Generalized negations and Intuitionistic Fuzzy Sets
A criticism to a widely used terminology

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Abstract

Intuitionistic Fuzzy Sets Theory is based on a wrong nominalistic (terminological) assumption. It is defined as "intuitionistic" a negation which does not satisfy usual properties of the intuitionistic Brouwer negation, but it is called with this term only a particular generalized notion of negation which indeed corresponds to the de Morgan negation.
This metatheoretical assumption is criticized and the role of different generalized negations is discussed.

Keywords: Brouwer negation, Kleene negation, intuitionistic fuzzy sets, HW algebras.

1 From Boolean negation to some generalized notions of negation

It is well known that in the particular case of a distributive lattice $\langle \Sigma, \wedge, \vee, 0, 1\rangle$, bounded by the *least* element 0 and the *greatest* element 1 and whose induced partial order relation is the usual one $a \leq b$ iff $a = a \wedge b$ (equivalently, $b = a \vee b$), a *standard orthocomplementation* is a mapping $' : \Sigma \mapsto \Sigma$ which satisfies the following conditions whatever be $a, b \in \Sigma$:

(oc-1) $a = a''$ (involution)

(oc-2a) $a \leq b$ implies $b' \leq a'$ (antimorphism)
(oc-2b) $a' \leq b'$ implies $b \leq a$ (dual antimorphism)
(oc-2c) $a' \wedge b' = (a \vee b)'$ ($\wedge$ de Morgan)
(oc-2d) $a' \vee b' = (a \wedge b)'$ ($\vee$ de Morgan)

(oc-3a) $a \wedge a' = 0$ (non contradiction)
(oc-3b) $a \vee a' = 1$ (excluded middle)

We have gathered these conditions in three groups since it is easy to prove that:

- under condition (oc-1) all conditions (oc-2a)–(oc-2d) are mutually equivalent;

- under conditions (oc-1) and (oc-2) also the two conditions (oc-3a) and (oc-3b) are equivalent.

In the following Sections we generalize this standard notion of orthocomplementation weakening some of the previous properties and leading, in this way, to the study of a certain number of *unusual orthocomplementations.* In particular we consider

(i) the *Kleene* orthocomplementation in which neither the non contradiction law nor the excluded middle law hold;

(ii) the *Brouwer* orthocomplementation in which condition ($oc-1$) is substituted by a weaker condition, moreover only one of the de Morgan laws is accepted, and the excluded middle law is not admitted.

Further, we will introduce some algebras to describe these two orthocomplementations, at a first step separately and then joining them in a unique structure.

2 de Morgan and Kleene lattices

Definition 2.1. A *de Morgan distributive lattice* is a structure $\langle \Sigma, \wedge, \vee, ', 0, 1\rangle$ where

- $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ is a distributive lattice, bounded by the least element 0 and the greatest element 1;
- $'$ is a unary operation on Σ, called *de Morgan* complement, that satisfies the following conditions for arbitrary $a, b \in \Sigma$:

(K1) $a = a''$
(K2) $(a \vee b)' = a' \wedge b'$

A *Kleene distributive lattice* is a de Morgan lattice which satisfies the further condition:

(K3) $a \wedge a' \leq b \vee b'$ (Kleene condition)

As said in section 1, under condition (K1) the de Morgan property (K2) is equivalent to the $\vee$ de Morgan law and both the antimorphism conditions, which, consequently, hold in any de Morgan (and thus also in any Kleene) lattice. Note that $1 = 0'$. However, both the excluded middle law and the non contradiction principle are not generally true. This means that contradictions are not necessarily false, while the disjunction of a proposition and its negation is not necessarily true. In particular, Kleene lattices represent an adequate abstract tool to model fuzzy concrete situations.

3 Brouwer lattices

Definition 3.1. A *distributive lattice with weak Brouwer complementation* is a structure $\langle \Sigma, \wedge, \vee, ^{\sim}, 0, 1\rangle$ where

- $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ is a distributive lattice, bounded by the least element 0 and the greatest element 1;
- $^{\sim}$ is a unary operation on Σ, called *weak Brouwer* or *weak intuitionistic* (in [6] also called *minimal*) complementation, that satisfies the following conditions:

(B1) $a \leq a^{\sim\sim}$ (i.e., $a = a \wedge a^{\sim\sim}$)
(B2) $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$

A *Brouwer distributive lattice* is a distributive lattice with weak Brouwer complementation which satisfies the further condition:

(B3) $a \wedge a^{\sim} = 0$ (non contradiction)

In this case $^{\sim}$ is called *Brouwer* or *intuitionistic* complement.

Note that $1 = 0^{\sim}$. Under condition (B1) the de Morgan law (B2) is equivalent to the contraposition law "if $a \leq b$ then $b^{\sim} \leq a^{\sim}$". In general, the dual de Morgan law $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$, the converse contraposition "$a^{\sim} \leq b^{\sim}$ implies $b \leq a$", and the excluded middle law are not verified.

Thus, the Brouwer complement is a good algebraic axiomatization of an intuitionistic negation: it does not satisfy both the excluded middle law and the double negation law, whereas the non contradiction principle is satisfied.

Indeed, as stated by Church [5]: "Brouwer proposes that the law of excluded middle should not be regarded as an admissible logical principle, and expresses, as a basis for his proposal, doubts concerning the truth of this law (...) He says also that to assert the law of excluded middle is equivalent to asserting the doubtful proposition that every proposed theorem can be either proved or disproved".

Moreover, quoting from [7], "intuitionistic sentence logic lacks some classical theorems, including double negation and excluded middle". On the other hand it satisfies the non contradiction principle "which is accepted by intuitionistic logicians: one cannot know at the same time a proposition and its negation" [6].

4 Weak Brouwer–de Morgan lattices and Brouwer–Kleene lattices

So far we have analyzed two disjoint notions of complementation. Now, we put them together obtaining new lattice structures ([4]).

Definition 4.1. A system $\langle \Sigma, \wedge, \vee, ', ^\sim, 0, 1\rangle$ is a *weak Brouwer de Morgan (wBD) lattice* iff

1. the substructure $\langle \Sigma, \wedge, \vee, ', 0, 1\rangle$ is a de Morgan lattice
2. the substructure $\langle \Sigma, \wedge, \vee, ^\sim, 0, 1\rangle$ is a weak Brouwer lattice
3. The two complementations are linked by the following *interconnection rule*: $a^{\sim\prime} = a^{\sim\sim}$.

A *Brouwer–Kleene (BK) lattice* is a system $\langle \Sigma, \wedge, \vee, ', ^\sim, 0, 1\rangle$ in which the substructure $\langle \Sigma, \wedge, \vee, ', 0, 1\rangle$ is a Kleene lattice and the substructure $\langle \Sigma, \wedge, \vee, ^\sim, 0, 1\rangle$ is a Brouwer lattice satisfying the above interconnection rule.

Making use of these two negations one can define a further unusual complementation, either in the case of a wBD lattice or of a BK lattice, defined as $a^\flat := a^{\prime\sim\prime}$ and called the *anti – Brouwer* negation. This negation satisfies the following properties in the case of a wBD lattice:

(AB1) $a^{\flat\flat} \leq a$;

(AB2) $a^\flat \vee b^\flat = (a \wedge b)^\flat$ [equivalently, $a \leq b$ implies $b^\flat \leq a^\flat$].

In the case of a BK lattice the following further condition holds:

(AB3) $a \vee a^\flat = 1$.

5 Łukasiewicz implication

We have seen that Kleene lattices have some desirable properties in a fuzzy context. A more complete environment to describe many valued situations is given by Wajsberg algebras ([9]), which axiomatize the Łukasiewicz implication.

Definition 5.1. A Wajsberg algebra is a structure $\langle \Sigma, \rightarrow_L, ', 1\rangle$ satisfying the following axioms

(W1) $1 \rightarrow_L a = a$

(W2) $(a \rightarrow_L b) \rightarrow_L ((b \rightarrow_L c) \rightarrow_L (a \rightarrow_L c)) = 1$

(W3) $(a \rightarrow_L b) \rightarrow_L b = (b \rightarrow_L a) \rightarrow_L a$

(W4) $(a' \rightarrow_L b') \rightarrow_L (b \rightarrow_L a) = 1$

The following result is straightforward.

Proposition 5.2. *Let $\langle \Sigma, \rightarrow_L, ', 1\rangle$ be a Wajsberg algebra. Then, once set $0 = 1'$, we can introduce the following operations*

$$a \wedge b := ((a' \rightarrow_L b') \rightarrow_L b')'$$
$$a \vee b := (a \rightarrow_L b) \rightarrow_L b$$
$$a' := a \rightarrow_L 0$$

It turns out that the obtained structure $\langle \Sigma, \wedge, \vee, ', 0, 1\rangle$ is a Kleene lattice. The partial order relation induced from the lattice structure ($a \leq b$ iff $a = a \wedge b$) is equivalently expressed as:

$$a \leq b \quad \textit{iff} \quad a \rightarrow_L b = 1$$

6 Heyting implication

Brouwer distributive lattices are a good algebraic axiomatization of an intuitionistic negation. A more complete and adequate structure to model intuitionistic logic is given by the notion of *Heyting algebra*([8]), where as a primitive operator, we have a Gödel implication.

Definition 6.1. A *Heyting algebra* is a structure $\langle \Sigma, \wedge, \vee, \rightarrow_G, 0\rangle$ which satisfies the following axioms:

(H1) $a \rightarrow_G a = b \rightarrow_G b$
(H2) $(a \rightarrow_G b) \wedge b = b$
(H3) $a \rightarrow_G (b \wedge c) = (a \rightarrow_G c) \wedge (a \rightarrow_G b)$
(H4) $a \wedge (a \rightarrow_G b) = a \wedge b$
(H5) $(a \vee b) \rightarrow_G c = (a \rightarrow_G c) \wedge (b \rightarrow_G c)$
(H6) $0 \wedge a = 0$

Proposition 6.2. *Let* $\langle \Sigma, \wedge, \vee, \rightarrow_G, 0\rangle$ *be a* Heyting algebra. *Then,setting* $1 := 0 \rightarrow_G 0$, *the following hold:*

1. $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ *is a bounded distributive lattice;*

2. $\rightarrow_G$ *is a* residual *operation, i.e., it satisfies the condition* $a \rightarrow_G b = max\{c \in \Sigma : a \wedge c \leq b\}$.

Let us note that a Brouwer complement $^\sim$ can be naturally defined in any Heyting algebra as $a^\sim := a \rightarrow_G 0$ and the structure $\langle \Sigma, \wedge, \vee, ^\sim, 0, 1\rangle$ is a Brouwer lattice where in particular $1 = 0^\sim$.

If a Heyting algebra is enriched with a de Morgan negation, the resulting structure is called symmetric Heyting algebra [8].

Definition 6.3. A *symmetric Heyting algebra* is a structure $\langle \Sigma, \wedge, \vee, \rightarrow_G, ', 0\rangle$ satisfying axioms $(H1)$– $(H5)$ and the further conditions:

(H7) $a'' = a$
(H8) $(a \wedge b)' = a' \vee b'$

7 Heyting–Wajsberg Algebras

As BK lattices are obtained by pasting the two structures of Brouwer and Kleene lattices, now we introduce HW algebras ([2]) as structures able to past together both Łukasiewicz and Gödel implications.

Definition 7.1. A system $\langle \Sigma, \rightarrow_L, \rightarrow_G, 0\rangle$ is a *Heyting Wajsberg* (HW) *algebra* if Σ is a non empty set, $0 \in \Sigma$, and $\rightarrow_L, \rightarrow_G$ are binary operators, such that, once defined

$$\begin{aligned} a \wedge b &:= ((a' \rightarrow_L b') \rightarrow_L b')' \\ a \vee b &:= (a \rightarrow_L b) \rightarrow_L b \\ a' &:= a \rightarrow_L 0 \\ a^\sim &:= a \rightarrow_G 0 \\ 1 &:= 0' \end{aligned}$$

the following are satisfied:

(HW1) $a \rightarrow_G a = 1$
(HW2) $(a \rightarrow_G b) \wedge b = b$
(HW3) $a \rightarrow_G (b \wedge c) = (a \rightarrow_G c) \wedge (a \rightarrow_G b)$
(HW4) $a \wedge (a \rightarrow_G b) = a \wedge b$
(HW5) $(a \vee b) \rightarrow_G c = (a \rightarrow_G c) \wedge (b \rightarrow_G c)$
(HW6) $1 \rightarrow_L a = a$
(HW7) $a \rightarrow_L (b \rightarrow_L c) = (a \rightarrow_L c)' \rightarrow_L b'$
(HW8) $a^{\sim\prime} \rightarrow_L a^{\sim\sim} = 1$
(HW9) $(a \rightarrow_G b) \rightarrow_L (a \rightarrow_L b) = 1$

On any HW algebra we can define a partial order relation according to one of the following mutually equivalent ways:

$$a \leq b \quad \text{iff} \quad a \wedge b = a \tag{7.1a}$$
$$\text{iff} \quad a \rightarrow_L b = 1 \tag{7.1b}$$
$$\text{iff} \quad a \rightarrow_G b = 1 \tag{7.1c}$$

From the above definition it is easy to prove that the primitive operator $\rightarrow_L$ behaves as a Łukasiewicz implication, that is the structure $\langle \Sigma, \rightarrow_L, ', 1\rangle$ is a Wajsberg algebra. In particular, the unary operator $a' := a \rightarrow_L 0$ is a Kleene complementation.
On the other hand, the primitive operator

$\rightarrow_G$ behaves as a Gödel implication, i.e., the structure $\langle \Sigma, \wedge, \vee, \rightarrow_G, 0 \rangle$ is a Heyting algebra. In particular, the unary operation $a^\sim = a \rightarrow_G 0$ is a *Brouwer* orthocomplementation.
Finally, from any HW algebra one can induce a lattice structure endowed with the Kleene and the Brouwer negation introduced above, which can be easily proved to be a distributive BK lattice.

Proposition 7.2. *Let $\langle \Sigma, \rightarrow_L, \rightarrow_G, 0 \rangle$ be a HW algebra. Then, by introducing the operations $\wedge$, $\vee$, and $'$, $^\sim$, and the element 1 as in Definition (7.1), we have that $(\Sigma, \wedge, \vee, ', ^\sim, 0, 1)$ is a distributive BK lattice.*

8 Fuzzy Sets as HW algebras

Let us consider a set of objects X, called the universe. A *fuzzy set* or *generalized characteristic functional* on X is defined as usual as a $[0,1]$–valued function on X: $f : X \mapsto [0,1]$. In the sequel, for any fixed $k \in [0,1]$ we denote by $\mathbf{k}$ the fuzzy set $\forall x \in X$, $\mathbf{k}(x) = k$.

Proposition 8.1. *Let $\mathcal{F}(X) = [0,1]^X$ be the collection of all fuzzy sets on the universe X. Let us define the operators*

$$(f_1 \rightarrow_L f_2)(x) := \min\{1, 1 - f_1(x) + f_2(x)\}$$

$$(f_1 \rightarrow_G f_2)(x) := \begin{cases} 1 & f_1(x) \leq f_2(x) \\ f_2(x) & \textit{otherwise} \end{cases}$$

and the identically 0 fuzzy set $\mathbf{0}$. Then the structure $\langle [0,1]^X, \rightarrow_L, \rightarrow_G, \mathbf{0} \rangle$ is a HW algebra.
In particular, $\mathcal{F}(X)$ is also a BK lattice with respect to the HW induced

- lattice operations
$$(f_1 \wedge f_2)(x) = \min\{f_1(x), f_2(x)\}$$
$$(f_1 \vee f_2)(x) = \max\{f_1(x), f_2(x)\}$$
whose corresponding partial order is the usual pointwise partial order on fuzzy sets:
$$f_1 \leq f_2 \quad \text{iff} \quad \forall x \in X,\ f_1(x) \leq f_2(x)$$
- complementations
$$f'(x) = 1 - f(x) \qquad \text{(Kleene)}$$
$$f^\sim(x) = \begin{cases} 1 & \text{if } f(x) = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Brouwer)}$$

9 Intuitionistic Fuzzy Sets

In the previous sections we analyzed two generalized notions of complementation and we gave them an algebraic structure. Now, we will see how these negations are related to Intuitionistic Fuzzy Sets (IFS). First, we give some definitions and some important properties about IFS.

Definition 9.1. Let X be a set of objects, called the universe. An *intuitionistic fuzzy set (IFS)* on X is any pair of fuzzy sets $A = \langle f_A, g_A \rangle \in \mathcal{F}(X) \times \mathcal{F}(X)$ such that for all $x \in X$ $f_A(x) + g_A(x) \leq 1$.

Definition 9.2. Let $\mathcal{IF}(X)$ be the collection of IFS on the universe X. Then we can define on it the two binary operations

$$\langle f_A, g_A \rangle \cap \langle f_B, g_B \rangle = \langle f_A \wedge f_B, g_A \vee g_B \rangle$$
$$\langle f_A, g_A \rangle \cup \langle f_B, g_B \rangle = \langle f_A \vee f_B, g_A \wedge g_B \rangle$$

It can be easily shown that $\mathcal{IF}(X)$ equipped with the above $\cap$, $\cup$ operations is a distributive lattice bounded by the least element $(\mathbf{0}, \mathbf{1})$ and the greatest element $(\mathbf{1}, \mathbf{0})$, whose induced partial order relation is:

$$\langle f_A, g_A \rangle \subseteq \langle f_B, g_B \rangle \quad \text{iff} \quad \forall x \in X, \quad f_A(x) \leq f_B(x) \text{ and } g_B(x) \leq g_A(x).$$

The unary operation defined for any arbitrary IFS $\langle f_A, g_A \rangle$ by $\langle f_A, g_A \rangle^- = \langle g_A, f_A \rangle$ is a de Morgan complementation. That is for any pair of IFSs $A = \langle f_A, g_A \rangle$ and $B = \langle f_B, g_B \rangle$ the following hold:

(K1) $(A^-)^- = A$
(K2) $(A \cup B)^- = A^- \cap B^-$

The Kleene condition (K3), i.e., $A \cap A^- \subseteq B \cup B^-$ for arbitrary IFSs A and B, is not generally valid. Let us consider for instance the two IFSs $A = \langle \mathbf{0.4}, \mathbf{0.5} \rangle$ and $B = \langle \mathbf{0}, \mathbf{0.2} \rangle$. Then $A \cap A^- = \langle \mathbf{0.4}, \mathbf{0.5} \rangle \not\subseteq \langle \mathbf{0.2}, \mathbf{0} \rangle = B \cup B^-$.

Further, as can be seen, the negation $^-$ does not satisfy the excluded middle law (for instance $(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \cup (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})^- = (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \neq (\mathbf{1}, \mathbf{0})$). However, this is not enough to justify the name of Intuitionistic Fuzzy Sets as stated by Atanassov: "The definition makes clear that for the so constructed new type of FS [i.e., fuzzy set] the logical law of the excluded middle law is not valid, similarly to the case in intuitionistic mathematics. Herefrom emerges the name of that set" ([1]).
We want to stress that, as we have previously remarked, another fundamental requirement which must be satisfied by an intuitionistic negation is the non contradiction principle, which in the Atanassov structure does not hold. For instance $(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \cap (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})^- = (\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}) \neq (\mathbf{0}, \mathbf{1})$.
Furthermore, and this is another drawback of the Atanassov claim about intuitionistic behavior of the above IFS complement $^-$, the double negation law (K1) is satisfied, contrary to the intuitionistic rejection of this principle.

Hence, the unitary operator $^-$ is not an algebraic realization of the Brouwer (intuitionistic) negation, but on the contrary of the de Morgan one.

10 Conclusion

We have defined some generalized notions of non standard negation and seen some of their possible algebrization. Then, we showed that the usual notion of negation on IFS is not an intuitionistic negation as stated by Atanassov in his founding work on IFS ([1]), indeed it is a de Morgan negation.
In the other work presented to this conference ([3]), we analyze the relationship between orthopair fuzzy sets and IFS and we show how IFS are related to the above introduced algebras.

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References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst. **20** (1986), 87–96.

[2] G. Cattaneo and D. Ciucci, *Heyting Wajsberg algebras as an abstract environment linking fuzzy and rough sets*, LNAI, vol. 2475, Springer, 2002, pp. 77–84.

[3] ———, *Intuitionistic fuzzy sets or orthopair fuzzy sets? a criticism to a widely used terminology*, EUSFLAT03, 2003.

[4] G. Cattaneo and G. Nisticò, *Brouwer-Zadeh posets and three valued Łukasiewicz posets*, Fuzzy Sets Syst. **33** (1989), 165–190.

[5] A. Church, *On the law of excluded middle*, Bull. Amer. Math. Soc. **34** (1928), 75–78.

[6] J.M. Dunn, *Relevance logic and entailment*, Handbook of Philosophical Logic (D. Gabbay and F. Guenther, eds.), vol. 3, Kluwer, 1986.

[7] S. Haack, *Philosophy of logics*, Cambridge University Press, 1978.

[8] A. A. Monteiro, *Sur les algèbres de Heyting symétriques*, Portugaliae Mathematica **39** (1980), 1–237.

[9] S. Surma, *Logical works*, Polish Academy of Sciences, Wroclaw, 1977.