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New operations defined over the intuitionistic fuzzy sets

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Abstract: New types of operations are defined over the intuitionistic fuzzy sets.

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Some operations (as $\cup$, $\cap$, $+$, $\cdot$) are defined over the Intuitionistic Fuzzy Sets (IFSs) in [1]. Here we shall introduce four new ones, and we shall show some of their basic properties.

Let a set E be fixed. An IFS A in E is an object having the form

$$A^* = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ to the set A, which is a subset of E, respectively, and for every $x \in E$:

$$0 \leqslant \mu_A(x) + \gamma_A(x) \leqslant 1.$$

For every two IFSs A and B are valid (see [1–5]) the following definitions (let $\alpha, \beta \in [0, 1]$):

$$A \subset B \quad \text{iff } (\forall x \in E)\ (\mu_A(x) \leqslant \mu_B(x)\ \&\ \gamma_A(x) \geqslant \gamma_B(x)),$$

$$A \supset B \quad \text{iff } B \subset A,$$

$$A = B \quad \text{iff } (\forall x \in E)\ (\mu_A(x) = \mu_B(x)\ \&\ \gamma_A(x) = \gamma_B(x)),$$

$$\bar{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle \mid x \in E\},$$

$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))\rangle \mid x \in E\},$$

$$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))\rangle \mid x \in E\},$$

$$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \gamma_A(x) \cdot \gamma_B(x)\rangle \mid x \in E\},$$

$$A \cdot B = \{\langle x, \mu_A(x) \cdot \mu_B(x), \gamma_A(x) + \gamma_B(x) - \gamma_A(x) \cdot \gamma_B(x)\rangle \mid x \in E\},$$

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\},$$

$$\Diamond A = \{\langle x, 1 - \gamma_A(x), \gamma_A(x)\rangle \mid x \in E\},$$

$$C(A) = \{\langle x, K, L\rangle \mid x \in E\} \quad \text{where } K = \max_{x \in E} \mu_A(x), \quad L = \min_{x \in E} \gamma_A(x),$$

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$$I(A)=\{\langle x, k, l\rangle \mid x\in E\} \quad \text{where } k=\min_{x\in E}\mu_A(x), \quad l=\max_{x\in E}\gamma_A(x),$$

$$D_\alpha(A)=\{\langle x, \mu_A(x)+\alpha\cdot\pi_A(x), \gamma_A(x)+(1-\alpha)\cdot\pi_A(x)\rangle \mid x\in E\},$$

$$F_{\alpha,\beta}(A)=\{\langle x, \mu_A(x)+\alpha\cdot\pi_A(x), \gamma_A(x)+\beta\cdot\pi_A(x)\rangle \mid x\in E\} \quad \text{where } \alpha+\beta\leqslant 1,$$

$$G_{\alpha,\beta}(A)=\{\langle x, \alpha\cdot\mu_A(x), \beta\cdot\gamma_A(x)\rangle \mid x\in E\},$$

$$H_{\alpha,\beta}(A)=\{\langle x, \alpha\cdot\mu_A(x), \gamma_a(x)+\beta\cdot\pi_A(x)\rangle \mid x\in E\},$$

$$H^*_{\alpha,\beta}(A)=\{\langle x, \alpha\cdot\mu_A(x), \gamma_A(x)+\beta\cdot(1-\alpha\cdot\mu_A(x)-\gamma_A(x))\rangle \mid x\in E\},$$

$$J_{\alpha,\beta}(A)=\{\langle x, \mu_A(x)+\alpha\cdot\pi_A(x), \beta\cdot\gamma_A(x)\rangle \mid x\in E\},$$

$$J^*_{\alpha,\beta}(A)=\{\langle x, \mu_A(s)+\alpha\cdot(1-\mu_A(x)-\beta\cdot\gamma_A(x)), \beta\cdot\gamma_A(x)\rangle \mid x\in E\},$$

$$\begin{aligned}X_{a,b,c,d,e,f}(A)=\{\langle x, {} & a\cdot\mu_A(x)+b\cdot(1-\mu_A(x)-c\cdot\gamma_A(x)),\\ & d\cdot\gamma_A(x)+e\cdot(1-f\cdot\mu_A(x)-\gamma_A(x))\rangle \mid x\in E\},\end{aligned}$$

where $a, b, c, d, e, f\in[0,1]$ and $a+e-e\cdot f\leqslant 1$, $b+d-b\cdot c\leqslant 1$.

Here we shall define the following new operations (cf. [6–10]):

$$A @ B=\{x, \tfrac{1}{2}(\mu_A(x)+\mu_B(x)), \tfrac{1}{2}(\gamma_A(x)+\gamma_B(x)) \mid x\in E\},$$

$$A \$ B=\{\langle x, \sqrt{\mu_A(x)\cdot\mu_B(x)}, \sqrt{\gamma_A(x)\cdot\gamma_B(x)}\rangle \mid x\in E\},$$

$$A \# B=\{\langle x, 2\cdot\mu_A(x)\cdot\mu_B(x)/(\mu_A(x)+\mu_B(x)), 2\cdot\gamma_A(x)\cdot\gamma_B(x)/(\gamma_A(x)+\gamma_B(x))\rangle \mid x\in E\}$$

for which we shall accept that if $\mu_A(x)=\mu_B(x)=0$, then $\mu_A(x)\cdot\mu_B(x)/(\mu_A(x)+\mu_B(x))=0$ and if $\gamma_A(x)=\gamma_B(x)=0$, then $\gamma_A(x)\cdot\gamma_B(x)/(\gamma_A(x)+\gamma_B(x))=0$;

$$A * B=\left\{\left\langle x, \frac{\mu_A(x)+\mu_B(x)}{2\cdot(\mu_A(x)\cdot\mu_B(x)+1)}, \frac{\gamma_A(x)+\gamma_B(x)}{2\cdot(\gamma_A(x)\cdot\gamma_B(x)+1)}\right\rangle \,\middle|\, x\in E\right\}.$$

Obviously, for every two IFSs A and B, $A @ B$, $A \$ B$, $A \# B$ and $A * B$ are an IFS.

Theorem 1. *For every three IFSs A, B and C:*
(a) $A @ B = B @ A$,
(b) $A \$ B = B \$ A$,
(c) $A \# B = B \# A$,
(d) $A * B = B * A$,
(e) $\overline{\bar{A} @ \bar{B}} = A @ B$,
(f) $\overline{\bar{A} \$ \bar{B}} = A \$ B$,
(g) $\overline{\bar{A} \# \bar{B}} = A \# B$,
(h) $\overline{\bar{A} * \bar{B}} = A * B$,
(i) $(A\cap B) @ C = (A @ C)\cap(B @ C)$,
(j) $(A\cup B) @ C = (A @ C)\cup(B @ C)$,
(k) $(A\cap B) \# C = (A \# C)\cap(B \# C)$,
(l) $(A\cup B) \# C = (A \# C)\cup(B \# C)$,
(m) $(A+B) @ C \subset (A @ C)+(B @ C)$,
(n) $(A\cdot B) @ C \supset (A @ C)\cdot(B @ C)$,
(o) $(A @ B)+C = (A+C) @ (B+C)$,
(p) $(A @ B)\cdot C = (A\cdot C) @ (B\cdot C)$.

Proof. (m) Initially we shall prove, that for every three numbers $a, b, c \in [0, 1]$ the following inequality is valid:

$$c \cdot (2 - a - b - c) + a \cdot b \geqslant 0. \tag{*}$$

When $c^2 \leqslant a \cdot b$, then

$$\begin{aligned} c \cdot (2 - a - b - c) + a \cdot b &= c \cdot (2 - a - b) - c^2 + a \cdot b \\ &\geqslant a \cdot b - c^2 \geqslant 0. \end{aligned}$$

When $c^2 > a \cdot b$, then

$$\begin{aligned} & c \cdot (2 - a - b - c) + a \cdot b \\ & \geqslant \sqrt{a \cdot b} \cdot (1 - a - b + \sqrt{a \cdot b}) \\ & \geqslant \begin{cases} \sqrt{a \cdot b} \cdot ((1 - a) + \sqrt{b} \cdot (\sqrt{a} - \sqrt{b})) \geqslant 0 & \text{if } a \geqslant b, \\ \sqrt{a \cdot b} \cdot ((1 - b) + \sqrt{a} \cdot (\sqrt{b} - \sqrt{a})) \geqslant 0 & \text{if } a < b. \end{cases} \end{aligned}$$

Let A, B and C be three given IFSs. Then

$$(A + B) @ C = \{\langle x, \tfrac{1}{2}(\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) + \mu_C(x)), \tfrac{1}{2}(\gamma_A(x) \cdot \gamma_B(x) + \gamma_C(x))\rangle \mid x \in E\}$$

$$\begin{aligned} & (A @ C) + (B @ C) = \\ & \quad \{\langle x, \tfrac{1}{2}(\mu_A(x) + \mu_C(x)) + \tfrac{1}{2}(\mu_B(x) + \mu_C(x)) - \tfrac{1}{4}(\mu_A(x) + \mu_C(x)) \cdot (\mu_B(x) + \mu_C(x)), \\ & \quad \tfrac{1}{4}(\gamma_A(x) + \gamma_C(x)) \cdot (\gamma_B(x) + \gamma_C(x))\rangle \mid x \in E\}. \end{aligned}$$

Using (*), from

$$\begin{aligned} & \tfrac{1}{2}(\mu_A(x) + \mu_C(x)) + \tfrac{1}{2}(\mu_B(x) + \mu_C(x)) - \tfrac{1}{4}(\mu_A(x) + \mu_C(x)) \cdot (\mu_B(x) + \mu_C(x)) \\ & \quad - \tfrac{1}{2}(\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) + \mu_C(x)) \\ & \quad = \tfrac{1}{4}(2 \cdot \mu_C(x) + \mu_A(x) \cdot \mu_B(x) - \mu_A(x) \cdot \mu_C(x) - \mu_B(x) \cdot \mu_C(x) - \mu_C(x)^2) \geqslant 0 \end{aligned}$$

and from

$$\begin{aligned} & \tfrac{1}{2}(\gamma_A(x) \cdot \gamma_B(x) + \gamma_C(x)) - \tfrac{1}{4}(\gamma_A(x) + \gamma_C(x)) \cdot (\gamma_B(x) + \gamma_C(x)) \\ & \quad = \tfrac{1}{4}(2 \cdot \gamma_C(x) + \gamma_A(x) \cdot \gamma_B(x) - \gamma_A(x) \cdot \gamma_C(x) - \gamma_B(x) \cdot \gamma_C(x) - \gamma_C(x)^2) \geqslant 0. \end{aligned}$$

we see that (m) is valid.

The other assertions are proved in a similar way. The other relations between the operations @, \$, #, * and the other above operations are not valid. By the same means the following theorem is proved.

Theorem 2. *For every two IFSs A and B and for every two numbers $\alpha, \beta \in [0, 1]$:*

(a) $\square(A @ B) = \square A @ \square B$,
(b) $\square(A \$ B) \supset \square A \$ \square B$,
(c) $\square(A \# B) \supset \square A \# \square B$,
(d) $\diamond(A @ B) = \diamond A @ \diamond B$,
(e) $\diamond(A \$ B) \subset \diamond A \$ \diamond B$,
(f) $\diamond(A \# B) \subset \diamond A \# \diamond B$,
(g) $D_\alpha(A @ B) = D_\alpha(A) @ D_\alpha(B)$,
(h) $F_{\alpha,\beta}(A @ B) = F_{\alpha,\beta}(A) @ F_{\alpha,\beta}(B)$, *for* $\alpha + \beta \leqslant 1$,
(i) $G_{\alpha,\beta}(A @ B) = G_{\alpha,\beta}(A) @ G_{\alpha,\beta}(B)$,
(j) $G_{\alpha,\beta}(A \$ B) = G_{\alpha,\beta}(A) \$ G_{\alpha,\beta}(B)$,
(k) $G_{\alpha,\beta}(A \# B) = G_{\alpha,\beta}(A) \# G_{\alpha,\beta}(B)$,
(l) $H_{\alpha,\beta}(A @ B) = H_{\alpha,\beta}(A) @ H_{\alpha,\beta}(B)$,

(m) $H^*_{\alpha,\beta}(A @ B) = H^*_{\alpha,\beta}(A) @ H^*_{\alpha,\beta}(B)$,
(n) $J_{\alpha,\beta}(A @ B) = J_{\alpha,\beta}(A) @ J_{\alpha,\beta}(B)$,
(p) $J^*_{\alpha,\beta}(A @ B) = J^*_{\alpha,\beta}(A) @ J^*_{\alpha,\beta}(B)$,
(q) $X_{a,b,c,d,e,f}(A @ B) = X_{a,b,c,d,e,f}(A) @ X_{a,b,c,d,e,f}(B)$,
(r) $C(A @ B) \subset C(A) @ C(B)$,
(s) $C(A \$ B) \subset C(A) \$ C(B)$,
(t) $I(A @ B) \supset I(A) @ I(B)$,
(u) $I(A \$ B) \supset I(A) \$ I(B)$.

It can be easily seen that the equalities

$$(A @ B) @ C = A @ (B @ C),$$

$$(A \$ B) \$ C = A \$ (B \$ C),$$

$$(A \# B) \# C = A \# (B \# C),$$

$$(A * B) * C = A * (B * C)$$

are not valid. Thus we define

$$A_1 @ A_2 = \underset{i=1}{\overset{2}{@}} A_i$$

and

$$\underset{i=1}{\overset{n}{@}} A_i = \left\{\left\langle x, \left(\sum_{i=1}^{n} \mu_{A_i}(x)\right) \Big/ n, \left(\sum_{i=1}^{n} \gamma_{A_i}(x)\right)\Big/ n\right\rangle \,\Big|\, x \in E\right\}.$$

Now, we have the following theorem.

Theorem 3. *For every $n+1$ IFSs $A_1, A_2, \ldots, A_n$ and B:*

(a) $\overline{\underset{i=1}{\overset{n}{@}} \bar{A}_i} = \underset{i=1}{\overset{n}{@}} A_i$,

(b) $\underset{i=1}{\overset{n}{@}} A_i + B = \underset{i=1}{\overset{n}{@}} (A_i + B)$,

(c) $\underset{i=1}{\overset{n}{@}} A_i \cdot B = \underset{i=1}{\overset{n}{@}} (a_i \cdot B)$.

Theorem 4. *For every n IFSs $A_1, A_2, \ldots, A_n$ and for every two numbers $\alpha, \beta \in [0, 1]$:*

(a) $\Box\left(\underset{i=1}{\overset{n}{@}} A_i\right) = \underset{i=1}{\overset{n}{@}} \Box A_i$,

(b) $\Diamond\left(\underset{i=1}{\overset{n}{@}} A_i\right) = \underset{i=1}{\overset{n}{@}} \Diamond A_i$,

(c) $D_\alpha\left(\underset{i=1}{\overset{n}{@}} A_i\right) = \underset{i=1}{\overset{n}{@}} D_\alpha(A_i)$,

(d) $F_{\alpha,\beta}\left(\underset{i=1}{\overset{n}{@}} A_i\right) = \underset{i=1}{\overset{n}{@}} F_{\alpha,\beta}(A_i)$, *for* $\alpha + \beta \leqslant 1$,

(e) $G_{\alpha,\beta}\left(\underset{i=1}{\overset{n}{@}} A_i\right) = \underset{i=1}{\overset{n}{@}} G_{\alpha,\beta}(A_i)$,

(f) $H_{\alpha,\beta}\left(\overset{n}{\underset{i=1}{@}} A_i\right) = \overset{n}{\underset{i=1}{@}} H_{\alpha,\beta}(A_i)$,

(g) $H^*_{\alpha,\beta}\left(\overset{n}{\underset{i=1}{@}} A_i\right) = \overset{n}{\underset{i=1}{@}} H^*_{\alpha,\beta}(A_i)$,

(h) $J_{\alpha,\beta}\left(\overset{n}{\underset{i=1}{@}} A_i\right) = \overset{n}{\underset{i=1}{@}} J_{\alpha,\beta}(A_i)$,

(i) $J^*_{\alpha,\beta}\left(\overset{n}{\underset{i=1}{@}} A_i\right) = \overset{n}{\underset{i=1}{@}} J^*_{\alpha,\beta}(A_i)$,

(j) $C\left(\overset{n}{\underset{i=1}{@}} A_i\right) \subset \overset{n}{\underset{i=1}{@}} C(A_i)$,

(k) $I\left(\overset{n}{\underset{i=1}{@}} A_i\right) \supset \overset{n}{\underset{i=1}{@}} I(A_i)$.

The proofs of these assertions are the same as in the above proof.

Theorem 5. *For every two IFSs A and B*:

(a) $A \cdot B \subset \left\{ \begin{matrix} A \cap B \subset \left\{ \begin{matrix} A \# B \\ A \$ B \\ A @ B \end{matrix} \right\} \subset A \cup B \\ A * B \end{matrix} \right\} \subset A + B$,

(b) $A \$ B \subset_{\Box} A @ B$,

(c) $A @ B \subset_{\Diamond} A \$ B$,

(d) $A \# B \subset_{\Box} A \$ B \subset_{\Box} A @ B$,

(e) $A @ B \subset_{\Diamond} A \$ B \subset_{\Diamond} A \# B$,

(f) $A * B \subset_{\Box} A @ B$,

(g) $A * B \subset_{\Diamond} A \# B$.

The validity of these inclusions follows from the validity of the following inequalities for arbitrary real numbers $a, b \in [0, 1]$:

$$a \cdot b \leqslant \min(a, b) \leqslant \frac{2 \cdot a \cdot b}{a + b} \leqslant \sqrt{a \cdot b} \leqslant \frac{a + b}{2} \leqslant \max(a, b) \leqslant a + b - a \cdot b,$$

$$a \cdot b \leqslant \frac{a + b}{2 \cdot (a \cdot b + 1)} \leqslant \frac{a + b}{2}.$$

These operations can be transfered to operations on ordinary fuzzy sets as follows:

$$A @ B = \{\langle x, \tfrac{1}{2}(\mu_A(x) + \mu_B(x))\rangle \mid x \in E\},$$

$$A \$ B = \{\langle x, \sqrt{\mu_A(x) \cdot \mu_B(x)}\rangle \mid x \in E\},$$

$$A \# B = \{\langle x, 2 \cdot \mu_A(x) \cdot \mu_B(x)/(\mu_A(x) + \mu_B(x))\rangle \mid x \in E\},$$

$$A * B = \left\{\left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2 \cdot (\mu_A(x) \cdot \mu_B(x) + 1)}\right\rangle \,\middle|\, x \in E\right\}.$$

Now, Theorem 5 obtains the form:

$$A \cdot B \subset \left\{ \begin{array}{c} A \cap B \subset A * B \subset A \$ B \subset A @ B \subset A \cup B \\ A * B \end{array} \right\} \subset A + B.$$

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