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On the Łukasiewicz operations over intuitionistic fuzzy sets

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Abstract: In [8] some properties of two operations – conjunction and disjunction from Łukasiewicz type – over intuitionistic fuzzy sets were studied. In the paper, a new proof of the results is given an some analogies of the results in MV algebras are given.
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1 Introduction

Let a set E be fixed. The Intuitionistic Fuzzy Set (IFS) A in E is defined by (see, e.g., [1]):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Let us define the *empty IFS and the unit IFS* (see [1]) by:

$$O^* = \{\langle x, 0, 1\rangle | x \in E\},$$

$$E^* = \{\langle x, 1, 0\rangle | x \in E\}.$$

Different relations and operations are introduced over the IFSs. Some of them are the following

$$\begin{array}{lll}
A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \& \nu_A(x) \geq \nu_B(x)), \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)), \\
A \subseteq B & \text{iff} & A \subset B \text{ or } A = B, \\
\overline{A} & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}, \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}, \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}.
\end{array}$$

In [6], the following two operations were introduced:

$$\begin{array}{lll}
A \oplus B & = & \{\langle x, \min(1, \mu_A(x) + \mu_B(x)), \max(0, \nu_A(x) + \nu_B(x) - 1)\rangle | x \in E\}, \\
A \odot B & = & \{\langle x, \max(0, \mu_A(x) + \mu_B(x) - 1), \min(1, \nu_A(x) + \nu_B(x))\rangle | x \in E\},
\end{array}$$

Curiously, the same operations were discussed in [3] by K. Atanassov and R. Tcvetkov, because by that moment they had not known of B. Riecan's paper [6]. While in [6] these two operations are not named, but just denoted, in [3] these operations are named *conjunction and disjunction from Łukasiewicz type.*

2 New proof

Theorem 1. For any IF sets A, B, C there holds

$$(A \cap B) \oplus C = (A \oplus C) \cap (B \oplus C),$$

$$(A \cup B) \odot C = (A \odot C) \cup (B \odot C).$$

Proof. We have

$$(A \cap B) \oplus C = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B) \oplus (\mu_C,\ \nu_C) =$$

$$= (((\mu_A \wedge \mu_B) + \mu_C) \wedge 1, ((\nu_A \vee \nu_B) + \nu_C - 1) \vee 0) =$$

$$= ((\mu_A + \mu_C) \wedge (\mu_B + \mu_C) \wedge 1, (\nu_A + \nu_C - 1) \vee (\nu_B + \nu_C - 1) \vee 0) =$$

$$= (A \oplus C) \cap (B \oplus C).$$

The second identity can be proved analogically. □

Theorem 2. For any IF sets A, B, C there holds

$$(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C),$$

$$(A \cap B) \odot C = (A \odot B) \cap (B \odot C).$$

Proof. We have

$$(A \cup B) \oplus C = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B) \oplus (\mu_C, \nu_C) =$$
$$= ((\mu_A \vee \mu_B + \mu_C) \wedge 1, ((\nu_A \wedge \nu_B) + \nu_C - 1) \vee 0) =$$
$$= (((\mu_A + \mu_C) \vee (\mu_B + \mu_C)) \wedge 1, ((\nu_A + \nu_C - 1) \wedge (\nu_B + \nu_C - 1)) \vee 0) =$$
$$= (((\mu_A + \mu_C) \wedge 1) \vee (\mu_A + \mu_C) \wedge 1), ((\nu_A + \nu_C - 1) \vee 0) \wedge ((\nu_B + \nu_C - 1) \vee 0)) =$$
$$= (A \oplus C) \cup (B \oplus C.$$

The second identity can be proved analogically. $\square$

3 MV-algebras

It was shown in [6] that any family of IF-sets can be imbedded to an MV-algebra. From the category point of view it was shown in [4]. We shall show that some analogies of previous results can be formulated and proved in any MV-algebra, too.

In [5] it has been shown that any MV-algebra can be presented as an interval $M = [0, u]$ in a lattice order group $(G, +, \leq, 0)$. The group has the following properties:

1. G is a commutative group $(G, +, 0)$,
2. G is a lattice $(G, \leq)$,
3. $a \leq b \Longrightarrow a + c \leq b + c$.

We shall use the notation $\vee, \wedge$ for lattice operations. Further we define the Łukasiewicz binary operations on M:

$$a \oplus b = (a + b) \wedge u,$$
$$a \odot b = (a + b - u) \vee 0.$$

Theorem 3. Let M be a MV-algebra. Then for any $a, b, c \in M$ there hold:

(i) $(a \vee b) \oplus c = (a \oplus c) \vee (b \oplus c)$,

(ii) $(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c)$,

(iii) $(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$,

(iv) $(a \vee b) \odot c = (a \odot c) \vee (b \odot c)$.

Proof. We shall use the folowing identities:

$$(f \vee g) + h = (f + h) \vee (g + h),$$

$$(f \wedge g) + h = (f + h) \wedge (g + h),$$

$$(f \wedge g) \vee h = (f \vee h) \wedge (g \vee h),$$

$$(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h).$$

Now, let $a, b, c \in M = [0, u] \subset G$. Then

$$(a \vee b) \oplus c = **a \vee b) + c) \wedge c =$$

$$= ((a + c) \vee (b + c)) \wedge u =$$

$$= ((a + c) \wedge u) \vee ((b + c) \wedge u) =$$

$$= (a \oplus b) \vee (b \oplus c),$$

hence (i) has been proved. Similarly (ii) can be proved:

$$(a \wedge b) \odot c = ((a \wedge b) + c - u) \vee 0 =$$

$$= ((a + c - u) \wedge (b + c - u)) \vee 0 =$$

$$= ((a + c - u) \vee 0) \wedge ((b + c - u) \vee 0) =$$

$$= (a \odot c) \wedge (b \odot c).$$

The identities (iii) and (iv) cn be proved without the lattice distributive law:

$$(a \wedge b) \oplus c = ((a \wedge b) + c) \wedge U =$$

$$= (a + c) \wedge (A + c) \wedge u =$$

$$= ((a + c) \wedge u) \wedge ((b + c) \wedge u) =$$

$$= (a \oplus c) \wedge (b \oplus c).$$

Similarly

$$(a \vee b) \odot c - ((sa \vee b) + c - u) \vee 0 =$$

$$= (a + c - u) \vee (b + c - u) \vee 0 =$$

$$= ((a + c - u) \vee 0) \vee ((b + c - u) \vee 0) =$$

$$= (a \odot c) \vee (b \odot c).$$

This completes the proof. □

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