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On the possibility for intuitionistic fuzzy interpretations of modal logic axioms

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ABSTRACT: Intuitionistic fuzzy interpretations of the axioms of the modal logics **S1, S2, S3, S4, S5, K, T** and **B** are discussed.

KEYWORDS: Axiom, Intuitionistic fuzzy logic, Modal logic

1. Introduction

Intuitionistic fuzzy logic is introduced in [1] and discussed in [2-5]. In a series of research the possibility for intuitionistic fuzzy interpretations of the axioms of different logic systems are discussed. Here we shall do this for modal logics **S1, S2, S3, S4, S5, K, T** and **B**.

We must note immediatelly, that in all research up to now (in the present, too) we have discussed only the representations from one (mentioned above) directions. The opposite direction, related to representing the intuitionistic fuzzy logics by other logics is not discussed. This will is an important open problem for the future.

2. Short remarks on intuitionistic fuzzy logic

Two real numbers, $\mu(p)$ and $\nu(p)$, are assigned to the proposition p with the following constraint to hold:

$$\mu(p) + \nu(p) \leq 1.$$

They correspond to the "truth degree" and to the "falsity degree" of p.

Let this assignment be provided by an evaluation function V, defined over a set of propositions S in such a way that:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

When values $V(p)$ and $V(q)$ of the proposition forms p and q are known, the evaluation function V can be extended also for the operations "negation" ($\neg$), "conjunction" (two forms: & and $\wedge$), "disjunction" (two forms: $\vee$ and $\sqcup$), and "implication" (two forms: $\supset$ and $\rightarrow$) through the definitions (cf. [1,5]) as follows:

$$V(\neg p) = \neg V(p) = \langle \nu(p), \mu(p)) \rangle,$$

$$V(p) \& V(q) = V(p \& q) = \langle min(\mu(p), \mu(q)), max(\nu(p), \nu(q)) \rangle,$$

$$V(p) \wedge V(q) = V(p \wedge q) = \langle \mu(p).\mu(q), \nu(p) + \nu(q) - \nu(p).\nu(q) \rangle,$$

$$V(p) \vee V(q) = V(p \vee q) = \langle max(\mu(p), \mu(q)), min(\nu(p), \nu(q))\rangle.$$

$$V(p) \sqcup V(q) = V(p \sqcup q) = \langle \mu(p) + \mu(q) - \mu(p).\mu(q), \nu(p).\nu(q)\rangle,$$

$$V(p) \supset V(q) = V(p \supset q)$$

$$= \langle 1 - (1 - \mu(q)).sg(\mu(p) - \mu(q)), \nu(q).sg(\mu(p) - \mu(q)).sg(\nu(q) - \nu(p))\rangle,$$

$$V(p) \to V(q) = V(p \to q) = \langle max(\nu(p), \mu(q)), min(\mu(p), \nu(q))\rangle,$$

where:

$$sg(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Let for every propositional form A it is a (standard) tautology if and only if

$$\mu(A) = 1 \text{ and } \nu(A) = 0.$$

The propositional form A such that $V(A) = \langle a, b\rangle$ is an *intuitionistic fuzzy tautology* (IFT) if and only if $a \geq b$.

When value $V(A)$ of the proposition form A is known, the valuation function V can be extended also for the modal operators "$\Box$" and "$\Diamond$", respectively, by means of the definitions (cf. [2,5]):

$$V(\Box p) = \Box V(p) = \langle \mu(p), 1 - \mu(p)\rangle,$$

$$V(\Diamond p) = \Diamond V(p) = \langle 1 - \nu(p), \nu(p)\rangle.$$

There are other intuitionistic fuzzy logics operations, that are not discussed here, because they are more complex and the work with them is more difficult.

3. On the intuitionistic fuzzy interpretations of the axioms of some modal logics

We shall construct intuitionistic fuzzy interpretations of the axioms of the modal logics **S1, S2, S3, S4, S5, K, T** and **B**.

Now, following [6,7], we shall introduce the axioms of the modal logics **S1, S2, S3, S4, S5, K, T** and **B**.

System S1: 1. If A is an axiom in the first order logic, then $\Box A$ is an axiom.

2. $\Box(\Box A \supset A)$,

3. $\Box((\Box(A \supset B) \& \Box(B \supset C)) \supset \Box(A \supset C))$.

Rules: 1. $\dfrac{A,\ A \supset B}{B}$ (Modus Ponens),

2. $\dfrac{\Box A}{A}$,

3. $\dfrac{\Box(A \supset B),\ \Box(B \supset A)}{\Box(\Box A \supset \Box B)}$.

System S2: S1 + $\{\Box(\Box A \supset \Box(A \vee B))\}$.

System S3: S2 + $\{\Box(\Box(A \supset B) \supset \Box(\Box A \supset \Box B))\}$.

System S4: S3 + $\{\Box(\Box(A \supset \Box\Box A)\}$.

System S5: S4 + $\{\Box(A \supset \Box\Diamond A)\}$.

System K: 1. All axioms in the first order logic.

2. $\Box(A \supset B) \supset (\Box A \supset \Box B)$.

Rules: 1. Modus Ponens,

2. $\dfrac{A}{\Box A}$.

System T: K + $\{\Box A \supset A\}$.

System B: T + $\{A \supset \Box \Diamond A\}$.

Let us assume everywhere below that

$$V(A) = \langle a, b\rangle,$$

$$V(B) = \langle c, d\rangle,$$

$$V(C) = \langle e, f\rangle,$$

where $a, b, c, d, e, f, a + b, c + d, e + f \in [0, 1]$.

Following [1,5] we shall mentione that all axioms of the first order logic from [8]

$$\begin{array}{ll}
(a) & A \supset A \\
(b) & A \supset (B \supset A) \\
(c) & (A\&B) \supset A \\
(d) & (A\&B) \supset B \\
(e) & A \supset (A \vee B) \\
(f) & B \supset (A \vee B) \\
(g) & A \supset (B \supset (A\&B)) \\
(h) & (A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)) \\
(i) & \neg\neg A \supset A \\
(j) & (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \\
(k) & (\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)
\end{array}$$

are valid for the $(max - min)$-implication. All of them except axilom (k) are valid for the sg-implication

In [1,5] it is shown that Modus Ponens is valid for the sg-implication, while it is not valid for the $(max - min)$-implication. On the other hand, the expression

$$(A\&(A \to B)) \to B$$

is an IFT and it is equipolen in the standard first order logic with Modus Ponens.

We must note that as rule $\dfrac{A}{\Box A}$ together with its equipolent expression $A \to \Box A$ are not valid for the $(max - min)$-implication.

Our main result is the following

Theorem The axioms and rules of the modal logics **S1, S2, S3, S4, S5, K, T** and **B** are tautologies for the sg-implication and IFTs for the $(max - min)$-implication, except the above mentioned cases.

Proof: The validity of Axiom 3 from **S1** is checked for the &−form of the conjunction and sg-form of implication, as follows:

$$X \equiv V(\Box((\Box(A \supset B)\&\Box(B \supset C)) \supset \Box(A \supset C)))$$

$$= \Box((\Box(\langle a, b\rangle \supset \langle c, d\rangle)\&\Box(\langle c, d\rangle \supset \langle e, f\rangle)) \supset \Box(\langle a, b\rangle \supset \langle e, f\rangle))$$

$$= \Box((\Box(\langle 1-(1-c).sg(a-c), d.sg(a-c).sg(d-b)\rangle)\&\Box(\langle 1-(1-e).sg(c-e),$$

$$f.sg(c-e).sg(f-d)\rangle)) \supset \Box(\langle 1-(1-e).sg(a-e), f.sg(a-e).sg(f-b)\rangle))$$

$$= \Box(((\langle 1-(1-c).sg(a-c), (1-c).sg(a-c)\rangle \& \langle 1-(1-e).sg(c-e),$$

$$(1-e).sg(c-e)\rangle)) \supset \langle 1-(1-e).sg(a-e), (1-e).sg(a-e)\rangle))$$

$$= \Box(\langle min(1-(1-c).sg(a-c), 1-(1-e).sg(c-e)), max((1-c).sg(a-c)(1-e).sg(c-e))\rangle$$

$$\supset \langle 1-(1-e).sg(a-e), (1-e).sg(a-e)\rangle))$$

$$= \Box\langle 1-(1-(1-(1-e).sg(a-e))).sg(min(1-(1-c).sg(a-c), 1-(1-e).sg(c-e))$$

$$-(1-(1-e).sg(a-e))), (1-e).sg(a-e).sg(min(1-(1-c).sg(a-c),$$

$$1-(1-e).sg(c-e))-(1-(1-e).sg(a-e)))$$

$$.sg((1-e).sg(a-e)-max((1-c).sg(a-c)(1-e).sg(c-e))\rangle.$$

From

$$Y = min(1-(1-c).sg(a-c), 1-(1-e).sg(c-e))-(1-(1-e).sg(a-e))$$

$$= min(1-(1-c).sg(a-c), 1-(1-e).sg(c-e))-1+(1-e).sg(a-e)$$

$$= 1-max((1-c).sg(a-c), (1-e).sg(c-e))-1+(1-e).sg(a-e)$$

$$= (1-e).sg(a-e)-max((1-c).sg(a-c), (1-e).sg(c-e))\ leq 0,$$

it follows that $sg(Y) = 0$ and, therefore,

$$X = \langle 1, 0\rangle,$$

i.e. Axiom 3 is a tautology in the intuitionistic fuzzy interpretation for the &−case of the conjunction and sg-form of implication.

For comparison, we shall prove that the same Axiom 3 from **S1** about the &−form of the conjunction and for the $(max-min)$-implication is an IFT:

$$V(\Box((\Box(A \to B)\&\Box(B \to C)) \to \Box(A \to C)))$$

$$= \Box((\Box(\langle a,b\rangle \to \langle c,d\rangle)\&\Box(\langle c,d\rangle \to \langle e,f\rangle)) \to \Box(\langle a,b\rangle \to \langle e,f\rangle))$$

$$= \Box((\Box\langle max(b,c), min(a,d)\rangle \& \Box\langle max(d,e), min(c,f)\rangle) \to \Box\langle max(b,e), min(a,f)\rangle)$$

$$= \Box((\langle max(b,c), 1-max(b,c)\rangle \& \langle max(d,e), 1-max(d,e)\rangle) \to \langle max(b,e), 1-max(b,e)\rangle)$$

$$= \Box(\langle min(max(b,c), max(d,e)), max(1-max(b,c), 1-max(d,e)\rangle$$

$$\to \langle max(b,e), 1-max(b,e)\rangle)$$

$$= \Box(\langle max(1-max(b,c), 1-max(d,e), b, e), min(max(b,c), max(d,e), 1-max(b,e)\rangle)$$

$$= \langle max(1-max(b,c), 1-max(d,e), b, e), 1-max(1-max(b,c), 1-max(d,e), b, e)\rangle.$$

Let

$$X \equiv max(1-max(b,c), 1-max(d,e), b, e)-(1-max(1-max(b,c), 1-max(d,e), b, e))$$

$$= 2.max(1-max(b,c), 1-max(d,e), b, e)-1.$$

If b or $e \geq \frac{1}{2}$, then

$$X \geq 2.\frac{1}{2} - 1 = 0.$$

Let us assume that $b, e < \frac{1}{2}$. If $c \leq b$ or $d \leq e$, then $1 - c \geq 1 - b > \frac{1}{2}$ or $1 - d \geq 1 - e > \frac{1}{2}$ and therefore,

$$X \geq 2.\frac{1}{2} - 1 = 0.$$

Let us assume that $c > b$ and $d > e$ (but by definition $c + d \leq 1$). Then

$$X = 2.max(1 - c, 1 - d, b, e) - 1.$$

If $1 - c$ or $1 - d \geq \frac{1}{2}$, then

$$X \geq 2.\frac{1}{2} - 1 = 0.$$

Let us assume that $1 - c, 1 - d < \frac{1}{2}$. Then $c + d > 1$, which is impossible. Therefore, always $X \geq 0$, i.e. Axiom 3 is an IFT in the intuitionistic fuzzy interpretation for the $\&-$case of the conjunction and for the $(max - min)$-implication.

Analogically, it is checked the validity of each other axioms and rules for the $\&-$ and $\wedge-$cases of the conjunction, for the $\vee-$ and $\sqcup-$cases of the disjunction and for $\supset$- and $\rightarrow$-cases of the implication.

4. Open problems and future intentions

As we noted above, the opposite interpretation is an open problem. Its solving will help to determine the place of the intuitionistic fuzzy logic between the other logics.

The appearance of the intuitionistic fuzzy logic generated a lot of interesting problems. The first of them is related to the two modal operators $\Box$ and $\Diamond$.

For every propositional form A the equality is valid (cf. [7]):

$$\Diamond A = \neg \Box \neg A.$$

Its validity of is directly checked for the intuitionistic fuzzy case. This important for the modal logic equality shows not only the relation, but also the close interrelationship between both modal operator; their dualism. On the other hand, in the intuitionistic fuzzy logic a lot of new modal type operators, extending the two classical operators, are defined. Some of them are the following (see [2,5]).

Let A be a fixed propositional form for which the truth-value function V is

$$V(A) = \langle a, b \rangle, \tag{1}$$

where $a, b \in [0, 1], a + b \leq 1$ and let $\alpha, \beta \in [0, 1]$. We define operators $F_{\alpha,\beta}$ (for $\alpha + \beta \leq 1$), $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$ and $J^*_{\alpha,\beta}$ by:

$$\begin{aligned}
&V(F_{\alpha,\beta}(A)) = \langle a + \alpha.(1 - a - b), b + \beta.(1 - a - b) \rangle, \text{ for } \alpha + \beta \leq 1,\\
&V(G_{\alpha,\beta}(A)) = \langle \alpha.a, \beta.b \rangle,\\
&V(H_{\alpha,\beta}(A)) = \langle \alpha.a, b + \beta.(1 - a - b) \rangle,\\
&V(H^*_{\alpha,\beta}(A)) = \langle \alpha.a, b + \beta.(1 - \alpha.a - b) \rangle,\\
&V(J_{\alpha,\beta}(A)) = \langle a + \alpha.(1 - a - b), \beta.b \rangle,\\
&V(J^*_{\alpha,\beta}(A)) = \langle a + \alpha.(1 - a - \beta.b), \beta.b \rangle.
\end{aligned}$$

Now, we can seen immediatelly that for each proposition form A:

$$\Box A = D_0(A) = F_{0,1}(A),$$

$$\Diamond A = D_1(A) = F_{1,0}(A).$$

Therefore, in the intuitionistic fuzzy logic there exist operators, that generalize both standard modal logic operators. Now the later operators can be interpreted only as boundary points of operator D_α ($\alpha \in [0,1]$) This is a particular case of operator $F_{\alpha,\beta}$ (for $\alpha, \beta, \alpha+\beta \in [0,1]$), because for each proposition form A and for each $\alpha \in [0,1]$:

$$D_\alpha(A) = F_{\alpha,1-\alpha}(A).$$

Are there analogues of the intuitionistic fuzzy logic operators in standard modal logic?

The answer to this question is not clear. It is possibly related to the fact that the intuitionistic fuzzy modal logic only with the intuitionistic fuzzy interpretations of two operators $\Box$ and $\Diamond$ is essentially larger that the fuzzy logic, because for every propositional form A than satisfies (1):

$$V(\Box A) = V(A) = V(\Diamond A),$$

i.e., these two operators have no sense in the case of fuzzy logic.

Of course, other interpretations of the modal operatots can be constructed for the case of fuzzy logic, but they will not have the simplicity of the above discussed intuitionistic fuzzy interpretations. On the other hand, as it has already been shown, there are other intuitionistic fuzzy operators that extend the modal operators.

References:

[1] Atanassov K., Two variants of intuitonistc fuzzy propositional calculus. Preprint IM-MFAIS-5-88, Sofia, 1988.

[2] Atanassov K., Two variants of intuitionistic fuzzy modal logic Preprint IM-MFAIS-3-89, Sofia, 1989.

[3] Atanassov K., Gargov G., Intuitionistic fuzzy logic. Comptes Rendus de l'Academie bulgare des Sciences, Tome 43, 1990, No. 3, 9-12.

[4] Gargov G., Atanassov K., Two results in intuitionistic fuzzy logic, Comptes Rendus de l'Academie bulgare des Sciences, Tome 45, 1992, No. 12, 29-31.

[5] Atanassov K., Intuitionistic Fuzzy Sets, Springer Physica-Verlag, Berlin, 1999.

[6] Sobolev S., Modal logic, in "Mathematical Enciclopedy", Vol. 3, Moskow, Soviet Union Enciclopedy, 1982 (in Russian).

[7] Feys R., Modal logics, Paris, 1965.

[8] Mendelson E., Introduction to mathematical logic, Princeton, NJ: D. Van Nostrand, 1964.