NIFS Vol. 15 (2009), No. 1, 1–12

A note on the Hausdorff distance between Atanassov's intuitionistic fuzzy sets

Eulalia Szmidt and Janusz Kacprzyk

Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01–447 Warsaw, Poland
E-mail: {szmidt, kacprzyk}@ibspan.waw.pl

Abstract

In this paper we address the problem of constructing the Hausdorf distance between A-IFSs based on the Hamming metric. We pay particular attention to the consistency of the metric used and the essence of the Hausdorff distances.
Keywords: Intuitionistic fuzzy sets, distances, Hausdorff metric.

1 Introduction

Distances are indispensable tool used both in theoretical considerations and for practical purposes in many areas. It is not possible to overestimate their importance also in the context of fuzzy sets (Zadeh [32]) or their generalizations. Distances are necessary while concluding about similarity, when making group decisions, assessing soft consensus, in pattern recognition, classifications, etc.

Among different sort of distances the Hausdorff distances (cf. Grünbaum [9]) play an important role in practical applications, especially in many visual tasks, such as image matching, image analysis, motion tracking, visual navigation of robots, computer-assisted surgery and so on (cf. e.g., Huttenlocher et al. [10], Huttenlocher and Rucklidge [11], Olson [13], Peitgen et al. [14], Rucklidge [16]-[20]).

Though the definition of the Hausdorff distances is simple, the calculations needed to solve the real problems are complex. In effect the efficiency of the algorithms for computing the Hausdorff distances is decisive and computing approximations are of most interest (e.g, Aichholzer [1], Atallah [2], Huttenlocher et al. [10], Preparata and Shamos [15], Rucklidge [20], Veltkamp [31]).

In the light of the practical importance of the Haudorff distances (resulting from their properties), the formulas proposed for calculating the distances should be reliable. It is the motivation of this paper. Namely, we consider the results of using the Hamming distances between A-IFSs calculated in two possible ways - taking into account two parameter representation (membership and non-membership values) of A-IFSs, and next - taking into account three parameter representation (membership, non-membership values, and

the hesitation margins) of A-IFSs. We will verify if the resulting distances fulfill the properties of the Hausdorff distances.

2 Brief introduction to A-IFSs

As opposed to a fuzzy set in X (Zadeh [32]) , given by

$$A^{'} = \{< x, \mu_{A'}(x) > | x \in X\} \tag{1}$$

where $\mu_{A'}(x) \in [0,1]$ is the membership function of the fuzzy set $A^{'}$, an A-IFS (Atanassov [3], [4]) A is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2}$$

where: $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{3}$$

and $\mu_A(x)$, $\nu_A(x) \in [0,1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively. (Two approaches to the assigning memberships and non-memberships for A-IFSs are proposed by Szmidt and Baldwin [23]).

Obviously, each fuzzy set may be represented by the following A-IFS

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \tag{4}$$

An additional concept for each A-IFS in X, that is not only an obvious result of (2) and (3) but which is also relevant for applications, we will call (Atanasov [4])

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

a *hesitation margin* of $x \in A$ which expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [4]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

Hesitation margins turn out to be relevant for applications - in image processing (cf. Bustince et al. [6], [7]) and classification of imbalanced and overlapping classes (cf. Szmidt and Kukier [26], [27], [28]).

2.1 Distances Between the A-IFSs

Distances between A-IFSs are calculated in the literature in two ways, using two parameters only (e.g., Atanassov [4]) or all three parameters (Szmidt and Kacprzyk [24], [25], Tasseva et al. [30], Atanassov et al. [5], Szmidt and Baldwin [21], [22], Deng-Feng [8], Tan and Zhang [29], Narukawa and Torra [12])) describing elements belonging to the sets. Both ways are proper from the point of view of pure mathematical conditions concerning distances (all properties are fulfilled in both cases). Unfortunately one cannot say that both ways are equal when assessing the results obtained by the two approaches.

In Szmidt and Kacprzyk [24], [25], Szmidt and Baldwin [21], [22], it is shown why in the calculation of distances between A-IFSs one should use all three parameters describing A-IFSs.

Examples of the distances between any two A-IFSs A and B in $X = \{x_1, x_2, \ldots, x_n\}$ while using three parameter representation (Szmidt and Kacprzyk [24], Szmidt and Baldwin [21], [22]):

- the normalized Hamming distance:

$$\begin{aligned} l_{IFS}(A,B) &= \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + \\ &+ |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|) \end{aligned} \tag{6}$$

- the normalized Euclidean distance:

$$\begin{aligned} e_{IFS}(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + \\ + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2)^{\frac{1}{2}} \end{aligned} \tag{7}$$

Both distances are from the interval [0,1].

The counterparts of the above distances while using two parameter representation of A-IFSs are:

- the normalized Hamming distance:

$$l'(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|) \tag{8}$$

- the normalized Euclidean distance:

$$q'(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)^{\frac{1}{2}} \tag{9}$$

3 The Hausdorff distance

The Hausdorff distance is *the maximum distance of a set to the nearest point in the other set*. More formal description is given by the following

Definition 1 Given two finite sets $A = \{a_1, ..., a_p\}$ and $B = \{b_1, ..., b_q\}$, the Hausdorff distance $H(A,B)$ is defined as:

$$H(A,B) = \max\{h(A,B), h(B,A)\} \tag{10}$$

where

$$h(A,B) = \max_{a\in A}\min_{b\in B} d(a,b) \tag{11}$$

– a and b are elements of sets A and B respectively,
– $d(a,b)$ is any metric between these elements,
– the two distances $h(A,B)$ and $h(B,A)$ (11) are called directed Hausdorff distances.

The function $h(A,B)$ (the directed Hausdorff distance from A to B) ranks each element of A based on its distance to the nearest element of B, and then the largest ranked such element (the most mismatched element of A) specifies the value of the distance. Intuitively, if $h(A,B) = c$, then each element of A must be within distance c of some element of B, and there also is some element of A that is exactly distance c from the nearest element of B (the most mismatched element). In general $h(A,B)$ and $h(B,A)$ can attain very different values (the directed distances are not symmetric).

Brute force algorithm of calculating the directed Hausdorff distances (11):

1. $h = 0$
2. for every element a_i of A,
   2.1 $shortest = \infty$;
   2.2 for every point b_j of B
   $d_{ij} = d(a_i, b_j)$
   if $d_{ij} < shortest$ then
   $shortest = d_{ij}$
   2.3 if $shortest > h$ then
   $h = shortest$

It follows from Definition 1 and the above algorithm that if A and B contain one element each (a_1 and b_1, respectively), the Hausdorff distance is just equal to $d(a_1, b_1)$. In other words, if a formula which is expected to express the Hausdorff distance gives for the separate elements the results not consistent with the used metric d (e.g., the Hamming distance, the Euclidean distance etc.), the considered formula is not a proper definition of the Hausdorff distance.

3.1 Results for the Hausdorff distance while using the two parameter Hamming distance

Due to the algorithm of calculating the directed Hausdorff distances, when aplying two parameter distances (8)-(9) for A-IFSs, we obtain:

$$d_h(A,B) = \frac{1}{n}\sum_{i=1}^{n} max\{|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|\} \tag{12}$$

$$q_h(A,B) = (\frac{1}{n}\sum_{i=1}^{n} \max\{(\mu_A(x_i) - \mu_B(x_i))^2,\ (\nu_A(x_i) - \nu_B(x_i))^2\})^{\frac{1}{2}} \tag{13}$$

If the above distances are properly calculated Hausdorff distances, in the case of degenerated, i.e., one-element sets $A = \{< x, \mu_A(x), \nu_A(x) >\}$ and $B = \{< x, \mu_B(x), \nu_B(x) >\}$, they should give the same results as the two parameter Hamming distance, and the Euclidean distance. It means that in the case of the two prameter Hamming distance, for one element IFSs, the following equations should give just the same results:

$$d(A,B) = \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) \tag{14}$$

$$d(A,B) = \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|\} \tag{15}$$

where (14) is the normalized two parameter Hamming distance, and (15) should be its counterpart Hausdorff metric.

We will verify on the simple examples if (14) and (15) gives the same results as it should be following the essence of of the Hausdorff measures.

Example 1

Let consider the following one-element A-IFSs: A, B, D, G, $E \in X = \{x\}$

$$\begin{aligned} A &= \{< x, 1, 0 >\}, \quad B = \{< x, 0, 1 >\}, \quad D = \{< x, 0, 0 >\}, \\ G &= \{< x, \frac{1}{2}, \frac{1}{2} >\}, \quad E = \{< x, \frac{1}{4}, \frac{1}{4} >\} \end{aligned} \tag{16}$$

The results obtained from (15) are:

$$\begin{aligned} d_h(A,B) &= \max\{|1-0|, |0-1|\} = 1 \\ d_h(A,D) &= \max\{|1-0|, |0-0|\} = 1 \\ d_h(B,D) &= \max\{|0-0|, |1-0|\} = 1 \\ d_h(A,G) &= \max\{|1-1/2|, |0-1/2|\} = 0.5 \\ d_h(A,E) &= \max\{|1-1/4|, |0-1/4|\} = 0.75 \\ d_h(B,G) &= \max\{|0-1/2|, |1-1/2|\} = 0.5 \\ d_h(B,E) &= \max\{|0-1/4|, |1-1/4|\} = 0.75 \\ d_h(D,G) &= \max\{|0-1/2|, |0-1/2|\} = 0.5 \\ d_h(D,E) &= \max\{|0-1/4|, |1-1/4|\} = 0.25 \\ d_h(G,E) &= \max\{|1/2-1/4|, |1/2-1/4|\} = 0.25 \end{aligned}$$

Their counterpart Hamming distances calculated from (14) are:

$$\begin{aligned} l'(A,B) &= 0.5(|1-0| + |0-1|) = 1 \\ l'(A,D) &= 0.5(|1-0| + |0-0||) = 0.5 \\ l'(B,D) &= 0.5(|0-0| + |1-0||) = 0.5 \\ l'(A,G) &= 0.5(|0-1/2| + |0-1/2|) = 0.5 \\ l'(A,E) &= 0.5(|1-1/4| + |0-1/4||) = 0.5 \\ l'(B,G) &= 0.5(|1-1/4| + |0-1/4|) = 0.5 \end{aligned}$$

$$
\begin{aligned}
l^{'}(B,E) &= 0.5(|1-1/4|+|0-1/4|) = 0.5\\
l^{'}(D,G) &= 0.5(|0-1/2|+|0-1/2|) = 0.5\\
l^{'}(D,E) &= 0.5(|0-1/4|+|0-1/4|) = 0.25\\
l^{'}(G,E) &= 0.5(|1/2-1/4|+|1/2-1/4|) = 0.25
\end{aligned}
$$

i.e. the values of the Hamming distances (14) used to propose the Hausdorff measures (15), and the values of the resulting Hausdorff distances (15) calculated for the separate elements are not consistent (as they should be). The differences:

$$d_h(A,D) \neq l^{'}(A,D) \tag{17}$$
$$H_h(B,D) \neq l^{'}(B,D) \tag{18}$$
$$H_h(A,E) \neq l^{'}(A,E) \tag{19}$$
$$H_h(B,E) \neq l^{'}(B,E) \tag{20}$$

Now we will show that the inconsistencies as showed above occur not only for (17) – (20) but for infinite number of other cases.

Let us verify the conditions under which the equation (14) and (15) give the consistent results, i.e., when for the separate elements we have

$$
\begin{aligned}
&\frac{1}{2}(|\mu_A(x)-\mu_B(x)|+|\nu_A(x)-\nu_B(x)|) =\\
=\ &\max\{|\mu_A(x)-\mu_B(x)|, |\nu_A(x)-\nu_B(x)|\}
\end{aligned} \tag{21}
$$

Having in mind that

$$\mu_A(x)+\nu_A(x)+\pi_A(x) = 1 \tag{22}$$

$$\mu_B(x)+\nu_B(x)+\pi_B(x) = 1 \tag{23}$$

from (22) and (23) we obtain

$$(\mu_A(x)-\mu_B(x))+(\nu_A(x)-\nu_B(x))+(\pi_A(x)-\pi_B(x)) = 0 \tag{24}$$

It is easy to verify that (24) is not fulfilled for all elements belonging to an A-IFSs but for some elements only. The following conditions guarantee that (21) is fulfilled

- for $\pi_A(x)-\pi_B(x) = 0$, from (24) we have

$$|\mu_A(x)-\mu_B(x)| = |\nu_A(x)-\nu_B(x)| \tag{25}$$

  and taking into account (25), we can express (21) in the following way:

$$
\begin{aligned}
0.5(|\mu_A(x)-\mu_B(x)| &+ |\mu_A(x)-\mu_B(x)|) =\\
&= \max\{|\mu_A(x)-\mu_B(x)|, |\mu_A(x)-\mu_B(x)|\}
\end{aligned} \tag{26}
$$

- if $\pi_A(x) - \pi_B(x) \neq 0$ but the same time

$$\mu_A(x) - \mu_B(x) = \nu_A(x) - \nu_B(x) = -\frac{1}{2}(\pi_A(x) - \pi_B(x)) \tag{27}$$

  quarantee that (21) boils down again to (26).

In other words, (21) is fulfilled (what means that the Hausdorff measure given by (15) is a natural counterpart of (14)) only for such elements belonging to an A-IFS, for which some additional conditions are given like: $\pi_A(x) - \pi_B(x) = 0$ or (27). But in general, for infinite numbers of elements, (21) is not valid.

In the above context it seems unfortunate trying to construct the Hausdorff distance using two parameter Hamming distance between A-IFSs.

We have made similar calculations considering the normalized two parameter Euclidean distance (13) which is not the counterpart (in the sense of the commonly used definition of Hausdorff distance (11)) of the normalized Euclidean distance

$$q^{'}(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)^{\frac{1}{2}} \tag{28}$$

3.2 A straightforward generalizations of the Hamming distance based on the Hausdorff metric (11)

Now we will show that applying the three parameter Hamming distance for A-IFSs, we obtain its correct (in the sense of Definition 1) counterpart in terms of max function, i.e. obtain a generalization of the Hamming distance based on the Hausdorff metric.

Namely, if we calculate the three parameter Hamming distance between two degenerated, i.e. one-element IFSs, A and B in the spirit of Szmidt and Kacprzyk [24], [25], Szmidt and Baldwin [21], [22], i.e., in the following way:

$$\begin{aligned} l_{IFS}(A,B) &= \frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)| + \\ &+ |\pi_A(x) - \pi_B(x)|) \end{aligned} \tag{29}$$

we can give a counterpart of the above distance in terms of max function:

$$\begin{aligned} H_3(A,B) &= \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, \\ &, \quad |\pi_A(x) - \pi_B(x)|\} \end{aligned} \tag{30}$$

If $H_3(A,B)$ (30) is properly calculated Hausdorff distance, the following condition should be fulfilled:

$$\begin{aligned} &\frac{1}{2}(|\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)|) + |\pi_A(x) - \pi_B(x)|) = \\ = \ &\max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} \end{aligned} \tag{31}$$

Let us verify if (31) is valid. Without loss of generality we can assume

$$\begin{aligned} \max & \{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} = \\ = & |\mu_A(x) - \mu_B(x)| \end{aligned} \tag{32}$$

For $|\mu_A(x) - \mu_B(x)|$ fulfilling (32), and because of (22) and (23), we conclude that both $\nu_A(x) - \nu_B(x)$, and $\pi_A(x) - \pi_B(x)$ are of the same sign (both values are either positive or negative). Therefore

$$|\mu_A(x) - \mu_B(x)| = |\nu_A(x) - \nu_B(x)| + |\pi_A(x) - \pi_B(x)| \tag{33}$$

Applyng (33) we can verify that (31) always is valid as

$$\begin{aligned} & 0.5\{|\mu_A(x) - \mu_B(x)| + |\mu_A(x) - \mu_B(x)|\} = \\ = & \max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} = \\ = & |\mu_A(x) - \mu_B(x)| \end{aligned} \tag{34}$$

Now we will use the above formulas (29) and (30) for the data used in Example 1. But now, as we also take into account the hesitation margins $\pi(x)$ (5), instead of (16) we use the "full description" of the data $\{< x, \mu(x), \nu(x), \pi(x) >\}$, i.e. employing all three functions (membership, non-membership and hesitation margin) describing the considered A-IFSs:

$$\begin{aligned} A &= \{< x, 1, 0, 0 >\}, \quad B = \{< x, 0, 1, 0 >\}, \quad D = \{< x, 0, 0, 1 >\}, \\ G &= \{< x, \frac{1}{2}, \frac{1}{2}, 0 >\}, \quad E = \{< x, \frac{1}{4}, \frac{1}{4}, \frac{1}{2} >\} \end{aligned} \tag{35}$$

and obtain from (30):

$$\begin{aligned} H_3(A, B) &= \max(|1-0|, |0-1|, |0-0|) = 1 \\ H_3(A, D) &= \max(|1-0|, |0-0|, |0-1|) = 1 \\ H_3(B, D) &= \max(|0-0|, |1-0|, |0-1|) = 1 \\ H_3(A, G) &= \max(|0-1/2|, |0-1/2|, |0-0|) = 0.5 \\ H_3(A, E) &= \max(|1-1/4|, |0-1/4|, |0-1/2|) = 0.75 \\ H_3(B, G) &= \max(|1-1/4|, |0-1/4|, |0-1/2|) = 0.75 \\ H_3(B, E) &= \max(|1-1/4|, |0-1/4|, |0-1/2|) = 0.75 \\ H_3(D, G) &= \max(|0-1/2|, |0-1/2|, |1-0|) = 1 \\ H_3(D, E) &= \max(|0-1/4|, |0-1/4|, |1-1/2|) = 0.5 \\ H_3(G, E) &= \max(|1/2-1/4|, |1/2-1/4|, |0-1/2|) = 0.5 \end{aligned}$$

Now we calculate the counterpart Hamming distances using (29) (with all three functions). The results are

$$l_{IFS}(A, B) = 0.5(|1-0| + |0-1| + |0-0|) = 1$$

$$
\begin{aligned}
l_{IFS}(A,D) &= 0.5(|1-0|+|0-0|+|0-1|) = 1 \\
l_{IFS}(B,D) &= 0.5(|0-0|+|1-0|+|0-1|) = 1 \\
l_{IFS}(A,G) &= 0.5(|0-1/2|+|0-1/2|+|0-0|) = 0.5 \\
l_{IFS}(A,E) &= 0.5(|1-1/4|+|0-1/4|+|0-1/2|) = 0.75 \\
l_{IFS}(B,G) &= 0.5(|1-1/4|+|0-1/4|+|0-1/2|) = 0.75 \\
l_{IFS}(B,E) &= 0.5(|1-1/4|+|0-1/4|+|0-1/2|) = 0.75 \\
l_{IFS}(D,G) &= 0.5(|0-1/2|+|0-1/2|+|1-0|) = 1 \\
l_{IFS}(D,E) &= 0.5(|0-1/4|+|0-1/4|+|1-1/2|) = 0.5 \\
l_{IFS}(G,E) &= 0.5(|1/2-1/4|+|1/2-1/4|+|0-1/2|) = 0.5
\end{aligned}
$$

As we can see, the Hausdorff distance (30) proposed in this paper (using memberships, non-memberships and hesitation margins) and the Hamming distance (29) give for one-element IFS sets fully consistent results.

In other words, for the normalized Hamming distance expressed in the spirit of (Szmidt and Kacprzyk [24], [25]) given by (6) we can give the following equivalent representation in terms of max function:

$$
\begin{aligned}
H_3(A,B) &= \frac{1}{n}\sum_{i=1}^{n} \max\left\{\left|\mu_A(x_i)-\mu_B(x_i)\right|, \left|\nu_A(x_i)-\nu_B(x_i)\right|, \right. \\
&\quad \left. \left|\pi_A(x_i)-\pi_B(x_i)\right|\right\} \qquad (36)
\end{aligned}
$$

Unfortunately, it can be easily verified that it is impossible to give the counterpart pairs of the formulas as (6)–(36) for $r > 1$ in the Minkowski r–metrics ($r = 1$ is the Hamming distances, $r = 2$ is the Euclidean distances, etc.)

For the details on other distances between A-IFSs we refer the interested reader to Szmidt and Kacprzyk [24] and especially [25]. More details are given in [5] and [30]. The counterpart results, but in respect to mass assignment theory, are given by Szmidt and Baldwin [21], [22].

4 Conclusions

A correct method of the calculating distances between A-IFSs based on the Hausdorff metric (being a counterpart of the Hamming distance) was proposed. The method employs all three functions describing A-IFSs. The proposed method is both mathematically valid and intuitively appealing (cf. [25]).

References

[1] Aichholzer O., Alt H. and Rote G. (1997) Matching shapes with a reference point. Int. J. of Computational Geometry and Applications 7, 349–363.

[2] Atallah M.J. (1983), A linear time algorithm for the Hausdorff distance between convex polygons. Information Processing Letters, Vol. 17, pp. 207-209.

[3] Atanassov K. (1983), Intuitionistic Fuzzy Sets. VII ITKR Session. Sofia (Deposed in Centr. Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[4] Atanassov K. (1999), Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Verlag.

[5] Atanassov K., Tasseva V, Szmidt E. and Kacprzyk J. (2005) On the geometrical interpretations of the intuitionistic fuzzy sets. In: Issues in the Representation and Processing of Uncertain and Imprecise Information. Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets, and Related Topics. (Eds. Atanassov K., Kacprzyk J., Krawczak M., Szmidt E.), EXIT, Warsaw 2005.

[6] Bustince H., Mohedano V., Barrenechea E., and Pagola M. (2006) An algorithm for calculating the threshold of an image representing uncertainty through A-IFSs. IPMU'2006, 2383–2390.

[7] Bustince H., Mohedano V., Barrenechea E., and Pagola M. (2005) Image thresholding using intuitionistic fuzzy sets. In: Issues in the Representation and Processing of Uncertain and Imprecise Information. Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets, and Related Topics. (Eds. Atanassov K., Kacprzyk J., Krawczak M., Szmidt E.), EXIT, Warsaw 2005.

[8] Deng-Feng Li (2005) Multiattribute decision making models and methods using intuitionistic fuzzy sets. Journal of Computer and System Sciences, 70, 73–85.

[9] Grünmaum B. (1967) Convex Polytopes, Wiley-Interscience, New York.

[10] Huttenlocher D., Klanderman G., and Rucklidge W. (1993) Comparing images using the Hausdorff distance. IEEE Trans. on Pattern Analysis and Machine Intelligence 15 (9), 850–863.

[11] Huttenlocher D. and Rucklidge W. (1993) A multi–resolution technique for computing images using the Hausdorff distance. Proc. Computer Vision and Pattern Recognition, New York, 705–708.

[12] Narukawa Y. and Torra V. (in press) Non-monotonic fuzzy measure and intuitionistic fuzzy set. Accepted for MDAI'06.

[13] Olson C. and Huttenlocher D. (1997) Automatic target recognition by matching oriented edge pixels. IEEE Trans. on Image Processing 6 (1), 103–113.

[14] Peitgen H.O., Jürgens H. and Saupe D. (1992) Introduction to Fractals and Chaos. Springer-Verlag New York.

[15] Preparata F.P. and Shamos M.I. (1985) Computational Geometry. An Introduction. Springer-Verlag New York.

[16] W. Rucklidge (1995) Lower bounds for the complexity of Hausdorff distance. Tech. report TR 94-1441, Dept. of computer science, Cornell University, NY. A similar title appeared in Proc. 5th Canad. Conf. on Comp. Geom. (CCCG'93, Waterloo, CA), 145–150.

[17] W. Rucklidge (1995) Efficient computation of the minimum Hausdorff distance for visual recognition. Ph.D. thesis, Dept. of computer science, Cornell University, NY.

[18] W. J. Rucklidge (1995) Locating objects using the Hausdorff distance. Proc. of 5th Int. Conf. on Computer Vision (ICCV'95, Cambridge, MA), 457–464.

[19] W. J. Rucklidge (1996) Efficient visual recognition using the Hausdorff distance. Lecture Notes in Computer Science, 1173, Springer-Verlag, NY.

[20] W. J. Rucklidge (1997). Efficiently locating objects using the Hausdorff distance. Int. J. of Computer Vision, 24(3), 251–270.

[21] Szmidt E. and Baldwin J. (2003) New similarity measure for intuitionistic fuzzy set theory and mass assignment theory. Notes on IFSs, 9 (3), 60–76.

[22] Szmidt E. and Baldwin J. (2004) Entropy for intuitionistic fuzzy set theory and mass assignment theory. Notes on IFSs, 10 (3), 15–28.

[23] Szmidt E. and Baldwin J. (2006) Intuitionistic Fuzzy Set Functions, Mass Assignment Theory, Possibility Theory and Histograms. 2006 IEEE World Congress on Computational Intelligence, 237–243.

[24] Szmidt E. and Kacprzyk J. (2000) Distances between intuitionistic fuzzy sets. Fuzzy Sets and Systems, Vol. 114, No. 3, 505–518.

[25] Szmidt E. and Kacprzyk J. (2006) Distances between intuitionistic fuzzy sets: straightforward approaches may not work. 3rd International IEEE Conference Intelligent Systems IS06, London, 716–721.

[26] Szmidt E. and Kukier M. (2006) Classification of imbalanced and overlaping classes using intuitionistic fuzzy sets. 3rd International IEEE Conference Intelligent Systems IS06, London, 722–727.

[27] Szmidt E., Kukier M. (2008) A new approach to cassification of imbalanced classes via Atanassov's intuitionistic fuzzy sets. In: Hsiao-Fan Wang (Ed.): Intelligent Data Analysis: Developing New Methodologies Through Pattern Discovery and Recovery. Idea Group, 65–102.

[28] Szmidt E., Kukier M. (2008) Atanassov's intuitionistic fuzzy sets in classification of imbalanced and overlapping classes. In: Panagiotis Chountas, Ilias Petrounias, Janusz Kacprzyk (Eds.): Intelligent Techniques and Tools for Novel System Architectures. Springer, Berlin Heidelberg 2008, 455–471. Seria: Studies in Computational Intelligence.

[29] Tan Ch. and Zhang Q. (2005) Fuzzy multiple attribute TOPSIS decision making method based on intuitionistic fuzzy set theory. Proc. IFSA 2005, 1602–1605.

[30] Tasseva V., Szmidt E. and Kacprzyk J. (2005) On one of the geometrical interpretations of the intuitionistic fuzzy sets. Notes on IFS, Vol. 11, No. 3, 21–27.

[31] Veltkamp R. (2001) Shape Matching: similarity measures and algorithms. Proc. Shape Modelling International, Italy, IEEE Press, 187–197.

[32] Zadeh L.A. (1965) Fuzzy sets. Information and Control, 8, 338–353.

[33] Zadeh L.A. (1975) The concept of a linguistic variable and its application to approximate reasoning. INform. Sciences, 8, 199–249.