A, B \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$.*

1. $K^{\omega}_{\alpha,\beta}(A \cap B) = K^{\omega}_{\alpha,\beta}(A) \cap K^{\omega}_{\alpha,\beta}(B)$
2. $K^{\omega}_{\alpha,\beta}(A \cup B) = K^{\omega}_{\alpha,\beta}(A) \cup K^{\omega}_{\alpha,\beta}(B)$

Proof. (1)

$$\begin{aligned} K^{\omega}_{\alpha,\beta}(A \cap B) &= \{\langle x, \alpha\mu_{A\cap B}(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_{A\cap B}(x) + \alpha\beta(1-\omega)\rangle : x \in X\} \\ &= \left\{\left\langle \begin{array}{c} x, \alpha\min\{\mu_A(x), \mu_B(x)\} + \omega(1-\alpha), \\ \alpha(1-\beta)\max\{\nu_A(x), \nu_B(x)\} + \alpha\beta(1-\omega) \end{array}\right\rangle : x \in X\right\} \\ &= \left\{\left\langle \begin{array}{c} x, \min\{\alpha\mu_A(x) + \omega(1-\alpha), \alpha\mu_B(x) + \omega(1-\alpha)\}, \\ \max\{\alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega), \alpha(1-\beta)\nu_B(x) + \alpha\beta(1-\omega)\} \end{array}\right\rangle x \in X\right\} \\ &= \left\{\left\langle x, \min\{\mu_{K^{\omega}_{\alpha,\beta}(A)}, \mu_{K^{\omega}_{\alpha,\beta}(B)}\}, \max\{\nu_{K^{\omega}_{\alpha,\beta}(A)}(x), \nu_{K^{\omega}_{\alpha,\beta}(B)}(x)\}\right\rangle : x \in X\right\} \\ &= K^{\omega}_{\alpha,\beta}(A) \cap K^{\omega}_{\alpha,\beta}(B) \end{aligned}$$

(2)

$$K^{\omega}_{\alpha,\beta}(A\cup B)=\{\langle x,\alpha\mu_{A\cup B}(x)+\omega(1-\alpha),\alpha(1-\beta)\nu_{A\cup B}(x)+\alpha\beta(1-\omega)\rangle\colon x\in X\}$$
$$=\left\{\left\langle \begin{matrix} x,\alpha\max\{\mu_A(x),\mu_B(x)\}+\omega(1-\alpha), \\ \alpha(1-\beta)\min\{\nu_A(x),\nu_B(x)\}+\alpha\beta(1-\omega) \end{matrix}\right\rangle\colon x\in X\right\}$$
$$=\left\{\left\langle \begin{matrix} x,\max\{\alpha\mu_A(x)+\omega(1-\alpha),\alpha\mu_B(x)+\omega(1-\alpha)\}, \\ \min\{\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega),\alpha(1-\beta)\nu_B(x)+\alpha\beta(1-\omega)\} \end{matrix}\right\rangle\colon x\in X\right\}$$
$$=\left\{\left\langle x,\max\{\mu_{K^{\omega}_{\alpha,\beta}(A)},\mu_{K^{\omega}_{\alpha,\beta}(B)}\},\min\{\nu_{K^{\omega}_{\alpha,\beta}(A)}(x),\nu_{K^{\omega}_{\alpha,\beta}(B)}(x)\}\right\rangle : x\in X\right\}$$
$$=K^{\omega}_{\alpha,\beta}(A)\cup K^{\omega}_{\alpha,\beta}(B)$$

□

Theorem 4. *Let X be a universal, $A,B\in IFS(X)$ and $\alpha,\beta,\omega\in[0,1]$ and $\alpha+\beta\leq 1$.*

1. $L^{\omega}_{\alpha,\beta}(A\cap B)=L^{\omega}_{\alpha,\beta}(A)\cap L^{\omega}_{\alpha,\beta}(B)$
2. $L^{\omega}_{\alpha,\beta}(A\cup B)=L^{\omega}_{\alpha,\beta}(A)\cup L^{\omega}_{\alpha,\beta}(B)$

Proof. It is clear from definition. □

Proposition 4. *Let X be a universal, $A\in IFS(X)$ and $\alpha,\beta,\omega,\theta\in[0,1]$ and $\alpha+\beta\leq 1$. If $\omega\leq\theta$ then,*

1. $L^{\omega}_{\alpha,\beta}(A)\sqsubseteq L^{\theta}_{\alpha,\beta}(A)$
2. $K^{\theta}_{\alpha,\beta}(A)\sqsubseteq K^{\omega}_{\alpha,\beta}(A)$

Proof. (1) If $\omega\leq\theta$ then,

$$\omega(1-\alpha)\leq\theta(1-\alpha)\Rightarrow\alpha\mu_A(x)+\omega(1-\alpha)\leq\alpha\mu_A(x)+\theta(1-\alpha)$$

and

$$\alpha\beta(1-\theta)\leq\alpha\beta(1-\omega)\Rightarrow\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\theta)\leq\alpha(1-\beta)\nu_A(x)+\alpha\beta(1-\omega)$$

So,

$$L^{\omega}_{\alpha,\beta}(A)\sqsubseteq L^{\theta}_{\alpha,\beta}(A)$$

(2) If $\omega\leq\theta$ then,

$$\alpha\beta(1-\theta)\leq\alpha\beta(1-\omega)\Rightarrow\alpha(1-\beta)\mu_A(x)+\alpha\beta(1-\theta)\leq\alpha(1-\beta)\mu_A(x)+\alpha\beta(1-\omega)$$

and

$$\omega(1-\alpha)\leq\theta(1-\alpha)\Rightarrow\alpha\nu_A(x)+\omega(1-\alpha)\leq\alpha\nu_A(x)+\theta(1-\alpha)$$

So,

$$K^{\theta}_{\alpha,\beta}(A)\sqsubseteq K^{\omega}_{\alpha,\beta}(A)$$

This completes the proof. □

References

[1] Atanassov, K. T., Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, Vol. 20, 1986, 87–96.

[2] Atanassov, K. T., Remark on Two Operations Over Intuitionistic Fuzzy Sets, *Int. J. of Unceratanity, Fuzziness and Knowledge Syst.*, Vol. 9, 2001, No. 1, 71–75.

[3] Atanassov, K. T., On the type of intuitionistic fuzzy modal operators, *Notes on Intuitionistic Fuzzy Sets*. Vol. 11, 2005, No. 5, 24–28.

[4] Atanassov, K. T., The most general form of one type of intuitionistic fuzzy modal operators, *Notes on Intuitionistic Fuzzy Sets*, Vol. 12, 2006, No. 2, 36–38.

[5] Atanassov, K. T., Some Properties of the operators from one type of intuitionistic fuzzy modal operators, *Advanced Studies on Contemporary Mathematics*, Vol. 15, 2007, No. 1, 13–20.

[6] Atanassov, K. T., The most general form of one type of intuitionistic fuzzy modal operators, Part 2, *Notes on Intuitionistic Fuzzy Sets*, Vol. 14, 2008, No. 1, 27–32.

[7] Atanassov, K. T., *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin, 2012.

[8] Çuvalcıoğlu, G., Some Properties of $E_{\alpha,\beta}$ operator, *Advanced Studies on Contemporary Mathematics*, Vol. 14, 2007, No. 2, 305–310.

[9] Çuvalcıoğlu, G., Expand the modal operator diagram with $Z^{\omega}_{\alpha,\beta}$, *Proc. Jangjeon Math. Soc.*, Vol. 13, 2010, No. 3, 403–412.

[10] Çuvalcıoğlu, G., On the diagram of One Type Modal Operators on Intuitionistic Fuzzy Sets: Last Expanding with $Z^{\omega,\theta}_{\alpha,\beta}$, *Iranian Journal of Fuzzy Systems*, Vol. 10, 2013, No. 1, 89–106.

[11] Çuvalcıoğlu, G., The extension of modal operators' diagram with last operators, *Notes on Intuitionistic Fuzzy Sets*, Vol. 19, 2013, No. 3, 56–61.

[12] Dencheva, K., Extension of intuitionistic fuzzy modal operators $\boxplus$ and $\boxtimes$, *Proc. of the Second Int. IEEE Symp. Intelligent Systems*, Varna, June 22–24 2004, Vol. 3, 21–22.

[13] Zadeh, L. A., Fuzzy Sets, *Information and Control*, Vol. 8, 1965, 338–353.