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Operators similar to operators defined over intuitionistic fuzzy sets

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Abstract: In the paper the so-called d_φ-intuitionistic fuzzy set (d_φ-*IFS*), over the non-empty universe E, are considered for the case when d_φ is $\mathcal{R}^2$-metric induced by an arbitrary fixed absolute normalized $\mathcal{R}^2$-norm φ. Using a bijective isomorphism between the class of all such sets and the class of all intuitionistic fuzzy sets over E, any operator acting over one of the mentioned classes produces a similar operator acting on the other. Some of the operators defined over the intuitionistic fuzzy sets and their corresponding similar operators are considered and studied in this paper and important limit theorems are established.
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1 Basic definitions and preliminary results

Here we recall some basic definitions and properties (see e.g. [1]):

Definition 1. Let $A \subset E$ and $\mu_A : E \to [0,1], \nu_A : E \to [0,1]$ are mappings such that for any $x \in E$ the inequality $\mu_A(x) + \nu_A(x) \leq 1$ holds. The set $\tilde{A} = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$ is called intuitionistic fuzzy set over E.

The mappings μ_A and ν_A are called membership and non-membership function. The map $\pi_A : E \to [0,1]$, that for $x \in E$ is introduced by $\pi_A(x) \stackrel{\text{def}}{=} 1 - \mu_A(x) - \nu_A(x)$, is called hesitancy function.

The class of all intuitionistic fuzzy sets over E is denoted by IFS(E).

Definition 2. An $\mathcal{R}^2$-norm φ is called normalized norm if the equality $\varphi((1,0)) = \varphi((0,1)) = 1$ holds. The class of all normalized $\mathcal{R}^2$-norms is denoted by N_2.

Definition 3. An $\mathcal{R}^2$-norm φ is called absolute norm if for any $(\mu,\nu) \in \mathcal{R}^2$ the equality $\varphi((\mu,\nu)) = \varphi((|\mu|,|\nu|))$ holds. The class of all absolute normalized $\mathcal{R}^2$-norms is denoted by AN_2.

Let $\varphi \in N_2$. Then φ induced $\mathcal{R}^2$-metric d_φ by the formula $d_\varphi((\mu_1,\nu_1),(\mu_2,\nu_2)) = \varphi((|\mu_1 - \mu_2|,|\nu_1 - \nu_2|))$.

Let d is $\mathcal{R}^2$-metric. In [4], for the first time, the notion d-intuitionistic fuzzy set (d-*IFS*) over E was introduced. Below we give the following

Definition 4. Let $\varphi \in N_2$, $A \subset E$ and $\mu_A : E \to [0,1], \nu_A : E \to [0,1]$ are mappings such that for any $x \in E$ the inequality $\varphi((\mu_A(x),\nu_A(x))) \leq 1$ holds. The set $\tilde{A} = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$ is called d_φ-intuitionistic fuzzy set (d_φ-*IFS*) over E. The mappings μ_A and ν_A are called membership and non-membership function. The map $\pi_A : E \to [0,1]$, that for $x \in E$ is introduced by $\pi_A(x) \stackrel{\text{def}}{=} 1 - \varphi((\mu_A(x),\nu_A(x)))$, is called hesitancy function.

The class of all d_φ-intuitionistic fuzzy sets over E is denoted by d_φ-IFS(E).

Let $\varphi \in AN_2$ and E is a universe set. As shown in [3], there is a bijection T between the classes d_φ-IFS(E) and IFS(E). More specifically, if $\mu : E \to [0,1]$ and $\nu : E \to [0,1]$ are the membership and non-membership functions for $A \in d_\varphi$-IFS(E) and

$$T(\mu,\nu) = (\mu^*,\nu^*), \tag{1}$$

where:

$$\mu^*(x) = \begin{cases} \frac{\mu(x)}{\mu(x)+\nu(x)}\varphi\left(\mu(x),\nu(x)\right), & \text{if } \mu(x)+\nu(x) > 0 \\ 0, \text{ if } \mu(x)+\nu(x) = 0; \end{cases} \tag{2}$$

$$\nu^*(x) = \begin{cases} \frac{\nu(x)}{\mu(x)+\nu(x)}\varphi\left(\mu(x),\nu(x)\right), & \text{if } \mu(x)+\nu(x) > 0 \\ 0, \text{ if } \mu(x)+\nu(x) = 0, \end{cases} \tag{3}$$

then $\mu^* : E \to [0,1]$ and $\nu^* : E \to [0,1]$ are the membership and non-membership functions for $B = T(A) \in$ IFS(E).

Respectively, if $\mu^* : E \to [0,1]$ and $\nu^* : E \to [0,1]$ are the membership and non-membership functions for a set $B \in$ IFS(E) and if

$$T^{-1}(\mu^*,\nu^*) = (\mu,\nu), \tag{4}$$

where:

$$\mu(x) = \begin{cases} \frac{\mu^*(x)(\mu^*(x)+\nu^*(x))}{\varphi(\mu^*(x),\nu^*(x))}, & \text{if } \mu^*(x)+\nu^*(x) > 0 \\ 0, \text{ if } \mu^*(x)+\nu^*(x) = 0; \end{cases} \tag{5}$$

$$\nu(x) = \begin{cases} \frac{\nu^*(x)(\mu^*(x)+\nu^*(x))}{\varphi(\mu^*(x),\nu^*(x))}, & \text{if } \mu^*(x)+\nu^*(x) > 0 \\ 0, \text{ if } \mu^*(x)+\nu^*(x) = 0, \end{cases} \tag{6}$$

then $\mu : E \to [0,1]$ and $\nu : E \to [0,1]$ are the membership and non-membership functions for $A = T^{-1}(B) \in d_\varphi$-IFS($E$).

Further we will use the mappings T and T^{-1} to obtain operators similar to operators defined over intuitionistic fuzzy sets.

2 Similar operators to operators defined over intuitionistic fuzzy sets

Let $L : \text{IFS}(E) \to \text{IFS}(E)$ is an operator. Then L induces an operator $\overset{\varphi}{L} : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$, which is similar to the operator L (under the similarity T), i.e.

$$\overset{\varphi}{L} \overset{\text{def}}{=} T^{-1}LT. \tag{7}$$

Remark 1. For $\varphi = \varphi_1$ (where φ_1 is the Manhattan norm) the operators L and $\overset{\varphi_1}{L}$ coincide, because T is the identity mapping.

The defining equality (7) allows us, using an operator L (acting over $\text{IFS}(E)$) to construct its respective analogue $\overset{\varphi}{L}$ (acting over $d_\varphi\text{-IFS}(E)$). Conversely, any operator $L : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$ induces operator $\underset{\varphi}{L} : \text{IFS}(E) \to \text{IFS}(E)$, which is given by: $\underset{\varphi}{L} \overset{\text{def}}{=} TLT^{-1}$.

Below we shall consider several examples.

Let $\alpha, \beta \in [0, 1]$ are real parameters. The operator $H_{\alpha,\beta} : \text{IFS}(E) \to \text{IFS}(E)$ is defined for $B = \{\langle x, \mu^*(x), \nu^*(x)\rangle | x \in E\} \in \text{IFS}(E)$ by (see [2, p. 83]):

$$H_{\alpha,\beta}(B) \overset{\text{def}}{=} \{\langle x, \alpha\mu^*(x), \nu^*(x) + \beta(1 - \mu^*(x) - \nu^*(x))\rangle | x \in E\}.$$

By substituting in (7) L with $H_{\alpha,\beta}$, we obtain the operator $\overset{\varphi}{H}_{\alpha,\beta} : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$. The direct computation and the defining relation

$$\frac{\varphi((z(x), w(x)))}{z(x) + w(x)} = 1, \text{ if } (z(x), w(x)) = (0, 0), \tag{8}$$

lead us to the following

Lemma 1. *For any $x \in E$ $\overset{\varphi}{H}_{\alpha,\beta}$ is given by the equation*

$$\overset{\varphi}{H}_{\alpha,\beta}(A) = \{\langle x, K(x)M(x), K(x)N(x)\rangle | x \in E\},$$

where:

$$\begin{aligned}
&A = \{\langle x, \mu(x), \nu(x)\rangle | x \in E\} \in d_\varphi\text{-IFS}(E);\\
&K(x) \overset{\text{def}}{=} \frac{M(x) + N(x)}{\varphi((M(x), N(x)))};\\
&M(x) \overset{\text{def}}{=} \alpha\mu(x)\frac{\varphi((\mu(x), \nu(x)))}{\mu(x) + \nu(x)};\\
&N(x) \overset{\text{def}}{=} \nu(x)\frac{\varphi((\mu(x), \nu(x)))}{\mu(x) + \nu(x)} + \beta(1 - \varphi((\mu(x), \nu(x)))).
\end{aligned}$$

Completely analogously, if we choose for L the operator $J_{\alpha,\beta} : \text{IFS}(E) \to \text{IFS}(E)$, acting by the formula (see [2, p. 83]):

$$J_{\alpha,\beta}(B) \overset{\text{def}}{=} \{\langle x, \mu^*(x) + \alpha(1 - \mu^*(x) - \nu^*(x)), \beta\nu^*(x)\rangle | x \in E\},$$

after calculation we obtain the following

Lemma 2. *For any $x \in E$ $\overset{\varphi}{J}_{\alpha,\beta}$ is given with the equation*

$$\overset{\varphi}{J}_{\alpha,\beta}(A) = \{\langle x, \tilde{K}(x)\tilde{M}(x), \tilde{K}(x)\tilde{N}(x)\rangle | x \in E\},$$

where:

$$A = \{\langle x, \mu(x), \nu(x)\rangle | x \in E\} \in d_\varphi\text{-IFS}(E);$$
$$\tilde{K}(x) \overset{\text{def}}{=} \frac{\tilde{M}(x) + \tilde{N}(x)}{\varphi((\tilde{M}(x), \tilde{N}(x)))};$$
$$\tilde{M}(x) \overset{\text{def}}{=} \mu(x)\frac{\varphi((\mu(x), \nu(x)))}{\mu(x) + \nu(x)} + \alpha(1 - \varphi((\mu(x), \nu(x))));$$
$$\tilde{N}(x) \overset{\text{def}}{=} \beta\nu(x)\frac{\varphi((\mu(x), \nu(x)))}{\mu(x) + \nu(x)}$$

and (8) *is valid.*

Another example is the operator $G_{\alpha,\beta} : \text{IFS}(E) \to \text{IFS}(E)$, acting by the formula (see [2, p. 82])

$$G_{\alpha,\beta}(B) \overset{\text{def}}{=} \{\langle x, \alpha\mu^*(x), \beta\nu^*(x)\rangle | x \in E\}.$$

The direct check, after putting $L = G_{\alpha,\beta}$, leads us to

Lemma 3. *For any $x \in E$ $\overset{\varphi}{G}_{\alpha,\beta}$ is given through the equality*

$$\overset{\varphi}{G}_{\alpha,\beta}(A) = \{\langle x, \alpha\mu(x)f(x), \beta\nu(x)f(x)\rangle | x \in E\},$$

where:

$$A = \{\langle x, \mu(x), \nu(x)\rangle | x \in E\} \in d_\varphi\text{-IFS}(E);$$
$$f(x) \overset{\text{def}}{=} \frac{\frac{\varphi((\mu(x),\nu(x)))}{\mu(x)+\nu(x)}}{\frac{\varphi((\alpha\mu(x),\beta\nu(x)))}{\alpha\mu(x)+\beta\nu(x)}}$$

and (8) *is valid .*

The last example considered in the paper (assuming $\alpha + \beta \in [0, 1]$) is for $L = F_{\alpha,\beta} : \text{IFS}(E) \to \text{IFS}(E)$, when $F_{\alpha,\beta}$ maps the set $B = \{\langle x, \mu^*(x), \nu^*(x)\rangle | x \in E\} \in \text{IFS}(E)$ into the set $F_{\alpha,\beta}(B) = \{\langle x, \mu^*(x) + \alpha\pi^*(x), \nu^*(x) + \beta\pi^*(x)\rangle | x \in E\}$, and π^* is the hesitancy function π_B of B. Between $\pi_{F_{\alpha,\beta}(B)}$ and π_B there exists an obvious relation

$$(\forall x \in E)\left(\pi_{F_{\alpha,\beta}(B)}(x) = (1 - \alpha - \beta)\pi_B(x)\right).$$

The direct computation gives

Lemma 4. *For any $x \in E$ $\overset{\varphi}{F}_{\alpha,\beta}$ is given by the equality*

$$\overset{\varphi}{F}_{\alpha,\beta}(A) = \{\langle x, \hat{\mu}(x), \hat{\nu}(x)\rangle | x \in E\}, \tag{9}$$

where:

$$\hat{\mu}(x) \overset{\text{def}}{=} \big(\mu(x)h(x) + \alpha\pi_{d_\varphi}(x)\big)\,\frac{\Delta_1(x)}{\Delta_2(x)}; \tag{10}$$

$$\hat{\nu}(x) \overset{\text{def}}{=} \big(\nu(x)h(x) + \beta\pi_{d_\varphi}(x)\big)\,\frac{\Delta_1(x)}{\Delta_2(x)} \tag{11}$$

and:

$$\begin{aligned}
&A = \{\langle x, \mu(x), \nu(x)\rangle | x \in E\} \in d_\varphi\text{-IFS}(E);\\
&h(x) \overset{\text{def}}{=} \frac{\varphi((\mu(x), \nu(x)))}{\mu(x) + \nu(x)};\\
&\Delta_1(x) \overset{\text{def}}{=} \varphi((\mu(x), \nu(x))) + (\alpha + \beta)\pi_{d_\varphi}(x);\\
&\Delta_2(x) \overset{\text{def}}{=} \varphi((\mu(x)h(x) + \alpha\pi_{d_\varphi}(x), \nu(x)h(x) + \beta\pi_{d_\varphi}(x)));\\
&\pi_{d_\varphi}(x) = 1 - \varphi((\mu(x), \nu(x)))
\end{aligned}$$

and (8) *is valid, as well as the defining equality*

$$\frac{\Delta_1(x)}{\Delta_2(x)} \overset{\text{def}}{=} 1, \tag{12}$$

for the case: $\alpha = \beta = 0; (\mu(x), \nu(x)) = (0, 0)$.

We will show that the operator $\overset{\varphi}{F}_{\alpha,\beta}$ is well defined. Since $\Delta_2(x_0) = 0$ is only possible if: $\alpha = \beta = 0; (\mu(x_0), \nu(x_0)) = (0, 0)$, we see that $\Delta_1(x_0) = 0$, i.e. $\overset{\varphi}{F}_{\alpha,\beta}$ is well defined because, according to (12), $\frac{\Delta_1(x_0)}{\Delta_2(x_0)} = 1$.

It can be checked directly that for any $x \in E$ it is valid

$$\varphi((\hat{\mu}(x), \hat{\nu}(x))) = \varphi((\mu(x), \nu(x))) + (\alpha + \beta)\pi_{d_\varphi}(x). \tag{13}$$

Hence

$$\pi_{d_\varphi;\overset{\varphi}{F}_{\alpha,\beta}(A)}(x) = (1 - \alpha - \beta)\pi_{d_\varphi}(x) \tag{14}$$

(where $\pi_{d_\varphi;\overset{\varphi}{F}_{\alpha,\beta}}$ is the hesitancy function for the set $\overset{\varphi}{F}_{\alpha,\beta}(A)$).

Remark 2. From (14) it is clear that for $\alpha+\beta > 0$ it is fulfilled $(\forall x \in E)(\pi_{d_\varphi;\overset{\varphi}{F}_{\alpha,\beta}(A)} < \pi_{d_\varphi}(x))$.

Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be infinite sequences of non-negative numbers, satisfying for each integer $n \geq 1$ the condition

$$\alpha_n + \beta_n \leq 1. \tag{15}$$

It can be checked directly that if $n \geq 1$ is an integer and $F_{\alpha_i,\beta_i} : \text{IFS}(E) \to \text{IFS}(E)$ are considered for: $\alpha = \alpha_i; \beta = \beta_i, i = 1, \ldots, n$, and the operators $\overset{\varphi}{F}_{\alpha_i,\beta_i} : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$ are defined with (7), with $L = F_{\alpha_i,\beta_i}, i = 1, \ldots, n$, then the equality

$$\overset{\varphi}{F}_{\alpha_n,\beta_n} \ldots \overset{\varphi}{F}_{\alpha_1,\beta_1} = T^{-1}F_{e_n,f_n}T, \tag{16}$$

is valid, where: the product of the operators in the left hand side of the above equality is to be understood as superposition; e_n and f_n are given by:

$$e_n = \alpha_1 + \sum_{k=2}^{n} \alpha_k \prod_{j=1}^{k-1}(1-\alpha_j-\beta_j); f_n = \beta_1 + \sum_{k=2}^{n} \beta_k \prod_{j=1}^{k-1}(1-\alpha_j-\beta_j); \tag{17}$$

the operator $F_{e_n,f_n} : \text{IFS}(E) \to \text{IFS}(E)$ coincides with the superposition $F_{\alpha_n,\beta_n} \dots F_{\alpha_1,\beta_1}$. On the other hand, according to (7), for $L = F_{e_n,f_n}$, we have

$$\overset{\varphi}{F}_{e_n,f_n} = T^{-1} F_{e_n,f_n} T. \tag{18}$$

Considering equalities (16) and (18), as well as (14), we reach the following important result:

Theorem 1. *For any integer $n > 1$ the operator equality*

$$\overset{\varphi}{F}_{\alpha_n,\beta_n} \dots \overset{\varphi}{F}_{\alpha_1,\beta_1} = \overset{\varphi}{F}_{e_n,f_n} \tag{19}$$

is valid and for any $A \in d_\varphi$-IFS(E), with hesitancy function π_{d_φ}, the relation

$$(\forall x \in E)\left(\pi_{d_\varphi;\overset{\varphi}{F}_{e_n,f_n}(A)}(x) = \pi_{d_\varphi}(x)\prod_{i=1}^{n}(1-\alpha_i-\beta_i)\right)$$

holds, where $\alpha_i, \beta_i \in [0,1] (i = 1, \dots, n)$, and e_n and f_n are given by (17)*.*

Definition 5. For any integer $n \geq 1$ and for $\alpha, \beta \in [0,1]$ (such that $\alpha + \beta \in [0,1]$) we introduce the operator $\left(\overset{\varphi}{F}_{\alpha,\beta}\right)^n : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$ with $\left(\overset{\varphi}{F}_{\alpha,\beta}\right)^n \overset{\text{def}}{=} \overset{\varphi}{F}_{\alpha,\beta}\left(\overset{\varphi}{F}_{\alpha,\beta}\right)^{n-1}$, where $\left(\overset{\varphi}{F}_{\alpha,\beta}\right)^0$ is the identity operator.

For $\alpha_i = \alpha, \beta_i = \beta, i = 1, \dots, n$, from (18) and from (19), based on Definition 5, it follows

Theorem 2. *For any integer $n \geq 1$ and for $\alpha, \beta \in [0,1]$ (such that $\alpha + \beta \in [0,1]$) the operator $\left(\overset{\varphi}{F}_{\alpha,\beta}\right)^n : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$ can be represented as $\left(\overset{\varphi}{F}_{\alpha,\beta}\right)^n = T^{-1}F_{e_n,f_n}T$, where:*

$$e_n = \alpha + \alpha\sum_{k=1}^{n}(1-\alpha-\beta)^{k-1}; f_n = \beta + \beta\sum_{k=1}^{n}(1-\alpha-\beta)^{k-1}$$

and for any $A \in d_\varphi$-IFS(E), with hesitancy function π_{d_φ}, is valid the relation

$$(\forall x \in E)\left(\pi_{d_\varphi;\left(\overset{\varphi}{F}_{e_n,f_n}\right)^n(A)}(x) = \pi_{d_\varphi}(x)(1-\alpha-\beta)^n\right). \tag{20}$$

Corollary 1. *For $\alpha + \beta > 0$ the representation $\left(\overset{\varphi}{F}_{\alpha,\beta}\right)^n = T^{-1}F_{e_n,f_n}T$ is valid, where:*

$$e_n = \frac{\alpha}{\alpha+\beta}(1-(1-\alpha-\beta)^n);\ f_n = \frac{\beta}{\alpha+\beta}(1-(1-\alpha-\beta)^n). \tag{21}$$

The sequences $\{e_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$, defined by (17), are convergent and their limits (denoted by e and f, respectively) satisfy the inequality $e + f \leq 1$. As a result, the existence of the limit operator $F_{e,f}$, satisfying the relation

$$\lim_{n\to\infty} F_{e_n,f_n} \overset{\text{def}}{=} F_{\lim\limits_{n\to\infty} e_n, \lim\limits_{n\to\infty} f_n} = F_{e,f}, \tag{22}$$

is established.

From (18), (22) and from Theorem 1 (see (19)) we obtain

Theorem 3. *Let the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ satisfy the condition*

$$(\forall m \geq 1)\,(\alpha_m, \beta_m \in [0,1], \alpha_m + \beta_m \in [0,1])$$

and the sequences $\{e_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$, are defined by (17)*. Then the sequence of operators $\overset{\varphi}{F}_{e_n,f_n} : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$ converges to the limit operator $\overset{\varphi}{F}_{e,f} : d_\varphi\text{-IFS}(E) \to d_\varphi\text{-IFS}(E)$, given by $\overset{\varphi}{F}_{e,f} = T^{-1}F_{e,f}T$, where: $e = \lim\limits_{n\to\infty} e_n; f = \lim\limits_{n\to\infty} f_n$.*

As a Corollary from Theorem 2, Corollary 1 and Theorem 3 we obtain:

Corollary 2. *Let $\alpha, \beta \in [0,1]$. If $\alpha + \beta > 0$, then it is fulfilled:*

(i) $\lim\limits_{n\to\infty} \left(\overset{\varphi}{F}_{\alpha,\beta}\right)^n = \overset{\varphi}{D}_{\frac{\alpha}{\alpha+\beta}}$,

where

$$\overset{\varphi}{D}_{\frac{\alpha}{\alpha+\beta}} \overset{\text{def}}{=} T^{-1}D_{\frac{\alpha}{\alpha+\beta}}T \tag{23}$$

(ii) For any $A = \{\langle x, \mu(x), \nu(x)\rangle | x \in E\} \in d_\varphi\text{-IFS}(E)$ it is fulfilled

$$(\forall x \in E)\left(\lim_{n\to\infty} \pi_{d_\varphi;\left(\overset{\varphi}{F}_{e_n,f_n}\right)^n(A)}(x) = 0\right). \tag{24}$$

Proof. From (21) and $1 - \alpha - \beta < 1$ it follows:

$$\lim_{n\to\infty} e_n = e = \frac{\alpha}{\alpha+\beta}; \lim_{n\to\infty} f_n = f = \frac{\beta}{\alpha+\beta}; f = 1 - e = 1 - \frac{\alpha}{\alpha+\beta}.$$

Hence:

$$\lim_{n\to\infty} \left(\overset{\varphi}{F}_{\alpha,\beta}\right)^n = \overset{\varphi}{F}_{e,1-e} = \overset{\varphi}{F}_{\frac{\alpha}{\alpha+\beta},1-\frac{\alpha}{\alpha+\beta}} = T^{-1}\left(F_{\frac{\alpha}{\alpha+\beta},1-\frac{\alpha}{\alpha+\beta}}\right)T = T^{-1}\left(D_{\frac{\alpha}{\alpha+\beta}}\right)T,$$

and due to (23) *(i)* is proved.

Similarly *(ii)* follows from (20), because of the inequality $1 - \alpha - \beta < 1$.

□

The operators considered here depend on two real parameters $\alpha, \beta \in [0,1]$. For $\overset{\varphi}{F}_{\alpha,\beta}$ an additional condition is imposed: $\alpha+\beta \in [0,1]$. If we consider these parameters as mappings: $\alpha : E \to [0,1]$, $\beta : E \to [0,1]$, then the respective point-wise operators: $H_{\alpha(x),\beta(x)}, J_{\alpha(x),\beta(x)}, G_{\alpha(x),\beta(x)}$, $F_{\alpha(x),\beta(x)}$, are obtained. They are defined over IFS(E), and for the last of them we impose also the restriction $(\forall x \in E)(\alpha(x) + \beta(x) \in [0,1])$. If L denotes any of the cited above point-wise operators, then L induces (by (7)) the corresponding point-wise operator $\overset{\varphi}{L}$, acting over d_φ-IFS(E). In such a way we get the point-wise operators: $\overset{\varphi}{H}_{\alpha(x),\beta(x)}$, $\overset{\varphi}{J}_{\alpha(x),\beta(x)}$, $\overset{\varphi}{G}_{\alpha(x),\beta(x)}$, $\overset{\varphi}{F}_{\alpha(x),\beta(x)}$, and the results obtained before are also valid for them. Just a small clarification is required. For Corollary 2 the additional condition $(\forall x \in E)(\alpha(x) + \beta(x) > 0)$ has to be imposed. Moreover, e_n, f_n, $(n \geq 1)$ are now not real numbers, but mappings from E to $[0,1]$ such that $(\forall x \in E)(e_n(x) + f_n(x) \in [0,1])$. Analogously e and f are also mappings from E to $[0,1]$, satisfying the condition: $(\forall x \in E)(e(x) + f(x) \leq 1)$.

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