Notes on Intuitionistic Fuzzy Sets
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Triangular norm-based intuitionistic fuzzy BE-algebras and filters

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Abstract: In this paper, intuitionistic fuzzy BE-algebra that a generalization of the BCK-algebra is introduced with respect to t-norms and t-conorms. The various algebraic properties of triangular norm-based intuitionistic fuzzy BE-algebras are studied in detail, and the characteristics of norm-based α-cuts are examined. Furthermore, the notion of intuitionistic fuzzy filters in the context of triangular BE-algebras is defined and analyzed. Theoretical results and properties related to these structures are discussed.

Keywords: Intuitionistic fuzzy sets, intuitionistic fuzzy BE-algebra, t-norms, intuitionistic fuzzy filter.
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1 Introduction

Fuzzy set theory, introduced by Lotfi A. Zadeh in 1965, revolutionized the way uncertainty and imprecision are handled in mathematical modeling and decision-making processes [20, 21]. By replacing the traditional two-valued logic of classical set theory with a membership function that assigns each element a degree of belonging within the unit interval $[0, 1]$, fuzzy sets provided a more flexible and realistic approach to dealing with vague and uncertain information. This foundational concept has been widely applied in various fields, including control systems, artificial intelligence, pattern recognition, and expert systems, where binary logic fails to capture the complexity of real-world phenomena.

Building upon Zadeh's work, Krassimir T. Atanassov introduced the concept of intuitionistic fuzzy sets (IFS) in 1983 as an extension of fuzzy set theory [2, 3]. In this generalization, the membership function is expanded to include both a degree of membership and a degree of non-membership, denoted as $\mu_A(x)$ and $\nu_A(x)$, respectively, with the constraint that their sum does not exceed one: $\mu_A(x) + \nu_A(x) \leq 1$. The additional component that emerges from this formulation is known as the hesitation degree or intuitionistic index, defined as: $\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$, where $\pi_A(x) \geq 0$. This hesitation degree represents the level of uncertainty or lack of knowledge about the membership of an element to a given set. Unlike classical fuzzy sets, which only account for partial truth, intuitionistic fuzzy sets provide a more nuanced representation of uncertainty by explicitly incorporating the hesitation degree. Due to their enhanced capacity for handling uncertainty, intuitionistic fuzzy sets have attracted significant research interest in both topological and algebraic structures, leading to the development of new mathematical frameworks and computational techniques. Many studies have explored operations, relations, and different aggregation techniques within intuitionistic fuzzy environments, enabling more sophisticated analysis and modeling.

Beyond theoretical advancements, intuitionistic fuzzy sets have found extensive applications in solving real-world problems, particularly in complex decision-making processes. They are widely utilized in medical diagnosis, engineering optimization, risk assessment, pattern recognition, supply chain management, and financial analysis. In health sciences, for instance, they help model uncertainty in diagnosing diseases based on incomplete or imprecise patient data. Similarly, in technological advancements, they assist in refining machine learning algorithms, optimizing expert systems, and improving decision support mechanisms. As research in intuitionistic fuzzy sets continues to evolve, their integration with other mathematical and computational tools, such as neural networks, genetic algorithms, and deep learning, is expected to further enhance their applicability in solving increasingly complex problems across diverse disciplines.

BCK-algebras and BCI-algebras are two classes of abstract algebras. BCI-algebras, as a category of logical algebras, serve as the algebraic representation of the set difference and its associated properties in set theory, as well as the implicational operator in logical systems. It is well-established that BCK-algebras class is a proper subset of the class of BCI-algebras. The concept of BE-algebra was first introduced by Kim and Kim as a generalization of the BCK-algebra, originally defined by K. Iseki and S. Tanaka [10]. A BE-algebra is an algebra $(X; *, 1)$ of type $(2, 0)$. Over time, this concept has been extended into the framework of fuzzy set theory, where its properties have been extensively studied [1, 16]. Various aspects of fuzzy BE-algebras have been explored, including fuzzy subalgebras, fuzzy topological $BE-$algebras, and fuzzy filters of BE-algebras. More recently, in 2019, the notion of intuitionistic fuzzy BE-algebras and their related substructures were introduced, further expanding the scope of research in this area [14]. The extension of fuzzy and intuitionistic fuzzy structures with triangular norms (t-norms) and their subsequent analysis has become a significant area of mathematical research, with applications across different domains [5, 6, 15, 17, 18].

In this study, a novel concept, namely t-norm-based intuitionistic fuzzy BE-algebra, is introduced. To illustrate this new structure, a distinctive example is provided. Additionally, key algebraic properties of this newly defined structure are examined, along with the role of triangular norm levels in shaping its behavior. Moreover, the concept of a t-norm-based intuitionistic fuzzy filter is formally defined and explored. The definitions and theoretical results presented in this study are reinforced with concrete examples, demonstrating the fundamental characteristics and properties of the proposed algebraic framework.

2 Preliminaries

Definition 1. [2] *An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$$

where the function $\mu_A(x)$ ($\mu_A : X \to [0,1]$) is called "degree of membership of x in A", $\nu_A(x)$ ($\nu_A : X \to [0,1]$) is called "degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{ for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 2. [2] *An intuitionistic fuzzy set (IFS) A is considered to be included in B (denoted as $A \sqsubseteq B$) if and only if, for all $x \in X$: $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 3. [2] *Let X be universal and $A, B \in IFS(X)$, then*

1. $A \sqcap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}$
2. $A \sqcup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\}$
3. $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$
4. $\Box(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$
5. $\Diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$

Definition 4. [14] *Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. If $B \in IFS(Y)$, then the pre-image of B under f is denoted by $f^{-1}(B)$:*

$$f^{-1}(B)(x) = \left(\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)\right)$$

where

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)) \textit{ and } \nu_{f^{-1}(B)}(x) = \nu_B(f(x))$$

Triangular norms (t-norms) were first introduced by Menger in 1942 in the context of probabilistic metric spaces to generalize the classical metric concept by incorporating uncertainty [12]. Over time, these norms have become fundamental in the development of fuzzy logic, fuzzy set theory, and multi-criteria decision-making, as they provide a mathematical framework for modeling conjunctions (AND operations) in systems involving uncertainty and vagueness. T-norms and t-conorms play a significant role in aggregation functions, fuzzy control systems, decision-making algorithms, and fuzzy probability theory. With the increasing complexity of real-world problems, researchers continue to develop and refine t-norm-based models to enhance the efficiency of computational intelligence and decision-support systems.

Algebraically, t-norms (also co-norms) are binary operations on unit interval $[0,1]$ and $([0,1], T, \leq)$ is an Abelian totally ordered semigroup with neutral element 1. T-norms have an important place in various areas of mathematics and computer science, especially in the domain of artificial intelligence.

Definition 5. [11] *A t-norm is an operation $T:[0,1]\times[0,1]\to[0,1]$ with following conditions for all $\alpha,\beta,\gamma\in[0,1]$:*

$(T1)$ $T(\alpha,1)=\alpha$

$(T2)$ $T(\alpha,\beta)=T(\beta,\alpha)$

$(T3)$ $T(\alpha,T(\beta,\gamma))=T(T(\alpha,\beta),\gamma)$

$(T4)$ $T(\alpha,\beta)\leq T(\alpha,\gamma)$ *whenever* $\beta\leq\gamma$.

The four basic t-norms are:

- $T_M(\alpha,\beta)=\min(\alpha,\beta)$ (minimum)
- $T_P(\alpha,\beta)=\alpha\beta$ (product)
- $T_L(\alpha,\beta)=\max(\alpha+\beta-1,0)$ (Lukasiewicz t-norm)
- $T_D(\alpha,\beta)=\begin{cases} 0, & \text{if } (\alpha,\beta)\in[0,1)^2 \\ \min(\alpha,\beta), & \text{otherwise} \end{cases}$ (drastic product)

It holds that $T(\alpha_1,\beta_1)\leq T(\alpha_2,\beta_2)$ whenever $\alpha_1\leq\alpha_2$ and $\beta_1\leq\beta_2$. Triangular conorms were introduced as dual operations of t-norms.

Definition 6. [11] *A t-conorm is an operation $S:[0,1]\times[0,1]\to[0,1]$ satisfying the following conditions for all $\alpha,\beta,\gamma\in[0,1]$, $(T1)$–$(T3)$ and*

$(S4)$ $S(\alpha,0)=\alpha$.

Here are the basic t-conorms:

- $S_M(\alpha,\beta)=\max(\alpha,\beta)$ (maximum)
- $S_P(\alpha,\beta)=\alpha+\beta-\alpha\beta$ (probabilistic sum)

- $S_L(\alpha, \beta) = \min(\alpha + \beta, 1)$ (Lukasiewicz t-conorm)
- $S_D(\alpha, \beta) = \begin{cases} 1, & \text{if } (\alpha, \beta) \in (0,1]^2 \\ \max(\alpha, \beta), & \text{otherwise} \end{cases}$ (drastic product)

It is clear that, $S(\alpha, \beta) = 1 - T(1-\alpha, 1-\beta)$.

The t-conorm is called the dual t-conorm of T and the t-norm is said to be the dual t-norm of S.

Since t-norms are associative, they can be extended to n-ary operations. With this extension, the basic properties of algebraic structures on finite and infinite sets, especially intersection, are examined.

Definition 7. [11] *For $n \in \mathbb{N}$, $T_n : \prod_{i \in I_n} [0,1] \to [0,1]$ is defined as:*

$$\begin{aligned} T_n(\alpha_1, \ldots, \alpha_n) &= T(\alpha_i, T_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)) \\ &= \mathop{T}_{i=1}^{n} \alpha_i \end{aligned}$$

for $i \in I_n$ where $T_2 = T$ and $T = id$ (identity).
Also T_∞ is defined as $\mathop{T}_{i=1}^{\infty} \alpha_i = \lim_{n \to \infty} T_n(\alpha_1, \ldots, \alpha_n)$.

Definition 8. [11] *By the intersection of $\{A_1, A_2, \ldots\}$ in the set X with respect to a t-norm T is meant the intuitionistic fuzzy subset*

$$\bigcap_T A_i(\alpha) = \{(x, T_\infty(\mu_1(\alpha), \mu_2(\alpha), \ldots), S_{T_\infty}(\nu_1(\alpha), \nu_2(\alpha), \ldots)) : \alpha \in X\}.$$

If the family of IF sets has finite elements,

$$(A_1 \cap_T A_2 \cap_T \cdots \cap_T A_n)(\alpha) = (T_n(\mu_1(\alpha), \ldots, \mu_n(\alpha)), S_{T_n}(\nu_1(\alpha), \ldots, \nu_n(\alpha))).$$

Fuzzy sets and intuitionistic fuzzy sets are divided into classes by level subsets. In this way, algebraic properties and relations of classes, such as equivalence classes, can be studied. In 2010, Janiš [9] brought a different approach to this issue and presented a constraint that can obtain better properties of cuts. Here, left continuous triangular norms are used.

Definition 9. [9] *Let $A \in IFS(X)$ and T be a left continuous t-norm. Then $A_{T,r}$ is a T-based r-cut of A, as defined by:*

$$A_{T,r} = \{a \in X : T(\mu_A(a), 1 - \nu_A(a)) \geq r\}$$

for every $r \in (0,1]$.

Definition 10. [9] *Let $A \in IFS(X)$ and (T, S_T) be a pair of mutually dual t-norms and t-conorms. Then $A_{T,S_T,r}$ is a (T, S_T)-based r-cut of A, as defined by:*

$$A_{T,S_T,r} = \{a \in X : T(\mu_A(a), S_T(\mu_A(a), \nu_A(a))) \geq r\}$$

for every $r \in [0,1]$.

Fuzzified algebraic structures are a topic of interest for researchers [4–6, 15, 17, 19]. Besides, the application of triangular norms to fuzzy and intuitionistic fuzzy algebraic structures are widely studied. Algebraic structures are fundamental concepts in mathematics that form the basis for many advanced theories. These structures, which include groups, rings, fields, and vector spaces, are not only central to pure mathematics but also serve as the foundation for a wide range of applications. For example, in computer science, algebraic structures are used in algorithms, cryptography, and data structures. In information science, they are critical for coding theory and signal processing. In theoretical physics, algebraic structures help describe symmetries and quantum mechanics. Moreover, in control engineering, they are essential for the analysis and design of systems. Since the introduction of fuzzy set theory, researchers have been extending abstract algebra concepts to fuzzy sets. Many studies have been put forward on fuzzy algebraic structures and their characteristic differences. With the definition of intuitionistic fuzzy set theory, one of the important extensions of fuzzy sets, by Atanassov, algebraic structures began to be studied in this theory. And also they redefined concepts by changing the minimum norm and standard negation.

In this paper, intuitionistic fuzzy BE-algebras will be examined under t-norms. Its relationship with crisp algebraic structures will be revealed through T-based level subsets. Its basic properties will be investigated and supported with examples. In addition, intuitionistic fuzzy filters of defined concept will be studied with respect to triangular norms. The BE-algebra concept is defined in 2006 as a generalization of BCK-algebra [10].

Definition 11. [10] *An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE-algebra if*

1. $\alpha * \alpha = 1$ *for all* $\alpha \in X$;
2. $\alpha * 1 = 1$ *for all* $\alpha \in X$;
3. $1 * \alpha = \alpha$ *for all* $\alpha \in X$;
4. $\alpha * (\beta * \gamma) = \beta * (\alpha * \gamma)$ *for all* $\alpha, \beta, \gamma \in X$.

A relation "$\leq$" on X is given by $\alpha \leq \beta \iff \alpha * \beta = 1$. A subalgebra of BE-algebra X is a non-empty subset that closed under the operation $*$.

Proposition 1. [10] *If $(X; *, 1)$ is a BE-algebra, then $\alpha * (\beta * \alpha) = 1$ for any $\alpha, \beta \in X$.*

A BE-algebra $(X; *, 1)$ is considered *transitive* if, for all $\alpha, \beta, \gamma \in X$, the following condition holds: $\beta * \gamma \leq (\alpha * \beta) * (\alpha * \gamma)$. Additionally, a BE-algebra $(X; *, 1)$ is called *self-distributive* if, for all $\alpha, \beta, \gamma \in X$, $\alpha * (\beta * \gamma) = (\alpha * \beta) * (\alpha * \gamma)$ is satisfied.

Definition 12. [10] *A filter of BE-algebra $(X; *, 1)$ is a subset F such that for all $\alpha, \beta \in X$:*

1. $1 \in F$,
2. *If* $\alpha * \beta \in F$ *and* $\alpha \in F$, *then* $\beta \in F$.

Definition 13. [13] *Let $(X; *, 1)$ be a BE-algebra. $F \subset X$ called an implicative filter if and only if:*

1. $1 \in F$,
2. *if* $\alpha * (\beta * \gamma) \in F$ *and* $\alpha * \beta \in F$, *then* $\alpha * \gamma \in F$, *for all* $\alpha, \beta \in X$.

It is evident that every filter in a self-distributive BE-algebra is an implicative filter. Additionally, every implicative filter qualifies as a filter in a BE-algebra. However, in general, the reverse of this statement does not hold true in a BE-algebra.

In 2011, Ahn *et al.* [1] fuzzified the concept BE-algebra. Around the same time, Rezaei and Saeid also introduced the fuzzy subalgebras of BE-algebras [16]. In the following years, fuzzy filters were studied by Dymek and Walendziak [7].

Definition 14. [1,16] *Let X be a BE-algebra and $\mu \in FS(X)$. μ is called a fuzzy BE-algebra of X if*

$$\mu(\alpha * \beta) \geq \min\{\mu(\alpha), \mu(\beta)\}$$

for all $\alpha, \beta \in X$.

If μ is a fuzzy BE-algebra of X, then $\mu(1) \geq \mu(\alpha)$, for all $\alpha \in X$.

Definition 15. [7] *Let X be a BE-algebra and $\mu \in FS(X)$. μ is called a fuzzy filter of X if*

$$\begin{aligned} \mu(1) &\geq \mu(\alpha) \text{ and} \\ \mu(\alpha) &\geq \mu(\beta * \alpha) \wedge \mu(\beta) \end{aligned}$$

for all $\alpha, \beta \in X$.

Definition 16. [13] *Let X be a BE-algebra and $\mu \in FS(X)$. μ is called a fuzzy implicative filter of X if*

$$\begin{aligned} \mu(1) &\geq \mu(\alpha) \text{ and} \\ \mu(\alpha * \gamma) &\geq \min\{\mu(\alpha * (\beta * \gamma)), \mu(\alpha * \beta)\} \end{aligned}$$

for all $\alpha, \beta, \gamma \in X$

BE-algebra on intuitionistic fuzzy sets and ideals were studied by different authors [8, 14]. Some algebraic properties were proved in these papers.

3 Intuitionistic fuzzy BE-algebras with (T, S_T)-norms

The extension of fuzzy algebraic concepts with the help of triangular norms has an important place in the literature, both in theoretical and applied fields such as decision making. In this study, the extension of the theoretical properties is studied.

In this section, main results will be presented. Here, the set X will mean BE-algebra.

Definition 17. *Let $A \in IFS(X)$ and Let (T, S_T) be at-norm pair. Then A called an intuitionistic fuzzy BE-algebra of X with respect to the $(T, S_T)-$norms if*

$$A(\alpha * \beta) \geq (T(\mu_A(\alpha), \mu_A(\beta)), S_T(\nu_A(\alpha), \nu_A(\beta)))$$

for all $\alpha, \beta \in X$.

Example 1. *Let $X = \{1, \alpha, \beta, \gamma\}$. With the following table:*

$*$	1	α	β	γ
1	1	α	β	γ
α	1	1	α	α
β	1	1	1	α
γ	1	1	α	1

X is a BE-algebra. Then $A = \{(1, 0.8, 0.1), (\alpha, 0.4, 0.5), (\beta, 0.6, 0.3), (\gamma, 0.5, 0.3)\}$ is an intuitionistic fuzzy BE-algebra with respect to (T_L, S_L)-norms. However, it is not an intuitionistic fuzzy BE-algebra since $A(\gamma * \beta) \ngeq A(\gamma) \wedge A(\beta)$.

Necessity and Possibility operators have a special role on intuitionistic fuzzy theory. Here we show that the introduced algebraic structure is preserved under the Necessity and Possibility operators.

Theorem 1. *Let $A \in IFS(X)$. If A is an IF BE-algebra of X with respect to the (T, S_T)-norms, then $\Box A$ and $\Diamond A$ are IF BE-algebras with respect to (T, S_T)-norms.*

Proof. It is clear from the hypothesis that:

$$\mu_A(\alpha * \beta) \geq T(\mu_A(\alpha), \mu_A(\beta)) \Rightarrow \mu_{\Box(A)}(\alpha * \beta) \geq T\left(\mu_{\Box(A)}(\alpha), \mu_{\Box(A)}(\beta)\right)$$

for all $\alpha, \beta \in X$, and

$$\begin{aligned} 1 - \mu_A(\alpha * \beta) &\leq 1 - T(\mu_A(\alpha), \mu_A(\beta)) = S_T(1 - \mu_A(\alpha), 1 - \mu_A(\beta)) \\ &\Rightarrow \nu_{\Box(A)}(\alpha * \beta) \leq S_T\left(\nu_{\Box(A)}(\alpha), \nu_{\Box(A)}(\beta)\right). \end{aligned}$$

$\Diamond A$ can be shown similarly. $\Box$

Corollary 1. *$\mu \in FS(X)$ is a fuzzy BE-algebra if and only if $A = (\mu_A, \mu_A^c)$ is an IF BE-algebra with respect to (T, S_T)-norms.*

Proof. It is clear from the above theorem. $\Box$

Let us study the intersections of norm-based intuitionistic fuzzy BE-algebras on the family of finite sets and the family of infinite sets separately.

Theorem 2. *Let $\{A_i\}_{i=1}^n$ be a finite family of IF BE-algebras with respect to (T, S_T)-norms. Then the intersection of A_i sets is an IF BE-algebra with respect to (T, S_T)-norms.*

Proof. Let $\alpha, \beta \in X$. Firstly,

$$\begin{aligned} \mu_{\cap_T A_i}(\alpha * \beta) &= T_n(\mu_1(\alpha * \beta), \ldots, \mu_n(\alpha * \beta)) \\ &= T(\mu_1(\alpha * \beta), T_{n-1}(\mu_2(\alpha * \beta), \ldots, \mu_n(\alpha * \beta))) \\ &\geq T(T(\mu_1(\alpha), \mu_1(\beta)), T(T_{n-1}(\mu_2(\alpha), \ldots, \mu_n(\alpha)), T_{n-1}(\mu_2(\beta), \ldots, \mu_n(\beta)))) \\ &= T(\mu_1(\beta), TT(\mu_1(\alpha), T_{n-1}(\mu_2(\alpha), \ldots, \mu_n(\alpha))), T_{n-1}(\mu_2(\beta), \ldots, \mu_n(\beta)))) \\ &= T(T_n(\mu_1(\beta), \mu_2(\beta), \ldots, \mu_n(\beta)), T_n(\mu_1(\alpha), \mu_2(\alpha), \ldots, \mu_n(\alpha))) \\ &= T((\mu_1 \cap_T \mu_2 \cap_T \cdots \cap_T \mu_n)(\alpha), (\mu_1 \cap_T \mu_2 \cap_T \cdots \cap_T \mu_n)(\beta)) \end{aligned}$$

and for dual conorm S_T, clearly shown that

$$\nu_{\cap_T A_i}(\alpha * \beta) \leq S_T\left((\nu_1 \cap_T \nu_2 \cap_T \cdots \cap_T \nu_n)(\alpha), (\nu_1 \cap_T \nu_2 \cap_T \cdots \cap_T \nu_n)(\beta)\right). \qquad \square$$

Theorem 3. *Let $\{A_i\}_{i\in I}$ be a family of IF BE-algebras with respect to continuous (T, S_T)-norms. Then infinite intersection of A_i sets is an intuitionistic fuzzy BE-algebra with respect to continuous $(T, S_T)-$norms.*

Proof. Let (T, S_T) are continuous dual norm pair. For $\alpha, \beta \in X$,

$$\bigcap_T A_i(\alpha * \beta) = \left(T_\infty(\mu_1(\alpha * \beta), \mu_2(\alpha * \beta), \ldots), S_{T_\infty}(\nu_1(\alpha * \beta), \nu_2(\alpha * \beta), \ldots)\right).$$

Considering the above theorem and continuity property;

$$\begin{aligned} T_\infty(\mu_1(\alpha * \beta), \mu_2(\alpha * \beta), \ldots) &= \lim_{n\to\infty} T_n(\mu_1(\alpha * \beta), \ldots, \mu_n(\alpha * \beta)) \\ &\geq \lim_{n\to\infty} T(T_n(\mu_1(\alpha), \ldots, \mu_n(\alpha)), T_n(\mu_1(\beta), \ldots, \mu_n(\beta))) \\ &= T\left(\lim_{n\to\infty} T_n(\mu_1(\alpha), \ldots, \mu_n(\alpha)), \lim_{n\to\infty} T_n(\mu_1(\beta), \ldots, \mu_n(\beta))\right) \\ &= T(T_\infty(\mu_1(\alpha), \mu_2(\alpha), \ldots), T_\infty(\mu_1(\beta), \mu_2(\beta), \ldots)). \end{aligned}$$

Similarly,

$$S_\infty(\nu_1(\alpha * \beta), \nu_2(\alpha * \beta), \ldots) \leq S_T(S_\infty(\nu_1(\alpha), \nu_2(\alpha), \ldots), S_\infty(\nu_1(\beta), \nu_2(\beta), \ldots)). \quad \square$$

Proposition 2. *Let $f : X \to Y$ be a BE-algebra homomorphism and B be an IF BE-algebra of Y with respect to (T, S_T)-norm. Then $f^{-1}(B)$ is an intuitionistic fuzzy BE-algebra of X with respect to (T, S_T)-norm.*

Proof. Let $\alpha, \beta \in X$,

$$\begin{aligned} \mu_{f^{-1}(B)}(\alpha * \beta) &= \mu_B(f(\alpha * \beta)) = \mu_B(f(\alpha) * f(\beta)) \\ &\geq T(\mu_B(f(\alpha)) * \mu_B(f(\beta))) \\ &= T\left(\mu_{f^{-1}(B)}(\alpha), \mu_{f^{-1}(B)}(\beta)\right) \end{aligned}$$

and

$$\begin{aligned} \nu_{f^{-1}(B)}(\alpha * \beta) &= \nu_B(f(\alpha * \beta)) = \nu_B(f(\alpha) * f(\beta)) \\ &\leq S_T(\nu_B(f(\alpha)) * \nu_B(f(\beta))) \\ &= S_T\left(\nu_{f^{-1}(B)}(\alpha), \nu_{f^{-1}(B)}(\beta)\right). \qquad \square \end{aligned}$$

Showing the relationships of fuzzified algebraic structures with structures in the crisp set is important for the consistency of the presented extension. In this study, instead of the classical level set concept on intuitionistic fuzzy sets, norm-based level sets were used. The results obtained for the case of r being 1 and for arbitrary values of r are presented.

Theorem 4. *Let $A \in IFS(X)$. If A is an IF BE-algebra of X with respect to (T, S_T)-norms, then $A_{T,1}$ is a crisp BE-subalgebra of X.*

Proof. Let $\alpha, \beta \in A_{T,1}$,

$$\begin{aligned}
T(\mu_A(\alpha * \beta), 1 - \nu_A(\alpha * \beta)) &\geq T(\mu_A(\alpha * \beta), 1 - S_T(\nu_A(\alpha), \nu_A(\beta))) \\
&\geq T(T(\mu_A(\alpha), \mu_A(\beta)), 1 - S_T(\nu_A(\alpha), \nu_A(\beta))) \\
&= T(T(\mu_A(\alpha), \mu_A(\beta)), T(1 - \nu_A(\alpha), 1 - \nu_A(\beta))) \\
&= T(T(T(\mu_A(\alpha), 1 - \nu_A(\alpha)), 1 - \nu_A(\beta)), \mu_A(\beta)) \\
&= T(T(\mu_A(\alpha), 1 - \nu_A(\alpha)), T(\mu_A(\beta)), 1 - \nu_A(\beta)) \\
&\geq T(1, 1) = 1
\end{aligned}$$

So, $\alpha * \beta \in A_{T,1}$. $A_{T,1}$ is a crisp BE-subalgebra of X. □

Proposition 3. *Let $A \in IFS(X)$. If A is a BE-algebra of X with respect to (T_m, S_{T_m})-norm, then $A_{T,r}$ is a crisp BE-subalgebra of X for everey $r \in [0, 1]$.*

Proof. It can be easily seen from the above proof. □

Theorem 5. *Let $A \in IFS(X)$. If A is a BE-algebra of X with respect to (T, S_T)-norms, then $A_{T,S_T,1}$ is a crisp BE-subalgebra of X.*

Proof. Let $\alpha * \beta \in A_{T,S_T,1}$.
Since $T(\mu_A(\alpha), S_T(\mu_A(\alpha), \nu_A(\alpha))) = 1$ and $T(\mu_A(\beta), S_T(\mu_A(\beta), \nu_A(\beta))) = 1$, then $\mu_A(\alpha) = \mu_A(\beta) = 1$. Now,

$$\begin{aligned}
T(\mu_A(\alpha * \beta), S_T(\mu_A(\alpha * \beta), \nu_A(\alpha * \beta))) &\geq T(\mu_A(\alpha * \beta), \mu_A(\alpha * \beta)) \\
&\geq T(T(\mu_A(\alpha), \mu_A(\beta)), T(\mu_A(\alpha), \mu_A(\beta))) \\
&= T(T(1, 1), T(1, 1)) \\
&= T(1, 1) = 1,
\end{aligned}$$

i.e., $\alpha * \beta \in A_{T,S_T,1.}$ $A_{T,S_T,1}$ is a crisp BE-subalgebra of X. □

3.1 Intuitionistic fuzzy filters of BE-algebras with respect to (T, S_T)-norms

The concept of filters on algebraic structures is an study area that has attracted great interest from researchers [7, 19]. Filters in BE-algebras are fundamental substructures that play an important role in defining various special classes of these structures. Also, since it is well known that filters are the core of perfect correspondences, many researchers have aimed to establish quotient BE-algebras by defining different types of filters in BE-algebras and to study certain properties of these structures.

Kim *et al.* [10] have investigated various properties of supersets in BE-algebras. Using the concept of supersets, many researchers have derived equivalent conditions for defining filters in

BE-algebras. They have also defined more comprehensive supersets in BE-algebras and studied their relations with filters.

In this section, t-norm-based intuitionistic fuzzy BE-filter is defined and specifically investigated.

Definition 18. *Let $A \in IFS(X)$, then A is called an intuitionistic fuzzy filters with respect to (T, S_T)-norms if it satisfies:*

1. $\mu_A(1) \geq \mu_A(\alpha), \nu_A(1) \leq \nu_A(\alpha)$ for all $\alpha \in X$,
2. $\mu_A(\alpha) \geq T(\mu_A(\beta * \alpha), \mu_A(\beta))$, $\nu_A(\alpha) \leq S_T(\nu_A(\beta * \alpha), \nu_A(\beta))$ for all $\alpha, \beta \in X$.

Example 2. *Let X be the BE-algebra from Example 1.*
$A = \{(1, 0.7, 0.2), (\alpha, 0.6, 0.3), (\beta, 0.5, 0.2), (\gamma, 0.4, 0.5)\}$ *is a filter of X with respect to (T_L, S_{T_L})-norm. Since $A(\gamma) \ngeq A(\alpha * \gamma) \wedge A(\alpha)$, then it is not a filter with (T_m, S_{T_M})-norm.*

The relation of norm pair based intuitionistic fuzzy BE-filters to the concept of filters on crisp sets is obtained with norm levels as follows. The results obtained for the special case of $r = 1$ are provided for the (min, max) norm pair in arbitrary cases of r.

Proposition 4. *Let $A \in IFS(X)$ be the IF filters of X with respect to (T_m, S_{T_m})-norm. For all $\alpha, \beta \in X$, if $\alpha \leq \beta$, then $A(\alpha) \leq A(\beta)$.*

Proof. If $\alpha \leq \beta$, then $\alpha * \beta = 1$. Since A an IF filters with respect to (T_m, S_{T_m})-norm, we have

$$\mu_A(\beta) \geq T_m(\mu_A(\alpha * \beta), \mu_A(\alpha)) \text{ and } \nu_A(\beta) \leq S_{T_m}(\nu_A(\alpha * \beta), \nu_A(\alpha)).$$

So,

$$\mu_A(\beta) \geq T_m(\mu_A(1), \mu_A(\alpha)) = \min\{\mu_A(1), \mu_A(\alpha)\} = \mu_A(\alpha)$$

and

$$\nu_A(\beta) \leq S_{T_m}(\nu_A(1), \nu_A(\alpha)) = \max\{\nu_A(1), \nu_A(\alpha)\} = \nu_A(\alpha).$$

□

Proposition 5. *Let $A \in IFS(X)$. If A is an IF BE-filter of X with respect to the (T, S_T)-norms, then $\Box A$ and $\Diamond A$ are IF BE-filters with respect to (T, S_T)-norms.*

Proof. For $\alpha \in X$,

$$\mu_{\Diamond(A)}(1) = 1 - \nu_A(1) \geq 1 - \nu_A(\alpha) = \mu_A(\alpha)$$

and

$$\nu_{\Diamond(A)}(1) = \nu_A(1) \leq \nu_A(\alpha).$$

Also,

$$\begin{aligned} \mu_{\Diamond(A)}(\alpha) &= 1 - \nu_A(\alpha) \\ &\geq 1 - S_T(\nu_A(\beta * \alpha), \nu_A(\beta)) \\ &= T(1 - \nu_A(\beta * \alpha), 1 - \nu_A(\beta)) \\ &= T\left(\mu_{\Diamond(A)}(\beta * \alpha), \mu_{\Diamond(A)}(\beta)\right) \end{aligned}$$

and $\nu_{\Diamond(A)}(\alpha) = \nu_A(\alpha) \leq S_T(\nu_A(\beta * \alpha), \nu_A(\beta))$.

$\Box A$ is similar. □

Theorem 6. *Let $A \in IFS(X)$. Then A is an IF filters of X with respect to (T, S_T)-norm if and only if $A_{T,1}$ is a filter of X.*

Proof. Assume that A is an IF filters of X with respect to (T, S_T)-norm. Let $\alpha, \alpha * \beta \in A_{T,1}$.

$$\begin{aligned} T(\mu_A(\beta), 1 - \nu_A(\beta)) &\geq T(T(\mu_A(\alpha * \beta), \mu_A(\alpha)), T(1 - \nu_A(\alpha * \beta), 1 - \nu_A(\alpha))) \\ &= T(T(\mu_A(\alpha * \beta), T(1 - \nu_A(\alpha * \beta), 1 - \nu_A(\alpha))), \mu_A(\alpha)) \\ &= T(T(\mu_A(\alpha * \beta), 1 - \nu_A(\alpha * \beta)), T(\mu_A(\alpha), 1 - \nu_A(\alpha))) \\ &\geq T(1, 1) = 1 \end{aligned}$$

so that $\beta \in A_{T,1}$.

Conversely, suppose that $A_{T,1}$ is a filter of X.

If $A(1) \geq A(\alpha)$ is not valid then $\mu_A(\alpha) > \mu_A(1)$ and $\nu_A(\alpha) < \nu_A(1)$.

Since $1 \in A_{T,1}$, then $T(\mu_A(1), 1 - \nu_A(1)) = 1$. It means that $\mu_A(1) = 1 \vee \nu_A(1) = 0$. If we examine these situations separately,

(i) If $\mu_A(1) = 1$ and $\mu_A(\alpha) > \mu_A(1)$ then $\mu_A(\alpha) > 1$.

(ii) If $\nu_A(1) = 0$ and $\nu_A(\alpha) < \nu_A(1)$, then $\nu_A(\alpha) < 0$. These situations gives a contradiction. So, $A(1) \geq A(\alpha)$ for all $\alpha \in X$.

On the other hand, assume that $A(\alpha) < T(A(\beta * \alpha), A(\beta))$ for all $\alpha * \beta \in X$. Let $\beta, \beta * \alpha \in A_{T,1}$, then

$$\begin{aligned} T(\mu_A(\beta * \alpha), \mu_A(\beta)) &= T(1,1) \ \vee \ S_T(\nu_A(\beta * \alpha), \nu_A(\beta)) = S(0,0) \\ &\Rightarrow T(1,1) > \mu_A(\alpha) \ \vee \ S(0,0) < \nu_A(\alpha) \\ &\Rightarrow 1 > \mu_A(\alpha) \ \vee \ 0 < \nu_A(\alpha) \end{aligned}$$

this is impossible, it means that $A(\alpha) \geq T(A(\beta * \alpha), A(\beta))$. □

Proposition 6. *Let $A \in IFS(X)$. Then A is an IF filters of X with respect to (T_m, S_{T_M})-norm if and only if $A_{T,r}$ is a filter of X for everey $r \in [0, 1]$.*

Proof. It is clear from above theorem. □

Theorem 7. *Let $A, B \in IFS(X)$ are IF filters of X with respect to $(T, S_T)-$norm. Then $A \cap_T B$ is an IF filter with respect to (T, S_T)-norm.*

Proof. For $\alpha \in X$,

$$\begin{aligned} \mu_{A \cap_T B}(1) &= T(\mu_A(1), \mu_B(1)) \geq T(\mu_A(\alpha), \mu_B(\alpha)) \\ &= \mu_{A \cap_T B}(\alpha). \end{aligned}$$

And also,

$$\begin{aligned} \mu_{A \cap_T B}(\alpha) &= T(\mu_A(\alpha), \mu_B(\alpha)) \geq T(T(\mu_A(\alpha * \beta), \mu_A(\beta)), T(\mu_B(\alpha * \beta), \mu_B(\beta))) \\ &= T(T(\mu_A(\alpha * \beta), T(\mu_B(\alpha * \beta), \mu_B(\beta))), \mu_A(\beta)) \\ &= T(T(T(\mu_A(\alpha * \beta), \mu_B(\alpha * \beta)), \mu_B(\beta)), \mu_A(\beta)) \\ &= T(\mu_{A \cap_T B}(\alpha * \beta), \mu_{A \cap_T B}(\beta)). \end{aligned}$$

So, the proof is completed. □

It is shown that the intersection of norm-based intuitionistic fuzzy BE-filters is also a norm-based IF filter. This result can be easily proven for finite (infinite) family of intuitionistic fuzzy BE-filters.

Definition 19. *Let $A \in IFS(X)$, then A is called an intuitionistic fuzzy implicative filter with respect to (T, S_T)-norms if it satisfies the following conditions:*

1. $\mu_A(1) \geq \mu_A(\alpha), \nu_A(1) \leq \nu_A(\alpha)$ for all $\alpha \in X$,
2. $\mu_A(\alpha * \beta) \geq T(\mu_A(\alpha * (\beta * \gamma)), \mu_A(\alpha * \beta)), \nu_A(\alpha * \beta) \leq S_T(\nu_A(\alpha * (\beta * \gamma)), \nu_A(\alpha * \beta))$ for all $\alpha, \beta, \gamma \in X$.

Theorem 8. *Let X be a self-distributive BE-algebra, then every $(T, S_T)-$intuitionistic fuzzy filter of X is an intuitionistic fuzzy implicative filter with respect to (T, S_T)-norm.*

Proof. Let $A \in IFS(X)$ be an intuitionistic fuzzy filter with respect to (T, S_T)-norm. So, $A(1) \geq A(\alpha)$ for all $\alpha \in X$. Then with self-distributive propertie; for $\alpha, \beta, \gamma \in X$,

$$\begin{aligned} A(\alpha * \gamma) &\geq \begin{pmatrix} T(\mu_A((\alpha * \beta) * (\alpha * \gamma)), \mu_A(\alpha * \gamma)), \\ S_T(\nu_A((\alpha * \beta) * (\alpha * \gamma)), \nu_A(\alpha * \gamma)) \end{pmatrix} \\ &= (T(\mu_A(\alpha * (\beta * \gamma)), \mu_A(\alpha * \gamma)), S_T(\nu_A(\alpha * (\beta * \gamma)), \nu_A(\alpha * \gamma))). \end{aligned}$$

So, A is a norm-based intuitionistic fuzzy implicative filter. □

Theorem 9. *Let $A \in IFS(X)$. If A is an IF implicative filter in X with respect to (T, S_T)-norm, then $A_{T,1}$ is an implicative filter in X.*

Proof. Let $\alpha * \beta, \alpha * (\beta * \gamma) \in A_{T,1}$ then

$$\begin{aligned} T(\mu_A(\alpha * \gamma), 1 - \nu_A(\alpha * \gamma)) &\geq T\begin{pmatrix} T(\mu_A(\alpha * (\beta * \gamma)), \mu_A(\alpha * \beta)), \\ T(1 - \nu_A(\alpha * (\beta * \gamma)), 1 - \nu_A(\alpha * \beta)) \end{pmatrix} \\ &= T\begin{pmatrix} T(\mu_A(\alpha * (\beta * \gamma)), 1 - \nu_A(\alpha * (\beta * \gamma))), \\ T(\mu_A(\alpha * \beta), 1 - \nu_A(\alpha * \beta)) \end{pmatrix} \\ &\geq T(1, 1) = 1 \end{aligned}$$

i.e., $\alpha * \gamma \in A_{T,1}$. Also, it is clear that $1 \in A_{T,1}$. □

4 Conclusion

The concept of (T, S_T)-intuitionistic fuzzy BE-algebra was introduced as an extension of intuitionistic fuzzy BE-algebra. In this context, various subalgebraic structures were explored, and their fundamental properties were thoroughly analyzed. Additionally, intuitionistic fuzzy filters on (T, S_T)-intuitionistic fuzzy BE-algebra were examined in detail, leading to a deeper understanding of their role within this algebraic system. Furthermore, significant results were derived by employing norm-based α-cuts, offering new insights into the structural behavior and characteristics of these algebraic constructs.

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