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On the extension of group-valued measures

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Abstract: Since any space of IF-sets can be embedded to an MV-algebra ([12]), and any MV-algebra can be presented as an interval in a lattice ordered group ([9]), it is interesting to study measures with respect to l-groups. In this paper the group-valued outer measures are studied. The main result is the Choquet lemma concerning lower continuity of the induced outer measure. The result is applied to the group-valued measure extension theorem.
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1 Introduction

In [10] the measure extension theorem for group-valued measure was proved. The considered domain was a family of subsets of a given set. The notion of group-valued outer measure defined on a family of functions has been introduced in [7] where it was proved that the measurable elements form a lattice. The theory has been development further in [8]. In this paper we present a complete theory of group-valued outer measures on the families of functions. In *Section 2* the family of measurable elements is investigated, in *Section 3* the notion of induced outer measure is introduced. *Section 4* contains the proof of Choquet lemma. The obtained results are applied in *Section 5* for the formulation and the proof of G-valued extension theorem.

2 Outer measure and μ^*-measurable elements

Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{F}$ be the set of all non-negative measurable functions. Let G be a complete ℓ-group, i.e. a structure $(G, +, \leq)$ such that $(G, +)$ is an Abelian group, $(G, \leq)$

is a complete lattice (i.e. any upper bounded subset of G has the supremum) and $a \leq b$ implies $a + c \leq b + c$ for any $c \in G$. Let 0 be the neutral element of G (i.e. $a + 0 = a$ for any $a \in G$), $G^+ = \{a \in G; a \geq 0\}$. Denote by ∞ an ideal element and

$$G^* = G^+ \cup \{\infty\},$$

where $a + \infty = \infty + a = \infty + \infty = \infty$ for any $a \in G$ and $a \leq \infty$, $\infty \leq \infty$ for any $a \in G$.

Definition 2.1. *A mapping $\mu^* : \mathcal{F} \to G^*$ is called an outer measure if it is satisfying the following conditions:*

1. $\mu^*(0) = 0$,
2. $f \leq g$ *implies* $\mu^*(f) \leq \mu^*(g)$,
3. $\mu^*(f + g) \leq \mu^*(f) + \mu^*(g)$ *for each* $f, g \in \mathcal{F}$.

Definition 2.2. *Let $\mu^* : \mathcal{F} \to G^*$ be an outer measure. A function $f \in \mathcal{F}$ is called μ^*-measurable element if it holds:*

$$\mu^*(h) = \mu^*(h \wedge f) + \mu^*(h - (h \wedge f))$$

for each $h \in \mathcal{F}$.

Remark 2.3. *In this paper the operations $\wedge$ and $\vee$ take precedence over the operations $+$, $-$, thus the notation $(h - h \wedge f)$ denotes $(h - (h \wedge f))$.*

Theorem 2.4. *Denote by $\mathcal{M}$ the set of all μ^*-measurable elements of $\mathcal{F}$. Then $\mathcal{M}$ form a lattice.*

Proof.
(i) We show that if f, g are the μ^*-measurable elements, then $f \wedge g$ is also the μ^*-measurable element. Because μ^* is subadditive it is sufficient to show an inequality:

$$\mu^*(h) \geq \mu^*(h \wedge f \wedge g) + \mu^*(h - h \wedge f \wedge g).$$

Let f, g be the μ^*-measurable elements. Then for any $h \in \mathcal{F}$ it holds:

$$\mu^*(h) = \mu^*(h \wedge f) + \mu^*(h - h \wedge f)$$

and $h \wedge f \in \mathcal{F}$ therefore:

$$\mu^*(h \wedge f) = \mu^*(h \wedge f \wedge g) + \mu^*(h \wedge f - h \wedge f \wedge g).$$

Then:

$$\mu^*(h) = \mu^*(h \wedge f \wedge g) + \mu^*(h \wedge f - h \wedge f \wedge g) + \mu^*(h - h \wedge f) \geq$$
$$\geq \mu^*(h \wedge f \wedge g) + \mu^*(h \wedge f - h \wedge f \wedge g + h - h \wedge f) =$$
$$= \mu^*(h \wedge f \wedge g) + \mu^*(h - h \wedge f \wedge g).$$

It proves that $f \wedge g$ is the μ^*-measurable element.
(ii) We show that if f, g are the μ^*-measurable elements, then $f \vee g$ is also the μ^*-measurable element. Since $h - h \wedge f = h \vee f - f$ we have:

$$\mu^*(h - h \wedge f) = \mu^*(h \vee f - f).$$

Therefore if f is μ^*-measurable then for any $h \in \mathcal{F}$:

$$\mu^*(h) - \mu^*(h \wedge f) = \mu^*(h \vee f) - \mu^*(f)$$

or

$$\mu^*(h) + \mu^*(f) = \mu^*(h \wedge f) + \mu^*(h \vee f).$$

Let $f, g, f \wedge g$ be the μ^*-measurable elements, then:

$$\mu^*(h \wedge f \wedge g) = \mu^*((h \wedge f) \wedge (h \wedge g)) =$$

$$= \mu^*(h \wedge f) + \mu^*(h \wedge g) - \mu^*(h \wedge (f \vee g))$$

and also:

$$\mu^*(h - h \wedge f \wedge g) = \mu^*((h - h \wedge f) \vee (h - h \wedge g)) =$$

$$= \mu^*(h - h \wedge f) + \mu^*(h - h \wedge g) - \mu^*((h - h \wedge f) \wedge (h - h \wedge g))$$

$$= \mu^*(h - h \wedge f) + \mu^*(h - h \wedge g) - \mu^*(h - h \wedge (f \vee g)).$$

Therefore:

$$\mu^*(h \wedge (f \vee g)) + \mu^*(h - h \wedge (f \vee g)) =$$

$$= \mu^*(h \wedge f) + \mu^*(h - h \wedge f) + \mu^*(h \wedge g) + \mu^*(h - h \wedge g) -$$

$$-\mu^*(h \wedge f \wedge g) - \mu^*(h - h \wedge f \wedge g) =$$

$$= \mu^*(h) + \mu^*(h) - \mu^*(h) = \mu^*(h).$$

It proves that $f \vee g$ is the μ^*-measurable element and all μ^*-measurable elements of $\mathcal{F}$ form a lattice. □

3 Induced outer measure

Definition 3.1. *Let H_0 be a set of non-negative real functions satisfying the following conditions:*

1. *if $f, g \in H_0$ then $f \vee g \in H_0$,*
2. *if $f, g \in H_0$ then $f \wedge g \in H_0$,*
3. *if $f, g \in H_0$ then $f + g \in H_0$,*
4. *if $f, g \in H_0$ then $f - f \wedge g \in H_0$,*

Assume that to any $f \in \mathcal{F}$ there exist the functions $f_i \in H_0$ $(i = 1, 2, \dots)$ such that $\sum\limits_{i=1}^{\infty} f_i \geq f$.

Definition 3.2. *A G-valued measure on H_0 is a function μ such that:*

1. $\mu(0) = 0$
2. *if* $f, f_i \in H_0$ $(i = 1, 2, \ldots)$ *and* $f = \sum_{i=1}^{\infty} f_i$ *then* $\mu(f) = \sum_{i=1}^{\infty} \mu(f_i)$.

In the group-valued case we shall need the following structure:

Definition 3.3. *Dedekind complete ℓ-group G is called to be of countable type, if to any bounded set $A \subset G$ there exists such a countable subset $B \subset A$ that*

$$\bigwedge A = \bigwedge B.$$

Definition 3.4. *Let G be a Dedekind complete ℓ-group of countable type. Let μ be a G-valued measure defined on H_0. Then for any $f \in \mathcal{F}$ we define:*

$$\mu^*(f) = \bigwedge \left\{ \sum_{i=1}^{\infty} \mu(f_i);\ f_i \in H_0, f \leq \sum_{i=1}^{\infty} f_i \right\}.$$

Theorem 3.5. *The function μ^* is an outer measure.*

Proof.

(1) We prove $\mu^*(0) = 0$. Evidently:

$$0 \leq \mu^*(0) \leq \mu(0) + \mu(0) + ... = 0,$$

hence $\mu^*(0) = 0$.

(2) We prove that $f \leq g$ implies $\mu^*(f) \leq \mu^*(g)$.

Let $f \leq g$. By comparing the sets we get:

$$\left\{ \sum_{i=1}^{\infty} \mu(f_i);\ f_i \in H_0,\ f \leq \sum_{i=1}^{\infty} f_i \right\} \supset \left\{ \sum_{i=1}^{\infty} \mu(g_i);\ g_i \in H_0,\ g \leq \sum_{i=1}^{\infty} g_i \right\}.$$

We can see that $\mu^*(f)$ is the infimum of the larger set, so it is also a lower bound of the smaller set. Therefore $\mu^*(f) \leq \mu^*(g)$.

(3) We prove $\mu^*(f + g) \leq \mu^*(f) + \mu^*(g)$. Let $f, g \in \mathcal{F}$, $f_i, g_i \in H_0$ $(i = 1, 2, \ldots)$, and $f \leq \sum_{i=1}^{\infty} f_i$, $g \leq \sum_{i=1}^{\infty} g_i$.

If $\mu^*(f) = \infty$ or $\mu^*(g) = \infty$ then the inequality holds.

Let $\mu^*(f) \neq \infty$ and $\mu^*(g) \neq \infty$. Then:

$$f + g \leq \sum_{i=1}^{\infty} f_i + \sum_{i=1}^{\infty} g_i$$

and

$$\mu^*(f + g) \leq \sum_{i=1}^{\infty} \mu(f_i) + \sum_{i=1}^{\infty} \mu(g_i).$$

Now we fix for a moment $\sum_{i=1}^{\infty} f_i$. Since the preceding inequality holds for any $\sum_{i=1}^{\infty} g_i$, we obtain:

$$\mu^*(f+g) - \sum_{i=1}^{\infty} \mu(f_i) \leq \sum_{i=1}^{\infty} \mu(g_i),$$

hence:

$$\mu^*(f+g) - \sum_{i=1}^{\infty} \mu(f_i) \leq \mu^*(g).$$

Similarly the relation:

$$\mu^*(f+g) - \mu^*(g) \leq \sum_{i=1}^{\infty} \mu(f_i)$$

for any f_i implies:

$$\mu^*(f+g) - \mu^*(g) \leq \mu^*(f).$$

This completes the proof. □

Proposition 3.6. *For any $f \in \mathcal{F}$ holds:*

$$\mu^*(f) = \bigwedge\{\bigvee_{n=1}^{\infty} \mu(g_n);\ g_n \in H_0,\ g_n \leq g_{n+1},\ f \leq \bigvee_{n=1}^{\infty} g_n\}.$$

Proof.
(1) Consider any $g_n \in H_0,\ g_n \leq g_{n+1},\ f \leq \bigvee_{n=1}^{\infty} g_n (n = 1, 2, ...)$. Since $g_n \leq g_{n+1}$, then $g_{n+1} - (g_{n+1} \wedge g_n) = g_{n+1} - g_n$.

Put $f_1 = g_1,\ f_2 = g_2 - g_1,\ f_3 = g_3 - g_2, ...$ Then $\sum_{i=1}^{n} f_i = g_n$ and $\sum_{i=1}^{n} \mu(f_i) = \mu(g_n)$. Therefore:

$$\mu^*(f) = \bigwedge\{\sum_{i=1}^{\infty} \mu(f_i)\} \leq \sum_{i=1}^{\infty} \mu(f_i) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(f_i) = \bigvee_{n=1}^{\infty} \mu(g_n)$$

and

$$\mu^*(f) \leq \bigwedge\{\bigvee_{n=1}^{\infty} \mu(g_n)\}.$$

(2) On the other hand consider any sequence $(f_i)_{i=1}^{\infty},\ f_i \in H_0,\ f \leq \sum_{i=1}^{\infty} f_i$. Put $g_n = \sum_{i=1}^{n} f_i$. Then $g_n \in H_0,\ g_n \leq g_{n+1}$ and $\bigvee_{n=1}^{\infty} g_n = \sum_{i=1}^{\infty} f_i \geq f$.

Therefore:

$$\bigvee_{n=1}^{\infty} \mu(g_n) = \bigvee_{n=1}^{\infty} \mu(\sum_{i=1}^{n} f_i) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(f_i) = \sum_{i=1}^{\infty} \mu(f_i)$$

and

$$\bigwedge\{\bigvee_{n=1}^{\infty} \mu(g_n);\ g_n \in H_0,\ g_n \leq g_{n+1},\ f \leq \bigvee_{n=1}^{\infty} g_n\} \leq \sum_{i=1}^{\infty} \mu(f_i).$$

From the last inequality we get:

$$\bigwedge\{\bigvee_{n=1}^{\infty}\mu(g_n);\ g_n\in H_0,\ g_n\le g_{n+1},\ f\le\bigvee_{n=1}^{\infty}g_n\}\le\mu^*(f).$$

This completes the proof. □

Proposition 3.7. *Denote by* H_0^+ *the set of all functions* $g=\bigvee_{n=1}^{\infty}g_n$ *such that* $g_n\in H_0$, $g_n\le g_{n+1}$ $(n=1,2,...)$. *The value* $\bigvee_{n=1}^{\infty}\mu(g_n)$ *does not depends on the sequence* $(g_n)_{n=1}^{\infty}$ *but only on the function* g.

Proof. Let $g_n\in H_0, g_n\le g_{n+1}, g_n\nearrow g$ and $h_n\in H_0, h_n\le h_{n+1}, h_n\nearrow g$ then:

$$\bigvee_{n=1}^{\infty}(g_n\wedge h_m)=(\bigvee_{n=1}^{\infty}g_n)\wedge h_m$$

$$g_n\wedge h_m\nearrow g\wedge h_m=h_m$$

and

$$\mu(g_n\wedge h_m)\nearrow\mu(h_m).$$

For each $m\in N$ holds:

$$\mu(h_m)=\bigvee_{n=1}^{\infty}\mu(g_n\wedge h_m)\le\bigvee_{n=1}^{\infty}\mu(g_n)$$

and

$$\bigvee_{m=1}^{\infty}\mu(h_m)\le\bigvee_{n=1}^{\infty}\mu(g_n).$$

Similarly can be proved opposite inequality.

We proved that the value $\bigvee_{n=1}^{\infty}\mu(g_n)$ does not depends on the sequence $(g_n)_{n=1}^{\infty}$ but only on the function g. □

Remark 3.8. *For any* $f\in\mathcal{F}$ *there exist a function* $g\in H_0^+$ *such that* $g\ge f$.

Remark 3.9. *Because* $\bigvee_{n=1}^{\infty}\mu(g_n)$ *does not depends on the sequence* $(g_n)_{n=1}^{\infty}$ *but only on the function* g, *we can define* $\mu^+:H_0^+\to G$ *by the formula:*

$$\mu^+(g)=\bigvee_{n=1}^{\infty}\mu(g_n).$$

Using this notation we can reformulate Proposition 3.6 *by the following way:*

$$\mu^*(f)=\bigwedge\{\mu^+(g);\ g\in H_0^+,\ g\ge f\}.$$

4 Choquet lemma

The Choquet lemma states that $f_n \nearrow f$ implies $\mu^*(f_n) \nearrow \mu^*(f)$. Since in general ℓ-groups cannot be used the usual ε-technique we shall need following structure:

Definition 4.1. *Dedekind σ-complete ℓ-group G is called to be weakly σ-distributive if for any bounded double sequence $(a_{i,j})$ such that $a_{i,j} \searrow 0 (j \to \infty, i = 1, 2, \dots)$ it is*

$$\bigwedge_{\varphi \in N^N} \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} = 0.$$

Proposition 4.2. *If G is a Dedekind complete ℓ-group of countable type then for any real non-negative function f there exists a bounded double sequence $a_{i,j} \searrow 0$ $(j \to \infty, i = 1, 2, \dots)$ such that for any $\varphi : N \to N$ there exists $g \in H_0^+, f \leq g$ such that:*

$$\mu^*(f) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \mu^+(g).$$

Proof. Since G is an ℓ-group of countable type, by *Remark 3.9* there exists a sequence $(h_n)_{n=1}^{\infty}$ of elements of H_0^+ such that $f \leq h_n$ and

$$\mu^*(f) = \bigwedge_{n=1}^{\infty} \mu^+(h_n).$$

Put $g_n = \bigwedge_{i=1}^{n} h_i$. Then $f \leq g_n,\ g_n \in H_0^+,\ g_{n+1} \leq g_n$ and:

$$\mu^*(f) = \bigwedge_{n=1}^{\infty} \mu^+(g_n).$$

Define $a_{i,j} = \mu^+(g_j) - \mu^*(f),\ j \to \infty,\ i = 1, 2, \dots$.

Then $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \dots)$. Let $\varphi : N \to N$. Then:

$$\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq a_{i,\varphi(i)} = \mu^+(g_{\varphi(i)}) - \mu^*(f).$$

We may put $g = g_{\varphi(i)}$ and we obtain the inequality:

$$\mu^*(f) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \mu^+(g).$$

This completes the proof. □

Theorem 4.3. *Let G be a Dedekind complete weakly σ-distributive ℓ-group of countable type. Let $f_n, f \in \mathcal{F},\ f_n \nearrow f$. Then $\mu^*(f_n) \nearrow \mu^*(f)$.*

Proof. We shall use two properties of μ^+ which are consequences of analogous properties of μ:

(1) $\mu^+(g_1) + \mu^+(g_2) = \mu^+(g_1 \vee g_2) + \mu^+(g_1 \wedge g_2)$ for any $g_1, g_2 \in H_0^+$;

(2) if $h_n \nearrow h, h_n \in H_0^+$, then also $h \in H_0^+$ and $\mu^+(h) = \bigvee_{n=1}^{\infty} \mu^+(h_n)$.

Let $f_n, f \in \mathcal{F},\ f_n \nearrow f$. Evidently $\mu^*(f_n) \leq \mu^*(f)$ and the equality holds if $\bigvee_{n=1}^{\infty} \mu^*(f_n) = \infty$. Therefore we can assume that $\bigvee_{n=1}^{\infty} \mu^*(f_n) \in G^+$.

By *Proposition 4.2* for any $n \in N$ there exists a bounded sequence $(a_{n,i,j})_{i,j}$ such that for any $\varphi : N \to N$ there exists $g_n \in H_0^+, f_n \leq g_n$ such that:

$$\mu^*(f_n) + \bigvee_{i=1}^{\infty} a_{n,i,\varphi(n+i)} \geq \mu^+(g_n).$$

By the Fremlin theorem (see [8], Theorem 3.2.3) there exists a bounded double sequence $(a_{i,j})_{i,j} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ and such that

$$a \wedge \Big(\sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{k,i,\varphi(i+k)} \Big) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for each $k \in N$.

By *Remark 3.8* for any $f \in \mathcal{F}$ there exist such $g \in H_0^+$ that $g \geq f$.

Put $h_n = (\bigvee_{i=1}^{n} g_i) \wedge g$. Then $h_n \in H_0^+, h_n \leq h_{n+1}$ and

$$f = \bigvee_{n=1}^{\infty} f_n \leq \bigvee_{n=1}^{\infty} h_n, \ \ \Big(\bigvee_{n=1}^{\infty} h_n = (\bigvee_{i=1}^{\infty} g_i) \wedge g \Big).$$

Therefore:

$$\mu^*(f_2) + \bigvee_{i=1}^{\infty} a_{2,i,\varphi(2+i)} \geq \mu^+(g_2) =$$

$$= \mu^+(g_1 \vee g_2) + \mu^+(g_1 \wedge g_2) - \mu^+(g_1) \geq$$

$$\geq \mu^+((g_1 \vee g_2) \wedge g) + \mu^*(f_1 \wedge f_2) - \mu^*(f_1) - \bigvee_{i=1}^{\infty} a_{1,i,\varphi(1+i)} =$$

$$= \mu^+(h_2) + \mu^*(f_1) - \mu^*(f_1) - \bigvee_{i=1}^{\infty} a_{1,i,\varphi(1+i)} =$$

$$= \mu^+(h_2) - \bigvee_{i=1}^{\infty} a_{1,i,\varphi(1+i)}.$$

Hence:

$$\mu^+(h_2) - \mu^*(f_2) \leq \bigvee_{i=1}^{\infty} a_{1,i,\varphi(1+i)} + \bigvee_{i=1}^{\infty} a_{2,i,\varphi(2+i)}$$

and similarly:

$$\mu^+(h_n) - \mu^*(f_n) \leq \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{k,i,\varphi(k+i)}.$$

At the same time:

$$\mu^+(h_n) - \mu^*(f_n) \leq \mu^+(h_n) \leq \mu^+(g).$$

Therefore:

$$\mu^+(h_n) - \mu^*(f_n) \leq \mu^+(g) \wedge \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{k,i,\varphi(k+i)}.$$

Let $a = \mu^+(g)$. Then we can use Fremlin theorem and it holds:

$$\mu^+(h_n) - \mu^*(f_n) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Therefore:

$$\bigvee_{n=1}^{\infty} \mu^*(f_n) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \bigvee_{n=1}^{\infty} \mu^+(h_n) = \mu^+(\bigvee_{n=1}^{\infty} h_n).$$

But by the *Remark 3.9*:

$$\mu^+(\bigvee_{n=1}^{\infty} h_n) \geq \mu^*(f).$$

Hence:

$$\bigvee_{n=1}^{\infty} a_{i,\varphi(i)} \geq \mu^*(f) - \bigvee_{n=1}^{\infty} \mu^*(f_n)$$

holds for any $\varphi : N \to N$ and G is weakly σ-distributive ℓ-group, therefore we obtain:

$$0 = \bigwedge_{\varphi} \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \mu^*(f) - \bigvee_{n=1}^{\infty} \mu^*(f_n).$$

This completes the proof. □

5 Measure extension theorem

Theorem 5.1. *Let G be a Dedekind complete weakly $\sigma-$distributive ℓ-group of countable type. Let $\mathcal{M} \subset \mathcal{F}$ be the family of all μ^*-measurable elements. Then $\mathcal{M}$ is a σ-complete lattice, $H_0 \subset \mathcal{M}$, and the restriction $\mu^*|\mathcal{M}$ is a measure.*

Proof. First we show that $\mathcal{M}$ is a σ-complete lattice.

We have already proved, that $\mathcal{M}$ is a lattice. So we have to show, that for any sequence $(f_n)_{n=1}^{\infty}$ of μ^*-measurable elements the functions $f = \bigvee_{n=1}^{\infty} f_n$ and $f' = \bigwedge_{n=1}^{\infty} f_n$ are also μ^*-measurable elements. Put $g_n = \bigvee_{i=1}^{n} f_i$ for each $n \in N$. Then $g_n \leq g_{n+1}$ and $\bigvee_{n=1}^{\infty} g_n = \bigvee_{n=1}^{\infty} f_n = f$ therefore $g_n \nearrow f$. Since g_n is μ^*-measurable for each $n \in N$ then for each $h \in \mathcal{F}$ holds:

$$\mu^*(h) = \mu^*(h \wedge g_n) + \mu^*(h - h \wedge g_n).$$

Since $f \geq g_n$ then $h \wedge f \geq h \wedge g_n$ and $h - h \wedge g_n \geq h - h \wedge f$. Therefore:

$$\mu^*(h - h \wedge g_n) \geq \mu^*(h - h \wedge f)$$

and

$$\mu^*(h) \geq \mu^*(h \wedge g_n) + \mu^*(h - h \wedge f).$$

Since $g_n \nearrow f$ also $h \wedge g_n \nearrow h \wedge f$ by Choquet lemma we obtain:

$$\lim_{n\to\infty} \mu^*(h \wedge g_n) = \mu^*(h \wedge f).$$

Therefore:

$$\mu^*(h) \geq \mu^*(h \wedge f) + \mu^*(h - h \wedge f).$$

Opposite inequality follows from subadditivity of outer measure μ^*.

This proves that f is also the μ^*-measurable element.

Now we show that $f' = \bigwedge_{n=1}^{\infty} f_n$ is a μ^*-measurable element.

Let $f' = \bigwedge_{n=1}^{\infty} f_n$. Put $g'_n = \bigwedge_{i=1}^{n} f_i$ for each $n \in N$. Then $g'_n \geq g'_{n+1}$ and $\bigwedge_{n=1}^{\infty} g'_n = \bigwedge_{n=1}^{\infty} f_n = f'$ therefore $g'_n \searrow f'$. Since g'_n is μ^*-measurable for each $n \in N$ then for each $h \in \mathcal{M}$ it holds:

$$\mu^*(h) = \mu^*(h \wedge g'_n) + \mu^*(h - h \wedge g'_n).$$

Since $g'_n \geq f'$, $(n = 1, 2, \ldots)$ then $h \wedge g'_n \geq h \wedge f'$ and

$$\mu^*(h \wedge g'_n) \geq \mu^*(h \wedge f').$$

Therefore:

$$\mu^*(h) \geq \mu^*(h \wedge f') + \mu^*(h - h \wedge g'_n).$$

Since $g'_n \searrow f$ then $h \wedge g'_n \searrow h \wedge f'$ and $h - h \wedge g'_n \nearrow h - h \wedge f'$ then by the Choquet lemma we obtain:

$$\lim_{n\to\infty} \mu^*(h - h \wedge g'_n) = \mu^*(h - h \wedge f')$$

therefore:

$$\mu^*(h) \geq \mu^*(h \wedge f') + \mu^*(h - h \wedge f').$$

Opposite inequality follows from subadditivity of the outer measure μ^*.

This proves that f' is the μ^*-measurable elements hence the lattice $\mathcal{M}$ is a σ-complete lattice.

Secondly, we show that $H_0 \subset \mathcal{M}$.

We already proved that for any $f \in \mathcal{F}$ it holds:

$$\mu^*(f) = \bigwedge\{\mu^+(g);\ g \in H_0^+,\ f \leq g\}.$$

Since μ^* is subadditive than for any $f \in \mathcal{F}$ and $h \in H_0$ it holds:

$$\mu^*(f) \leq \mu^*(f \wedge h) + \mu^*(f - f \wedge h).$$

By the *Proposition 4.2* for any real non-negative function f there exists a bounded double sequence $a_{i,j} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ such that for any $\varphi : N \to N$ there exists $g \in H_0^+$, $f \leq g$ and that:

$$\mu^*(f) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \mu^+(g).$$

But:

$$\mu^{+}(g) = \lim_{n\to\infty} \mu(g_n) = \lim_{n\to\infty} [\mu(g_n \wedge h) + \mu(g_n - g_n \wedge h)] =$$

$$= \lim_{n\to\infty} \mu(g_n \wedge h) + \lim_{n\to\infty} \mu(g_n - g_n \wedge h) = \mu^{+}(g \wedge h) + \mu^{+}(g - g \wedge h).$$

Now we show that:

$$\mu^{+}(g \wedge h) + \mu^{+}(g - g \wedge h) \geq \mu^{*}(f \wedge h) + \mu^{*}(f - f \wedge h).$$

Since $g \geq f$ then $g \wedge h \geq f \wedge h$ and $\mu^{+}(g \wedge h) \geq \mu^{*}(f \wedge h)$.

Let us use following notation:

$$(g - h)^{+} = 0 \vee (g - h).$$

If $(f - h)^{+} = 0$ and $g \geq f$ then $(g - h)^{+} \geq (f - h)^{+}$.
If $(f - h)^{+}(x) = f(x) - h(x) > 0$ then

$$(g - h)^{+}(x) = g(x) - h(x) \geq f(x) - h(x) = (f - h)^{+}(x).$$

Therefore:

$$(g - h)^{+} \geq (f - h)^{+}$$

and then:

$$0 \vee (g - h) \geq 0 \vee (f - h),$$

$$(g - g) \vee (g - h) \geq (f - f) \vee (f - h),$$

$$g - (g \wedge h) \geq f - (f \wedge h).$$

Then also:

$$\mu^{+}(g - g \wedge h) \geq \mu^{*}(f - f \wedge h).$$

Hence:

$$\mu^{*}(f) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \mu^{*}(f \wedge h) + \mu^{*}(f - f \wedge h)$$

for each $\varphi : N \to N$. Therefore:

$$\mu^{*}(f) = \mu^{*}(f \wedge h) + \mu^{*}(f - f \wedge h)$$

for each $h \in H_0$.

Thirdly, we show that $\mu^{*}|\mathcal{M}$ is a measure.

Because of the Choquet lemma it is sufficient to prove that $\mu^{*}|\mathcal{M}$ is an additive. Let $g, h \in \mathcal{M}$. Then it holds:

$$\mu^{*}(f) = \mu^{*}(f \wedge g) + \mu^{*}(f - f \wedge g).$$

Put $f = g + h$ then:

$$\mu^{*}(g + h) = \mu^{*}((g + h) \wedge g) + \mu^{*}((g + h) - ((g + h) \wedge g)) = \mu^{*}(g) + \mu^{*}(h).$$

This completes the proof. $\square$

Theorem 5.2. *Let $\sigma(H_0)$ be a σ-complete lattice generated by H_0. Then $\sigma(H_0)$ is a σ-complete lattice closed under the operation $+$ and the operation $(f, g) \mapsto f - f \wedge g$.*

Proof.

1. Let $f \in H_0$ and fix it for a moment. Define:

$$A = \{g;\ f + g \in \sigma(H_0)\}.$$

   Then $H_0 \subset A$ and A is a σ-complete lattice. Therefore also $\sigma(H_0) \subset A$.

2. Now we take $g \in \sigma(H_0)$ fixed and denote:

$$B = \{f;\ f + g \in \sigma(H_0)\}.$$

   We can see that for any $f \in H_0$ it holds $f + g \in \sigma(H_0)$. Therefore $H_0 \subset B$. Since B is a σ-complete lattice then also $\sigma(H_0) \subset B$.

3. Let $f \in H_0$ and fix it for a moment. Define:

$$C = \{g;\ f - f \wedge g \in \sigma(H_0)\}.$$

   Then $H_0 \subset C$ and C is a σ-complete lattice. Therefore also $\sigma(H_0) \subset C$.

4. Now we take $g \in \sigma(H_0)$ fixed and denote:

$$D = \{f;\ f - f \wedge g \in \sigma(H_0)\}.$$

   We can see that for any $f \in H_0$ it holds $f - f \wedge g \in \sigma(H_0)$. Therefore $H_0 \subset D$. Since D is a σ-complete lattice then also $\sigma(H_0) \subset D$.

We have proved that for any $f, g \in \sigma(H_0)$ also $f + g,\ f - f \wedge g \in \sigma(H_0)$. □

Theorem 5.3. *$\mu^*|\ \sigma(H_0) = \overline{\mu}$ is a measure on the σ-complete lattice $\sigma(H_0)$.*

Proof. Because $H_0 \subset \mathcal{M}$ and the set $\mathcal{M}$ of all μ^*-measurable elements is σ-complete, then $\sigma(H_0) \subset \mathcal{M}$ and therefore $\mu^*|\ \sigma(H_0) = \overline{\mu}$ is a restriction of the measure $\mu^*|\mathcal{M}$ on the lattice $\sigma(H_0)$. □

Theorem 5.4. *There exists exactly one measure $\bar{\mu}$ on $\sigma(H_0)$ that is an extension of $\mu : H_0 \to G$.*

Proof. Let $\nu : \sigma(H_0) \to G^*$ be a measure which extending μ. Put

$$\mathcal{K} = \{f \in \sigma(H_0); \nu(f) = \bar{\mu}(f)\}.$$

Then $H_0 \subset \mathcal{K}$ and $\mathcal{K}$ is σ-complete lattice, therefore $\sigma(H_0) \subset \mathcal{K}$. Therefore for any function $f \in \sigma(H_0)$ holds also $f \in \mathcal{K}$ an therefore $\nu(f) = \bar{\mu}(f)$ for each $f \in \sigma(H_0)$. □

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