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The Inclusion–Exclusion principle for general IF-states

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Abstract: Any real state on intuitionistic fuzzy sets (IF-sets) can be represented by integrals. L. Ciungu in [3] proved that for any real state on IF-sets and for a pair of binary operations which satisfy some special conditions holds an Inclusion–Exclusion principle. In [10], J. Považan proved that also any state on IF-sets with values from the arbitrary Riesz space we can represented by integrals. But could we consider Inclusion–Exclusion principle for any IF-state? In this paper we will prove this property for general case in very similar way as for real.
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1 Introduction

Let $(\Omega, \mathcal{S}, P)$ be a classical probability space where Ω is a nonempty set, $\mathcal{S}$ is a σ-algebra of subsets of Ω and P is a probability measure over $\mathcal{S}$.

Intuitionistic fuzzy sets which are an extension of fuzzy sets defined by L. Zadeh, was introduced by K. Atanassov in [1].

An IF-set A in Ω is given by an ordered triple $A = \{x, \mu_A(x), \nu_A(x); x \in \Omega\}$, where $\mu_A, \nu_A : \Omega \to [0, 1]$ satisfy the condition $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in \Omega$. Function μ_A is called membership and ν_A non-membership function. An IF-event A is defined as an IF-subset of Ω such that μ_A, ν_A are Borel measurable. We will denote by $\mathcal{F}$ the family of all IF-events.

Probability $\mathcal{P}_0 : \mathcal{F} \to \mathcal{J}$ on IF-events, where $\mathcal{J}$ is a family of all compact subintervals of $[0, 1]$, was first defined constructively by P. Grzegorzewski and E. Mrówka in [6] as follows:

$$\mathcal{P}_0(A) = \left[\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP \right].$$

In [11, 15] B. Riečan defined probability $\mathcal{P}$: $\mathcal{F} \to \mathcal{J}$ by axioms:

1. $\mathcal{P}((0,1)) = (0,0); \quad \mathcal{P}((1,0)) = (1,1)$
2. if $A \odot B = (0,1)$, then $\mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$
3. if $A_n \nearrow A$, then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$,

where $\oplus$, $\odot$ are Łukasiewicz operations on IF-events used by Riečan in [11]:

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0)$$
$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\mu_A + \mu_B) \wedge 1).$$

Operation $\oplus$ is called the Łukasiewicz sum and $\odot$ the Łukasiewicz product.

One of the most important result is the representation theorem of probability on IF-events provided in [14] where B. Riečan proved that for any probability $\mathcal{P} : \mathcal{F} \to \mathcal{J}$

$$\mathcal{P}(A) = \left[f\left(\int \mu_A dP, \int \nu_A dP \right), g\left(\int \mu_A dP, \int \nu_A dP \right) \right]$$

exist $\alpha, \beta \in [0,1]$, $\alpha \leq \beta$ such that

$$\mathcal{P} = \left((1-\alpha) \int \mu_A dP + \alpha \left(1 - \int \nu_A dP \right), (1-\beta) \int \mu_A dP + \beta \left(1 - \int \nu_A dP \right) \right).$$

Next very important result was proved in [4] and [9], and says that for any real state $m : \mathcal{F} \to [0,1]$ there exist probability measures $\mathcal{P}$, $\mathcal{Q}$ and $\alpha \in [0,1]$ such that:

$$m(A) = \int_\Omega \mu_A d\mathcal{P} + \alpha(1 - \int_\Omega \mu_A + \nu_A d\mathcal{Q}).$$

J. Považan in [10] makes a small change in this formulation and thus extended this idea for general states with values from arbitrary Riesz space.

Next step was to prove some properties of IF-state. We are interested in the Inclusion–Exclusion property. In [6, 7], P. Grzegorzewski proved two forms of Inclusion–Exclusion principle on IF-events for the pairs of operations $(\cup, \cap)$ and $(\boxplus, \boxdot)$, where:

$$A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$$
$$A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$
$$A \boxplus B = (\mu_A + \mu_B - \mu_A \mu_B, \nu_A \nu_B)$$
$$A \boxdot B = (\mu_A \mu_B, \nu_A + \nu_B - \nu_A \nu_B),$$

assuming $\mathcal{P}(A) = \left[\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP \right]$.

In [5] B. Riečan and L.C. Ciungu proved Inclusion–Exclusion principle for the pair of operations $(\boxplus, \boxdot)$ and for an IF-state $m : \mathcal{F} \to [0,1]$ such that $m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$.

In [3] there was proved the Inclusion–Exclusion principle on IF-events for more general case, i.e., for two binary operations with some special properties. This operations includes Grzegorzewski's approach as a special case.

In this paper we will continue in this direction and prove this principle for the general measure.

2 Binary operations

The classical Inclusion–Exclusion principle for measures is defined by equality:

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i) - \sum_{i<j}^{n} \mu(A_i \cap A_j) + \cdots + (-1)^{n+1}\mu(A_1 \cap A_2 \cap \cdots \cap A_n)$$

for any natural n. In what follows we extend this property for IF-sets.

If we want to be as general as possible, we have to introduce two operations for which we will consider Inclusion–Exclusion principle.

Definition 2.1. *Let $(\Box, \triangle)$ be two binary commutative and associative operations on $\mathbb{R}$. An ordered pair $(\Box, \triangle)$ is an Inclusion–Exclusion pair (IE pair), if the following conditions are satisfied:*

1. $a \Box b = a + b - a \triangle b$

2. $(a \Box b) \triangle c = a \triangle c + b \triangle c - a \triangle b \triangle c.$

Moreover, if $(\Box, \triangle)$ is an IE pair, then there holds:

3. $(a \triangle b) \Box c = a \Box c + b \Box c - a \Box b \Box c.$

This equation can be proved easily by condition 1:

$$\begin{aligned} a \Box c + b \Box c - a \Box b \Box c &= a + c - a \triangle c + b + c - b \triangle c \\ &- a - b - c + a \triangle b + a \triangle c + b \triangle c - a \triangle b \triangle c = \\ &= c + a \triangle b - a \triangle b \triangle c = (a \triangle b) \Box c. \end{aligned}$$

For example, let $a \Box b = a + b - a \triangle b$ and $a \triangle b = ab$. Then

$$\begin{aligned} a \Box b &= a + b - a \triangle b, \\ (a \Box b) \triangle c &= (a + b - ab) \triangle c = \\ &= ac + ab - abc = a \triangle b + b \triangle c - a \triangle b \triangle c, \end{aligned}$$

and moreover

$$\begin{aligned} &a \Box c + b \Box c - a \Box b \Box c = a + c - ac + b + c - bc - a - b - c + ab + ac + bc - abc = \\ &= ab + c - abc = (a \triangle b) \Box c. \end{aligned}$$

These three conditions can be extended for any natural n.

Lemma 2.1. *Let* $(\Box, \triangle)$ *be an IE pair,* $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$. *Then*

$$(\Box_{i=1}^{n} a_i)\triangle b = \sum_{i=1}^{n}(a_i \triangle b) - \sum_{i<j}^{n}(a_i \triangle a_j \triangle b) + \cdots + (-1)^{n+1} \triangle_{i=1}^{n} a_i \triangle b.$$

Proof. Evidently, the equality holds for $n = 1$. Assume that there is n, such that the assertion holds. We want to prove the formula is true for $n + 1$. By the Property 2 we obtain:

$$(\Box_{i=1}^{n+1} a_i)\triangle b = (\Box_{i=1}^{n} a_i \triangle b) + (a_{n+1} \triangle b) - (\Box_{i=1}^{n} a_i)\triangle a_{n+1} \triangle b.$$

By induction assumption we have:

$$(\Box_{i=1}^{n} a_i)\triangle b = \sum_{i=1}^{n}(a_i \triangle b) - \sum_{i<j}^{n}(a_i \triangle a_j \triangle b) + \cdots + (-1)^{n+1} \triangle_{i=1}^{n} a_i \triangle b$$

and similarly

$$(\Box_{i=1}^{n} a_i)\triangle a_{n+1} \triangle b = \sum_{i=1}^{n}(a_i \triangle a_{n+1} \triangle b) - \sum_{i<j}^{n}(a_i \triangle a_j \triangle a_{n+1} \triangle b) + \cdots +$$
$$+ (-1)^{n+1} \triangle_{i=1}^{n} a_i \triangle a_{n+1} \triangle b.$$

Using these equalities we have:

$$(\Box_{i=1}^{n+1} a_i)\triangle b = \sum_{i=1}^{n+1}(a_i \triangle b) - \sum_{i<j}^{n+1}(a_i \triangle a_j \triangle b) + \cdots + (-1)^{n+1} \triangle_{i=1}^{n+1} a_i \triangle b.$$

This completes the proof. □

Lemma 2.2.

$$(\triangle_{i=1}^{n} a_i)\Box b = \sum_{i=1}^{n}(a_i \Box b) - \sum_{i<j}^{n}(a_i \Box a_j \Box b) + \cdots + (-1)^{n+1} \Box_{i=1}^{n} a_i \Box b.$$

Proof. This lemma can be proved by induction similarly to the previous one, using third property of IE pair. □

Theorem 2.1. *Let* $(\Box, \triangle)$ *be an IE pair of binary operations and* $a_1, a_2, \ldots, a_n \in \mathbb{R}$. *Then there holds:*

$$\Box_{i=1}^{n} a_i = \sum_{i=1}^{n}(a_i) - \sum_{i<j}^{n}(a_i \triangle a_j) + \cdots + (-1)^{n+1}(a_1 \triangle a_2 \triangle \ldots \triangle a_n)$$

and

$$\triangle_{i=1}^{n} a_i = \sum_{i=1}^{n}(a_i) - \sum_{i<j}^{n}(a_i \Box a_j) + \cdots + (-1)^{n+1}(a_1 \Box a_2 \Box \ldots \Box a_n).$$

Proof. First we will prove the first equality. For $n = 2$ the assertion holds. Let us assume that the assertion is true for some $n \in \mathbb{N}$. We shall prove it for $n + 1$. By definition of IE pair we have:

$$\Box_{i=1}^{n+1} a_i = \Box_{i=1}^{n} a_i + a_{n+1} - \Box_{i=1}^{n} a_i \triangle a_{n+1}.$$

Using the induction assumption and Lemma 2.1, we obtain:

$$\Box_{i=1}^{n+1} a_i = \sum_{i=1}^{n}(a_i) - \sum_{i<j}^{n}(a_i \triangle a_j) + \cdots + (-1)^{n+1}(a_1 \triangle a_2 \triangle \ldots \triangle a_n) + a_{n+1}$$
$$- \sum_{i=1}^{n}(a_i \triangle a_{n+1}) + \sum_{i<j}^{n}(a_i \triangle a_j \triangle a_{n+1}) - \cdots + (-1)^{n}(a_1 \triangle a_2 \triangle \ldots \triangle a_{n+1}) =$$
$$= \sum_{i=1}^{n+1}(a_i) - \sum_{i<j}^{n+1}(a_i \triangle a_j) + \cdots + (-1)^{n+1}(a_1 \triangle a_2 \triangle \ldots \triangle a_{n+1}).$$

Thus we proved the first equality.

Similarly we can prove the second equality. Let it hold for some n and we shall prove it for $n + 1$. By the first property ($a \triangle b = a + b - a \Box b$) of IE pairs we have:

$$\triangle_{i=1}^{n+1} a_i = (\triangle_{i=1}^{n} a_i) \triangle a_{n+1} = (\triangle_{i=1}^{n} a_i) + a_{n+1} - \triangle_{i=1}^{n} a_i \triangle a_{n+1}.$$

Then we have:

$$\triangle_{i=1}^{n+1} a_i = \sum_{i=1}^{n}(a_i) + \sum_{i<j}^{n}(a_i \Box a_j) + \cdots + (-1)^{n+1}(a_1 \Box a_2 \Box \ldots \Box a_n) + a_{n+1}$$
$$- \sum_{i=1}^{n}(a_i \Box a_{n+1}) + \sum_{i<j}^{n}(a_i \Box a_j \Box a_{n+1}) - \cdots + (-1)^{n}(a_1 \Box a_2 \Box \ldots \Box a_{n+1}) =$$
$$= \sum_{i=1}^{n+1}(a_i) - \sum_{i<j}^{n+1}(a_i \Box a_j) + \cdots + (-1)^{n+1}(a_1 \Box a_2 \Box \ldots \Box a_{n+1}).$$

This completes the proof. □

3 Representation theorem for general states

As we said at the beginning, let $(\Omega, \mathcal{S}, P)$ be a classical probability space, A be an IF-event in Ω, and we denote $\mathcal{F}$ the family of all IF-events. As we mentioned, in this paper we will consider not only real states but also states with values in an arbitrary Riesz space.

Definition 3.1. *An Abelian partially ordered group $(L, +, \leq)$ is called l-group if $(L, \leq)$ is a lattice and for $\forall a, b, c \in L$ holds:*

$$(a \leq b) \Rightarrow (a + c \leq b + c). \tag{1}$$

Definition 3.2. *An ordered quadruplet $(V, +, \cdot, \leq)$ is called Riesz space iff $(V, +, \cdot)$ is vector (linear) space over real numbers, $(V, +, \leq)$ is an l-group and it is defined operation $\cdot : \mathbb{R} \times V \to V$, such that $\forall x, y \in V$ and $\forall \alpha \in \mathbb{R}$, $\alpha \geq 0$ there holds:*

$$x \leq y \Rightarrow \alpha x \leq \alpha y. \tag{2}$$

For example, let $n \in \mathbb{N}$. Then $\mathbb{R}^n$ with classical vector addition and multiplication by scalars is a Riesz space under a partial ordering $\preceq$. Let $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$. Then the partial ordering $\preceq$ is defined by

$$x \preceq y \Leftrightarrow x_i \leq y_i, \quad i = 1, 2, \dots, n.$$

Definition 3.3. *Let V be a Riesz space, $\mathcal{S}$ be a σ-algebra of subsets of Ω. Then a set-mapping $\mu : \mathcal{S} \to \{v \in V; v \geq 0\}$ is called a measure iff*

$$\mu(A) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(A_i)$$

for any

$$A = \bigcup_{i=1}^{\infty} A_i \quad A_i \in \mathcal{S}(i = 1, 2, ...), \quad A_i \cap A_j = \emptyset(i \neq j).$$

We shall use the special IF events:

$$0_\Omega = \{(x, 0, 1); x \in \Omega\},$$
$$1_\Omega = \{(x, 1, 0); x \in \Omega\}.$$

Similarly to pairs of operations $(\cup, \cap)$, $(\boxplus, \boxdot)$ defined by K. Atanassov in [1], we can introduce two operations for the Inclusion–Exclusion pair of operations on IF-events:

$$A \Box B = (\mu_A \Box \mu_B, \nu_A \triangle \nu_B)$$
$$A \triangle B = (\mu_A \triangle \mu_B, \nu_A \Box \nu_B).$$

and more generally, for any $n \in \mathbb{N}$:

$$\Box_{i=1}^{n} A_i = (\Box_{i=1}^{n} \mu_{A_1}, \triangle_{i=1}^{n} \nu_{A_i}) = (\mu_{\Box_{i=1}^{n} A_i}, \nu_{\triangle_{i=1}^{n} A_i})$$
$$\triangle_{i=1}^{n} = (\triangle_{i=1}^{n} \mu_{A_1}, \Box_{i=1}^{n} \nu_{A_i}) = (\mu_{\triangle_{i=1}^{n} A_i}, \nu_{\Box_{i=1}^{n} A_i}),$$

and we assume that

$$\Box_{i=1}^{n} \mu_{A_1} + \triangle_{i=1}^{n} \nu_{A_i} \leq 1, \qquad \triangle_{i=1}^{n} \mu_{A_1} + \Box_{i=1}^{n} \nu_{A_i} \leq 1$$

for any given $n \in \mathbb{N}$.

Definition 3.4. *Let $\mathcal{F}$ be a collection of IF subsets of set Ω which is closed under Łukasiewicz operations and monotone limits and $[0, u]$ is an interval in a Riesz space.*

A mapping $m : \mathcal{F} \to [0, u]$ is called a state iff

1. $m(0_\Omega) = 0$, $m(1_\Omega) = 1$
2. $m(A \oplus B) = m(A) + m(B) + m(A \odot B)$
3. *if* $A_n \nearrow A$ *then* $m(A_n) \nearrow m(A)$

L. Ciungu and B. Riečan proved in [5] that any real state can be represented by integrals. In [10], J. Považan proved that any vector valued state can be represented by integrals.

Theorem 3.1. *Let V be a Riesz space, $u \in V$, $0 < u$ be a fixed positive element. Then for any state $m : \mathcal{F} \to [0, u]$ there exist measures $P, Q : \mathcal{S} \to [0, u]$ for which*

$$m(A) = \int_\Omega \mu_A dP + \int_\Omega (1 - \mu_A - \nu_A) dQ.$$

This theorem is very important in next section, especially in the proof of Inclusion–Exclusion property.

4 Inclusion–Exclusion property

Theorem 4.1. *Let V be a Riesz space, $u \in V$, $u > 0$ is a fixed positive element, A_i be IF-events, $m : F \to [0, u]$ be a state, where F is family of all IF-events and $(\square, \triangle)$ be an IE pair of operations. Then it holds:*

$$m(\square_{i=1}^n A_i) = \sum_{i=1}^n m(A_i) - \sum_{i<j}^n m(A_i \triangle A_j) + \ldots (-1)^{n+1} m(A_1 \triangle A_2 \triangle \ldots \triangle A_n).$$

Proof. The first sum on the right side of the equality can be rewriten as:

$$\sum_{i=1}^n m(A_i) = \sum_{i=1}^n \int_\Omega \mu_{A_i} dP + \sum_{i=1}^n \int_\Omega 1 - \mu_{A_i} - \nu_{A_i} dQ =$$
$$= \int_\Omega \sum_{i=1}^n \mu_{A_i} dP + \int_\Omega \sum_{i=1}^n 1 - \mu_{A_i} - \nu_{A_i} dQ.$$

Similarly

$$\sum_{i<j}^n m(A_i \triangle A_j) = \int_\Omega \sum_{i<j}^n \mu_{A_i} \triangle \mu_{A_j} dP + \int_\Omega \sum_{i<j}^n (1 - \mu_{A_i} \triangle \mu_{A_j} - \nu_{A_i} \square \nu_{A_j}) dQ,$$

$$\sum_{i<j<k}^n m(A_i \triangle A_j \triangle A_k) = \int_\Omega \sum_{i<j<k}^n \mu_{A_i} \triangle \mu_{A_j} \triangle \mu_{A_k} dP$$
$$+ \int_\Omega \sum_{i<j<k}^n (1 - \mu_{A_i} \triangle \mu_{A_j} \triangle \mu_{A_k} - \nu_{A_i} \square \nu_{A_j} \square \nu_{A_k}) dQ,$$

$$\vdots$$

$$m(A_1 \triangle A_2 \triangle \ldots \triangle A_n) = \int_\Omega \mu_{A_1} \triangle \mu_{A_2} \triangle \ldots \triangle \mu_{A_n} dP$$
$$+ \int_\Omega (1 - \mu_{A_1} \triangle \mu_{A_2} \triangle \ldots \triangle \mu_{A_n} - \nu_{A_1} \square \nu_{A_2} \square \ldots \square \nu_{A_n}) dQ.$$

Put $A = \square_{i=1}^n A_i$ and get together previous equalities:

$$\sum_{i=1}^{n} m(A_i) - \sum_{i<j}^{n} m(A_i \triangle A_j) + \sum_{i<j<k}^{n} m(A_i \triangle A_j \triangle A_k) - \cdots +$$

$$+ (-1)^{n+1} m(A_1 \triangle A_2 \triangle \ldots \triangle A_n) = \int_\Omega \sum_{i=1}^{n} (\mu_{A_i}) - \sum_{i<j}^{n} (\mu_{A_i} \triangle \mu_{A_j})$$

$$+ \sum_{i<j<k}^{n} (\mu_{A_i} \triangle \mu_{A_j} \triangle \mu_{A_k}) + \cdots + (-1)^{n+1} (\mu_{A_1} \triangle \mu_{A_2} \triangle \ldots \triangle \mu_{A_n}) dP$$

$$+ \int_\Omega \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots + (-1)^{n+1} \binom{n}{n} dQ - \int_\Omega \sum_{i=1}^{n} (\mu_{A_i}) - \sum_{i<j}^{n} (\mu_{A_i} \triangle \mu_{A_j})$$

$$+ \sum_{i<j<k}^{n} (\mu_{A_i} \triangle \mu_{A_j} \triangle \mu_{A_k}) + \cdots + (-1)^{n+1} (\mu_{A_1} \triangle \mu_{A_2} \triangle \ldots \triangle \mu_{A_n}) dQ$$

$$- \int_\Omega \sum_{i=1}^{n} (\nu_{A_i}) - \sum_{i<j}^{n} (\nu_{A_i} \square \nu_{A_j}) + \sum_{i<j<k}^{n} (\nu_{A_i} \square \nu_{A_j} \square \nu_{A_k}) + \cdots +$$

$$+ (-1)^{n+1} (\nu_{A_1} \square \nu_{A_2} \square \ldots \square \nu_{A_n}) dQ.$$

Because of Theorems 2.1 and the property

$$\square_{i=1}^n A_i = (\square_{i=1}^n \mu_{A_i}, \triangle_{i=1}^n \nu_{A_i}) = (\mu_{\square_{i=1}^n A_i}, \nu_{\triangle_{i=1}^n A_i}),$$

we can rewrite the previous sum as follows:

$$\int_\Omega \mu_A dP + \int_\Omega (-1)(-1+1)^n + 1 dQ - \int_\Omega \mu_A dQ - \int_\Omega \nu_A dQ =$$
$$= \int_\Omega \mu_1 dP + \int_\Omega 1 - \mu_A - \nu_A dQ = m(A),$$

which completes the proof. □

5 Conclusion

The Inclusion–Exclusion principle for measures on classical sets is an simple property. P. Grzegorzewski introduced this principle for probability (based on his concept of probability) on IF-events and B. Riečan for IF-states. Grzegorzewski approach was developed by C. L. Ciungu, J. Kelemenová, M. Kuková and B. Riečan. For example, J. Kelemenová generalized this principle for strongly additive states on IF-sets. In [8] M. Kuková and M. Navara investigated for which fuzzy operations the Inclusion–Exclusion principle holds and proved that only continuous fuzzy operations which satisfy Inclusion–Exclusion principle are the Gödel ones. In this paper we continue by proving the Inclusion–Exclusion principle for vector measures. A next step could be investigation of the properties of probability on IF-events and properties of IF-state based on some others operations.

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