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Properties of intuitionistic fuzzy implication $\rightarrow_{171}$

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Abstract: Some basic properties of the intuitionistic fuzzy implication $\rightarrow_{171}$ are formulated and checked.
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1 Introduction

In [4] a new operation implication was introduced and some of its properties were studied. It obtained number 171. So it was denoted as $\rightarrow_{171}$.

In intuitionistic fuzzy propositional calculus (see [1–3, 8, 9]), if x is a variable then its truth-value is represented by the ordered couple $V(x) = \langle a, b\rangle$, so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x. In [5], we called this couple an *intuitionistic fuzzy pair" (IFP).*

Below we assume that for the two variables x and y the equalities: $V(x) = \langle a, b\rangle$ and $V(y) = \langle c, d\rangle$ $(a, b, c, d, a + b, c + d \in [0, 1])$ hold.

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1]) by:

$$x \text{ is an IFT if and only if for } V(x) = \langle a, b\rangle \text{ holds: } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$. As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0\rangle$.

2 Main results

The intuitionistic fuzzy implication $\to_{171}$ has the form (see [4])

$$V(x \to y) = \langle a, b\rangle \to \langle c, d\rangle = \langle \overline{\text{sg}}(\max(a, d) - \max(b, c)), \text{sg}(\max(a, d) - \max(b, c))\rangle,$$

where we use functions sg and $\overline{\text{sg}}$ defined by,

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases},$$

$$\overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}.$$

In [4] we proved that the new implication is defined correctly.

As we showed in [4], the new intuitionistic fuzzy implication generates the intuitionistic fuzzy negation $\neg_6$ (see [3]): $\neg\langle a, b\rangle = \langle \overline{\text{sg}}(1 - b), \text{sg}(1 - b)\rangle$.

For this negation in [4], we see directly that it satisfies the first and the third of the following three properties and does not satisfies the second one:
Property P1: $A \to \neg\neg A$ is a tautology (an IFT),
Property P2: $\neg\neg A \to A$ is a tautology (an IFT),
Property P3: $\neg\neg\neg A = \neg A$.

First, we can check directly that $\langle 0, 1\rangle \to_{171} \langle 0, 1\rangle = \langle 1, 0\rangle$, $\langle 0, 1\rangle \to_{171} \langle 1, 0\rangle = \langle 1, 0\rangle$, $\langle 1, 0\rangle \to_{171} \langle 0, 1\rangle = \langle 0, 1\rangle$, $\langle 1, 0\rangle \to_{171} \langle 1, 0\rangle = \langle 1, 0\rangle$,

i.e., the new implication has the beheviour of the standard classical logic implication.

Second, we give the 17 axioms of the intuitionistic logic (see, e.g. [7]) If A, B and C are arbitrary propositional forms, then:

(IL1) $A \to A$,

(IL2) $A \to (B \to A)$,

(IL3) $A \to (B \to (A \& B))$,

(IL4) $(A \to (B \to C)) \to (B \to (A \to C))$,

(IL5) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(IL6) $A \to \neg\neg A$,

(IL7) $\neg(A \& \neg A)$,

(IL8) $(\neg A \vee B) \to (A \to B)$,

(IL9) $\neg(A \vee B) \to (\neg A \& \neg B)$,

(IL10) $(\neg A \& \neg B) \to \neg(A \vee B)$,

(IL11) $(\neg A \vee \neg B) \to \neg(A \& B)$,

(IL12) $(A \to B) \to (\neg B \to \neg A)$,

(IL13) $(A \to \neg B) \to (B \to \neg A)$,

(IL14) $\neg\neg\neg A \to \neg A$,

(IL15) $\neg A \to \neg\neg\neg A$,

(IL16) $\neg\neg(A \to B) \to (A \to \neg\neg B)$,

(IL17) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 1. The intuitionistic fuzzy implication $\to_{171}$ satisfies axioms (IL1), (IL6), (IL7), (IL9), ..., (IL16) as tautologies.

Proof: Let us check the validity of (IL7). Let $V(A) = \langle a, b\rangle$ for $a, b \in [0, 1]$, so that $a + b \leq 1$.

Then

$$\begin{aligned} V(\neg(A \& \neg A)) &= \neg(\langle a, b\rangle \& \neg\langle a, b\rangle) \\ &= \neg(\langle a, b\rangle \& \langle \overline{\text{sg}}(1-b), \text{sg}(1-b)\rangle) \\ &= \neg\langle \min(a, \overline{\text{sg}}(1-b)), \max(b, \text{sg}(1-b))\rangle \\ &= \langle \overline{\text{sg}}(1 - \max(b, \text{sg}(1-b))), \text{sg}(1 - \max(b, \text{sg}(1-b)))\rangle. \end{aligned}$$

Let $X \equiv \overline{\text{sg}}(1 - \max(b, \text{sg}(1-b)))$.

If $b = 1$, then

$$X = \overline{\text{sg}}(1 - \max(1, 0)) = \overline{\text{sg}}(1-1) = \overline{\text{sg}}(0) = 1.$$

If $b < 1$, then $\text{sg}(1-b) = 1$ and

$$X = \overline{\text{sg}}(1 - \max(b, 1)) = \overline{\text{sg}}(1-1) = \overline{\text{sg}}(0) = 1.$$

Therefore, $X = 1$. Analogously, we obtain that $\text{sg}(1 - \max(b, \text{sg}(1-b))) = 0$.

Hence, (IL7) is a tautology. All other assertions are proved analogously. □

Corollary 1. The intuitionistic fuzzy implication $\to_{171}$ satisfies axioms (IL1), (IL6), (IL7), (IL9), ..., (IL16) as IFTs.

Third, we give the list of Kolmogorov's axioms of logic (see, e.g., [6]).

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 2. The intuitionistic fuzzy implication $\to_{171}$ satisfies only axioms (K2) and (K5) as tautologies.

Corollary 2. The intuitionistic fuzzy implication $\to_{171}$ satisfies only axioms (K2) and (K5) as IFTs.

Fourth, we give the list of Łukasiewicz–Tarski's axioms of logic (see, e.g., [6])

(LT1) $A \to (B \to A)$,

(LT2) $(A \to B) \to ((B \to C) \to (A \to C))$,

(LT3) $\neg A \to (\neg B \to (B \to A))$,

(LT4) $((A \to \neg A) \to A) \to A$.

Theorem 3. The intuitionistic fuzzy implication $\to_{171}$ satisfies only axiom (L3) as a tautology.
Corollary 3. The intuitionistic fuzzy implication $\to_{171}$ satisfies only axiom (L3) as an IFT.

3 Conclusion

In next research other new implications will be introduced and studied. All they show that intuitonistic fuzzy sets and logics in the sense, described in [2, 3] correspond to the ideas of Brouwer's intuitionism. It is note worthy that the search of new implications and negations is important for constructing of rules for multicriteria and intercriteria analyses and theie evaluations.

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