# Two examples for the use of 3-dimensional intuitionistic fuzzy index matrices 

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#### Abstract

On the basis of the definition of the concepts of the 2- and 3-dimensional intuitionistic fuzzy index matrix, in the paper, two examples are given. In the first example, intuitionistic fuzzy expert evaluations are described for the case, when a group of experts evaluate themselves (as experts). In the second example, the results of the book sales in different bookshops in different towns are discussed.


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## 1 Introduction

The concept of an Index Matrix (IM) was introduced in [1] and discusssed in more details in [2, 3]. Here, following [2], the basic definition of IM are given.

Let $I$ be a fixed set of indices and $\mathcal{R}$ be the set of all real numbers. By IM with index sets $K$ and $L(K, L \subset I)$, we mean the object

$$
\left[K, L,\left\{a_{k_{i}, l_{j}}\right\}\right] \equiv \begin{array}{c|cccc} 
& l_{1} & l_{2} & \ldots & l_{n} \\
\hline k_{1} & a_{k_{1}, l_{1}} & a_{k_{1}, l_{2}} & \ldots & a_{k_{1}, l_{n}} \\
k_{2} & a_{k_{2}, l_{1}} & a_{k_{2}, l_{2}} & \ldots & a_{k_{2}, l_{n}} \\
\vdots & & & & \\
k_{m} & a_{k_{m}, l_{1}} & a_{k_{m}, l_{2}} & \ldots & a_{k_{m}, l_{n}}
\end{array}
$$

where $K=\left\{k_{1}, k_{2}, \ldots, k_{m}\right\}, L=\left\{l_{1}, l_{2}, \ldots, l_{n}\right\}$, and for $1 \leq i \leq m$, and $1 \leq j \leq n: a_{k_{i}, l_{j}} \in \mathcal{R}$. In [7], the following IM operations are defined over the IM

$$
\begin{aligned}
& \rho_{(\max )}\left(A, k_{0}\right) \\
& =\begin{array}{c|ccc} 
& l_{1} & \ldots & l_{n} \\
k_{0} & \left\langle\max _{1 \leq i \leq m} \mu_{k_{i}, l_{1}}, \min _{1 \leq i \leq m} \nu_{k_{i}, l_{1}}\right\rangle & \ldots & \left\langle\max _{1 \leq i \leq m} \mu_{k_{i}, l_{n}}, \min _{1 \leq i \leq m} \nu_{k_{i}, l_{n}}\right\rangle
\end{array}, \\
& \rho_{(\text {ave })}\left(A, k_{0}\right) \\
& =\begin{array}{c|ccc} 
& l_{1} & \ldots & l_{n} \\
\hline k_{0} & \left\langle\frac{1}{m} \sum_{i=1}^{m} \mu_{k_{i}, l_{1}}, \frac{1}{m} \sum_{i=1}^{m} \nu_{k_{i}, l_{1}}\right\rangle & \ldots & \left\langle\frac{1}{m} \sum_{i=1}^{m} \mu_{k_{i}, l_{n}}, \frac{1}{m} \sum_{i=1}^{m} \nu_{k_{i}, l_{n}}\right\rangle
\end{array}, \\
& \rho_{(\min )}\left(A, k_{0}\right) \\
& =\begin{array}{c|ccc} 
& l_{1} & \cdots & l_{n} \\
k_{0} & \left\langle\min _{1 \leq i \leq m} \mu_{k_{i}, l_{1}}, \max _{1 \leq i \leq m} \nu_{k_{i}, l_{1}}\right\rangle & \ldots & \left\langle\min _{1 \leq i \leq m} \mu_{k_{i}, l_{n}}, \max _{1 \leq i \leq m} \nu_{k_{i}, l_{n}}\right\rangle
\end{array}, \\
& \begin{array}{c|c} 
& l_{0} \\
\hline k_{1} & \left\langle\max _{1 \leq j \leq n} \mu_{k_{1}, l_{j}}, \min _{1 \leq j \leq n} \nu_{k_{1}, l_{j}}\right\rangle
\end{array} \\
& \left.\sigma_{(\max )}\left(A, l_{0}\right)=\begin{array}{c}
\vdots \\
k_{i}
\end{array} \underset{\left\langle\max _{1 \leq j \leq n}\right.}{ } \mu_{k_{i}, l_{j}}, \min _{1 \leq j \leq n} \nu_{k_{i}, l_{j}}\right\rangle, \\
& \begin{array}{c|c}
\vdots & \vdots \\
k_{m} & \left\langle\max _{1 \leq j \leq n} \mu_{k_{m}, l_{j}}, \min _{1 \leq j \leq n} \nu_{k_{m}, l_{j}}\right\rangle
\end{array} \\
& \begin{array}{c|c} 
& l_{0} \\
\hline k_{1} & \left\langle\frac{1}{n} \sum_{j=1}^{n} \mu_{k_{1}, l_{j}}, \frac{1}{n} \sum_{j=1}^{n} \nu_{k_{1}, l_{j}}\right\rangle
\end{array} \\
& \sigma_{\mathrm{ave}}\left(A, l_{0}\right)=\begin{array}{c}
\vdots \\
k_{i}
\end{array} \quad\left\langle\frac{1}{n} \sum_{j=1}^{n} \mu_{k_{i}, l_{j}}, \frac{1}{n} \sum_{j=1}^{n} \nu_{k_{i}, l_{j}}\right\rangle, \\
& k_{m}\left\langle\left\langle\frac{1}{n} \sum_{j=1}^{n} \mu_{k_{m}, l_{j}}, \frac{1}{n} \sum_{j=1}^{n} \nu_{k_{m}, l_{j}}\right\rangle\right.
\end{aligned}
$$

$$
\sigma_{(\min )}\left(A, l_{0}\right)=\begin{array}{c|c} 
& l_{0} \\
\hline k_{1} & \left\langle\min _{1 \leq j \leq n} \mu_{k_{1}, l_{j}}, \max _{1 \leq j \leq n} \nu_{k_{1}, l_{j}}\right\rangle \\
\vdots & \vdots \\
k_{i} & \left\langle\min _{1 \leq j \leq n} \mu_{k_{i}, l_{j}}, \max _{1 \leq j \leq n} \nu_{k_{i}, l_{j}}\right\rangle \\
\vdots & \vdots \\
k_{m} & \left\langle\min _{1 \leq j \leq n} \mu_{k_{m}, l_{j}}, \max _{1 \leq j \leq n} \nu_{k_{m}, l_{j}}\right\rangle
\end{array},
$$

## 2 First example

Let us have a set of experts $E=\left\{E_{1}, \ldots, E_{e}\right\}$ and a discrete time set $T=\left\{T_{1}, \ldots, T_{t}\right\}$ during which the experts participate in investigations. Let it not be obligatory for each expert to participate in each investigation. Let us assume that before each investigation, the experts who will participate in it, evaluate all participants (we discuss both cases: including or excluding themselves). Therefore, before the $j$-th investigation we obtain the list $E_{j}=\left\{E_{j, 1}, \ldots, E_{j, s_{j}}\right\} \subset E$ of the experts, who will participate in the investigation and we can construct the IM

$$
S_{j}=\left[E_{j}, E_{j},\left\{a_{i_{1}, i_{2}, T_{j}}\right\}\right]=\begin{array}{c|ccc}
T_{j} & E_{j, 1} & \ldots & E_{j, s_{j}} \\
\hline E_{j, 1} & a_{1,1, T_{j}} & \ldots & a_{1, s_{j}, T_{j}} \\
\vdots & \vdots & \ldots & \vdots \\
E_{j, s_{j}} & a_{s_{j}, 1, T_{j}} & \ldots & a_{s_{j}, s_{j}, T_{j}}
\end{array} .
$$

In the case, when the experts do not evaluate themselves, elements $a_{1,1, T_{J}}=\ldots=a_{s_{j}, s_{j}, T_{j}}=\perp$, where symbol $\perp$ shows lack of estimation. In the opposite case, these estimations are from the same scale, as the rest evaluations. This scale can be, e.g., with natural numbers in interval $0,1,2, \ldots, 100$, real numbers in interval $[0,1]$ (for these two cases the numbers determine the degrees of preference), Intuitionistic Fuzzy Pairs (IFPs, see [8]), i.e., pairs of real numbers $m$ and $n$, such that each one of these numbers is in interval $[0,1]$ and they satisfy inequality $m+n \leq 1$ (for the third case the IFPs determine the degrees of preference and of the non-preference), etc. Using different methods, discussed, e.g., in [4], we can transform the first two cases to the third one. By this reason, below we use expert estimations in the forms of IFPs.

Having in mind that $T_{j} \in T$ for $1 \leq j \leq t$, we see that we have a set of IMs with the form

$$
S=\left\{\left.\begin{array}{c|ccc}
T_{j} & E_{j, 1} & \ldots & E_{j, s_{j}} \\
\hline E_{j, 1} & a_{1,1, T_{j}} & \ldots & a_{1, s_{j}, T_{j}} \\
\vdots & \vdots & \ldots & \vdots \\
E_{j, s_{j}} & a_{s_{j}, 1, T_{j}} & \ldots & a_{s_{j}, s_{j}, T_{j}}
\end{array} \right\rvert\, T_{j} \in T\right\}
$$

Now, we can use the "inflating operator" that is defined for index sets $K \subset P$ and $L \subset Q$ by

$$
{ }^{(P, Q)} A={ }^{(P, Q)}\left[K, L,\left\{a_{k_{i}, l_{j}}\right\}\right]=\left[P, Q,\left\{b_{p_{r}, q_{s}}\right\}\right],
$$

where

$$
b_{p_{r}, q_{s}}= \begin{cases}a_{k_{i}, l_{j}}, & \text { if } p_{r}=k_{i} \in K \text { and } q_{s}=l_{j} \in L \\ \perp, & \text { otherwise }\end{cases}
$$

Each element of set $S$ can be modified by the inflating operator to the form

$$
{ }^{(E, E)} S_{j}=^{(E, E)}\left[E_{j}, E_{j},\left\{a_{i_{1}, i_{2}, T_{j}}\right\}\right]=\begin{array}{c|ccc}
T_{j} & E_{1} & \ldots & E_{e} \\
\hline E_{1} & b_{1,1, T_{j}} & \ldots & b_{1, e, T_{j}} \\
\vdots & \vdots & \ldots & \vdots \\
E_{e} & b_{e, 1, T_{j}} & \ldots & b_{e, e, T_{j}}
\end{array},
$$

where

$$
b_{p_{r}, q_{s}, T_{j}}= \begin{cases}a_{k_{i}, l_{j}, T_{j}}, & \text { if } p_{r}=k_{i} \in E \text { and } q_{s}=l_{j} \in E \\ \perp, & \text { otherwise }\end{cases}
$$

Therefore, we can construct the set

$$
S^{*}=\left\{{ }^{(E, E)} S_{j} \mid 1 \leq j \leq t\right\} .
$$

We can study the elements of set $S^{*}$ from different points of view.
Let $\mathcal{I}$ be a fixed set. By Intuitionistic Fuzzy IM (IFIM, see [3]) with index sets $K$ and $L$ $(K, L \subset \mathcal{I})$, we denote the object:

$$
\begin{aligned}
& A=\left[K, L,\left\{\left\langle\mu_{k_{i}, l_{j}}, \nu_{k_{i}, l_{j}}\right\rangle\right\}\right] \\
& \begin{array}{c|ccccc} 
& l_{1} & \ldots & l_{j} & \ldots & l_{n} \\
\hline k_{1} & \left\langle\mu_{k_{1}, l_{1}}, \nu_{k_{1}, l_{1}}\right\rangle & \ldots & \left\langle\mu_{k_{1}, l_{j}}, \nu_{k_{1}, l_{j}}\right\rangle & \ldots & \left\langle\mu_{k_{1}, l_{n}}, \nu_{k_{1}, l_{n}}\right\rangle \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
k_{i} & \left\langle\mu_{\left.k_{i}, l_{1}, \nu_{k_{i}, l_{1}}\right\rangle} \ldots\right. & \left\langle\mu_{\left.k_{i}, l_{j}, \nu_{k_{i}, l_{j}}\right\rangle} \ldots\right. & \left\langle\mu_{\left.k_{i}, l_{n}, \nu_{k_{i}, l_{n}}\right\rangle}^{\vdots}\right. & \vdots & \ldots \\
k_{m} & \left\langle\mu_{\left.k_{m}, l_{1}, \nu_{k_{m}, l_{1}}\right\rangle} \ldots\right. & \left\langle\mu_{\left.k_{m}, l_{j}, \nu_{k_{m}, l_{j}}\right\rangle} \ldots\right. & \ldots & \left\langle\mu_{\left.k_{m}, l_{n}, \nu_{k_{m}, l_{n}}\right\rangle},\right.
\end{array}
\end{aligned}
$$

where for every $1 \leq i \leq m, 1 \leq j \leq n,\left\langle\mu_{k_{i}, l_{j}}, \nu_{k_{i}, l_{j}}\right\rangle$ is an IFP, i.e.,

$$
0 \leq \mu_{k_{i}, l_{j}}, \nu_{k_{i}, l_{j}}, \mu_{k_{i}, l_{j}}+\nu_{k_{i}, l_{j}} \leq 1 .
$$

Because the elements of IMs ${ }^{(E, E)} S_{j}$ are IFPs, we see that each of these matrices is an IFIMs and for each time-moment $T_{j}$ and for $1 \leq i_{1}, i_{2} \leq e$ :

$$
b_{i_{1}, i_{2}, T_{j}}=\left\langle\mu_{k_{i_{1}}, k_{i_{2}}, T_{j}}, \nu_{k_{i_{1}}, k_{i_{2}}, T_{j}}\right\rangle .
$$

Obviously, we must define $\perp$ as the $\operatorname{IFP}\langle 0,1\rangle$.
Now, we can construct new IFIMs, using aggregation operators over IMs, described in Section 2.

In this case, it is important to note that in the new IFIMs, the operations between their elements are defined over IFPs and the elements with value $\perp$ are omitted. The new IFIMs are:

$$
\begin{aligned}
& C_{\left(\max , T_{j}\right)}=\rho_{(\max )}\left({ }^{(E, E)} S_{j}, k_{0}\right) \\
& =\begin{array}{c|ccc}
T_{j} & E_{1} & \ldots & E_{e} \\
\hline k_{0} & \left\langle\max _{1 \leq i \leq e} \mu_{i, 1, T_{j}}, \min _{1 \leq i \leq e} \nu_{i, 1, T_{j}}\right\rangle & \ldots & \left\langle\max _{1 \leq i \leq e} \mu_{i, e, T_{j}}, \min _{1 \leq i \leq e} \nu_{i, e, T_{j}}\right\rangle
\end{array},
\end{aligned}
$$

$$
\begin{aligned}
& C_{\left(\mathrm{ave}, T_{j}\right)}=\rho_{(\mathrm{ave})}\left({ }^{(E, E)} S_{j}, k_{0}\right) \\
& =\begin{array}{c|ccc}
T_{j} & E_{j, 1} & \ldots & E_{j, s_{j}} \\
\hline k_{0} & \left\langle\frac{1}{m_{j, 1}} \sum_{i=1}^{e} \mu_{i, 1, T_{j}}, \frac{1}{m_{j, 1}} \sum_{i=1}^{e} \nu_{i, 1, T_{j}}\right\rangle & \ldots & \left\langle\frac{1}{m_{j, e}} \sum_{i=1}^{e} \mu_{i, e, T_{j}}, \frac{1}{m_{j, e}} \sum_{i=1}^{e} \nu_{i, e, T_{j}}\right\rangle
\end{array},
\end{aligned}
$$

where $m_{j, i}$ is the number of non- $\perp$-elements in $i$-th column (of the $j$-th matrix),

\[

\]

The elements of these IMs give us information about the evaluations that the separate experts have received from their colleagues. When for the $k$-th expert $E_{k}$ we know the average evaluation that his/her colleagues gave for him/her, knowing the values from IM $C_{\left(\max , T_{j}\right)}$ we can determine the experts that are well-disposed to him/her. They are the experts, who gave for $E_{k}$ higher evaluations than the average evaluation. On the other hand, knowing the values from $\operatorname{IM} C_{\left(\min , T_{j}\right)}$ we can determine the experts that are ill-disposed to him/her. They are the experts, who gave for $E_{k}$ lower evaluations than the average evaluation.

In the standard case, these procedures are realized only to obtain information about the evaluations of the experts, i.e., to obtain their scores (see, e.g., [6]). For the expert scores we can use the IFPs from IFIM $C_{\left(\text {ave }, T_{j}\right)}$. The additional information can be used, too.

Similarly, if we like to see to whom the expert $E_{k}$ is well-disposed and to whom is ill-disposed, we can construct the IMs:

$$
\begin{aligned}
& \begin{array}{c|c}
T_{j} & l_{0} \\
\hline E_{1} & \left\langle\max _{1 \leq k \leq e} \mu_{1, k, T_{j}}, \min _{1 \leq k \leq e} \nu_{1, k, T_{j}}\right\rangle
\end{array} \\
& D_{\left(\max , T_{j}\right)}=\sigma_{\max }\left({ }^{(E, E)} S_{j}, l_{0}\right)=\begin{array}{c|l}
\vdots & \vdots \\
E_{i} & \left\langle\max _{1 \leq k \leq e} \mu_{i, k, T_{j}}, \min _{1 \leq k \leq e} \nu_{i, k, T_{j}}\right\rangle,
\end{array} \\
& E_{e}\left\langle\left\langle\max _{1 \leq k \leq e} \mu_{e, k, T_{j}}, \min _{1 \leq k \leq e} \nu_{e, k, T_{j}}\right\rangle\right. \\
& D_{\left(\text {ave }, T_{j}\right)}=\sigma_{\text {ave }}\left({ }^{(E, E)} S_{j}, l_{0}\right)=\begin{array}{c|c}
T_{j} & l_{0} \\
\hline E_{1} & \left\langle\frac{1}{m_{j, 1}} \sum_{k=1}^{e} \mu_{1, k, T_{j}}, \frac{1}{m_{j, i}} \sum_{k=1}^{e} \nu_{1, k, T_{j}}\right\rangle \\
\vdots & \vdots \\
E_{i} & \left\langle\frac{1}{m_{j, i}} \sum_{k=1}^{e} \mu_{i, k, T_{j}}, \frac{1}{m_{j, i}} \sum_{k=1}^{e} \nu_{i, k, T_{j}}\right\rangle, \\
\vdots & \vdots \\
E_{e} & \left\langle\frac{1}{m_{j, e}} \sum_{k=1}^{e} \mu_{e, k, T_{j}}, \frac{1}{m_{j, i}} \sum_{k=1}^{e} \nu_{e, k, T_{j}}\right\rangle
\end{array},
\end{aligned}
$$

$$
D_{\left(\min , T_{j}\right)}=\sigma_{\min }\left({ }^{(E, E)} S_{j}, l_{0}\right)=\begin{array}{c|c}
T_{j} & l_{0} \\
\hline E_{1} & \left\langle\min _{1 \leq k \leq e} \mu_{1, k, T_{j}}, \max _{1 \leq k \leq e} \nu_{1, k, T_{j}}\right\rangle \\
\vdots & \vdots \\
E_{i} & \left\langle\min _{1 \leq k \leq e} \mu_{i, k, T_{j}}, \max _{1 \leq k \leq e} \nu_{i, k, T_{j}}\right\rangle \\
\vdots & \vdots \\
E_{e} & \left\langle\min _{1 \leq k \leq e} \mu_{e, k, T_{j}}, \max _{1 \leq k \leq e} \nu_{e, k, T_{j}}\right\rangle
\end{array},
$$

As it is discussed in [5], if for $s(1 \leq s \leq n)$ :

$$
A_{s}=\left[K^{s}, L^{s},\left\{a_{k_{i}, l_{j}}^{s}\right\}\right]=\begin{array}{c|ccccc} 
& l_{s, 1} & \ldots & l_{s, j} & \ldots & l_{s, n_{s}} \\
\hline k_{s, 1} & a_{k_{s, 1}, l_{s, 1}} & \ldots & a_{k_{s, 1}, l_{s, j}} & \ldots & a_{k_{s, 1}, l_{s, n}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
k_{s, i} & a_{k_{s, i}, l_{s, 1}} & \ldots & a_{k_{s, i}, l_{s, j}} & \ldots & a_{k_{s, i}, l_{s, n s}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
& k_{s, m} & a_{k_{s, m}, l_{s, 1}} & \ldots & a_{k_{s, m}, l_{s, j}} & \ldots \\
a_{k_{s, m}, l_{s, n}}
\end{array},
$$

then the operation "composition" is defined by

$$
b\left\{A_{s} \mid 1 \leq s \leq n\right\}=\left[\bigcup_{s=1}^{n} K^{s}, \bigcup_{s=1}^{n} L^{s},\left\{\left\langle c_{1, t_{1, u}, v_{1, w}}, c_{2, t_{2, u}, v_{2, w}}, \ldots, c_{n, t_{n, u}, v_{n, w}}\right\rangle\right\}\right],
$$

where for $r(1 \leq r \leq n)$ :

$$
c_{r, t_{u}, v_{w}}= \begin{cases}a_{r, k_{i}, l_{j}}, & \text { if } t_{u}=k_{i} \in K^{r} \text { and } v_{w}=l_{j} \in L^{r} \\ \perp, & \text { otherwise }\end{cases}
$$

Now, we can apply operator $b$ over set $S^{*}$. Then, to each pair of experts $\left\langle E_{i_{1}}, E_{i_{2}}\right\rangle$, we can juxtapose the vector

$$
\left\langle\left\langle\mu_{k_{i_{1}}, k_{i_{2}}, T_{1}}, \nu_{k_{i_{1}}, k_{i_{2}}, T_{1}}\right\rangle, \ldots,\left\langle\mu_{k_{i_{1}}, k_{i_{2}}, T_{t}}, \nu_{k_{i_{1}}, k_{i_{2}}, T_{t}}\right\rangle\right\rangle,
$$

that represents the opinion of expert $E_{i_{1}}$ for expert $E_{i_{2}}$ in time. Therefore, we can observe the changes of the opinion of the first expert to his/her colleague. So, we can check whether there is a correct or tendentious opinion of expert $E_{i_{1}}$ for expert $E_{i_{2}}$ in time.

## 3 Second example

Let us have bookshops $B_{1}, B_{2}, \ldots, B_{b}$ in different towns $C_{1}, C_{2}, \ldots, C_{c}$. Obviously, some bookshops can be in one company and in different towns. Let us interested in the sales of the books with titles $A_{1}, A_{2}, \ldots, A_{a}$.

First, we can construct an 3D-IM with elements - real (natural) numbers, e.g., with the form

$$
M=\left[A, B, C,\left\{d_{k_{i}, l_{j}, h_{g}}\right\}\right]
$$

$$
\begin{aligned}
& \equiv\left\{\left.\begin{array}{c|ccccc}
C_{g} & B_{1} & \ldots & B_{j} & \ldots & B_{n} \\
\hline A_{1} & d_{A_{1}, B_{1}, C_{g}} & \vdots & d_{A_{1}, B_{j}, C_{g}} & \ldots & d_{A_{1}, B_{n}, C_{g}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{i} & d_{A_{i}, B_{1}, C_{g}} & \vdots & d_{A_{i}, B_{j}, C_{g}} & \ldots & d_{A_{i}, B_{n}, C_{g}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{m} & d_{A_{m}, B_{1}, C_{g}} & \ldots & d_{A_{m}, B_{j}, C_{g}} & \ldots & d_{A_{m}, B_{n}, C_{g}}
\end{array} \quad \right\rvert\, C_{g} \in C\right\} \\
& \equiv\left\{\begin{array}{c|ccccc}
C_{1} & B_{1} & \ldots & B_{j} & \ldots & B_{n} \\
\hline A_{1} & d_{A_{1}, B_{1}, C_{1}} & \vdots & d_{A_{1}, B_{j}, C_{1}} & \ldots & d_{A_{1}, B_{n}, C_{1}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{i} & d_{A_{i}, B_{1}, C_{1}} & \vdots & d_{A_{i}, B_{j}, C_{1}} & \ldots & d_{A_{i}, B_{n}, C_{1}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{m} & d_{A_{m}, B_{1}, C_{1}} & \ldots & d_{A_{m}, B_{j}, C_{1}} & \ldots & d_{A_{m}, B_{n}, C_{1}}
\end{array},\right. \\
& \begin{array}{c|ccccc}
C_{2} & B_{1} & \ldots & B_{j} & \ldots & B_{n} \\
\hline A_{1} & d_{A_{1}, B_{1}, C_{2}} & \vdots & d_{A_{1}, B_{j}, C_{2}} & \ldots & d_{A_{1}, B_{n}, C_{2}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{i} & d_{A_{i}, B_{1}, C_{2}} & \vdots & d_{A_{i}, B_{j}, C_{2}} & \ldots & d_{A_{i}, B_{n}, C_{2}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{m} & d_{A_{m}, B_{1}, C_{2}} & \ldots & d_{A_{m}, B_{j}, C_{2}} & \ldots & d_{A_{m}, B_{n}, C_{2}}
\end{array}, \\
& \left.\begin{array}{c|ccccc}
C_{f} & B_{1} & \ldots & B_{j} & \ldots & B_{n} \\
\hline A_{1} & d_{A_{1}, B_{1}, C_{f}} & \vdots & d_{A_{1}, B_{j}, C_{f}} & \ldots & d_{A_{1}, B_{n}, C_{f}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{i} & d_{A_{i}, B_{1}, C_{f}} & \vdots & d_{A_{i}, B_{j}, C_{f}} & \ldots & d_{A_{i}, B_{n}, C_{f}} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
A_{m} & d_{A_{m}, B_{1}, C_{f}} & \ldots & d_{A_{m}, B_{j}, C_{f}} & \ldots & d_{A_{m}, B_{n}, C_{f}}
\end{array}\right\},
\end{aligned}
$$

where $A=\left\{A_{1}, A_{2}, \ldots, A_{a}\right\}, B=\left\{B_{1}, B_{2}, \ldots, B_{b}\right\}, C=\left\{C_{1}, C_{2}, \ldots, C_{c}\right\}$, and for $1 \leq i \leq a$, $1 \leq j \leq b, 1 \leq g \leq c: d_{A_{i}, B_{j}, C_{g}} \geq 0$ is a natural number, representing the total number of sold books with title $A_{i}$ in bookshop $B_{j}$ in town $C_{g}$.

Second, we can modify the present IM, changing its elements $d_{A_{i}, B_{j}, C_{g}}$ with the IFPs

$$
\left\langle m_{A_{i}, B_{j}, C_{g}}, n_{A_{i}, B_{j}, C_{g}}\right\rangle,
$$

where $m_{A_{i}, B_{j}, C_{g}}$ is the quantity of sold books from $A_{i}$-th title, divided by the total quantity of this book, received in bookshop $B_{j}$ in town $C_{g}$ and $n_{A_{i}, B_{j}, C_{g}}$ is the quantity of the same book in the
bookshop warehouse. Therefore, $m_{A_{i}, B_{j}, C_{g}}+n_{A_{i}, B_{j}, C_{g}} \leq 1$ and number $1-m_{A_{i}, B_{j}, C_{g}}-n_{A_{i}, B_{j}, C_{g}}$ corresponds to the number of non-sold books that stay on the shelves in the bookshop, but are not in its warehouse. Therefore, the above standard 3D-IM is transformed to 3D-IFIM.

## 4 Conclusion

The second example is interesting, because it is a good illustration not only of the possibility to transform a standard 3D-IM to a 3D-IFIM, but on its basis we can construct a 4D-IM or 4D-IFIM. For this aim, we add a fourth component in the IM-definition, e.g. - finite time $T=\left\{T_{1}, T_{2}, \ldots, T_{t}\right\}$. So, we obtain two new IMs with the forms

$$
M_{\text {standart IM }}=\left[A, B, C, T,\left\{d_{A_{i}, B_{j}, C_{g}, t_{u}}\right\}\right]
$$

and

$$
M_{I F I M}=\left[A, B, C, T,\left\{\left\langle m_{A_{i}, B_{j}, C_{g}, t_{u}}, n_{A_{i}, B_{j}, C_{g}, t_{u}}\right\rangle\right\}\right],
$$

where for the above discussed $d_{A_{i}, B_{j}, C_{g}}$ and for $1 \leq u \leq t: d_{A_{i}, B_{j}, C_{g}, t_{u}}$ is a natural number and $\left\langle m_{A_{i}, B_{j}, C_{g}, t_{u}}, n_{A_{i}, B_{j}, C_{g}, t_{u}}\right\rangle$ is an IFP.

## References

[1] Atanassov, K., Generalized index matrices, Comptes rendus de l'Academie Bulgare des Sciences, Vol. 40, 1987, No. 11, 15-18.
[2] Atanassov, K., On index matrices, Part 1: Standard cases. Advanced Studies in Contemporary Mathematics, Vol. 20, 2010, No. 2, 291-302.
[3] Atanassov, K., On index matrices, Part 2: Intuitionistic fuzzy case. Proceedings of the Jangjeon Mathematical Society, Vol. 13, 2010, No. 2, 121-126.
[4] Atanassov, K., On Intuitionistic Fuzzy Sets Theory, Springer, Berlin, 2012.
[5] Atanassov, K., Extended intuitionistic fuzzy index matrices, Advanced Studies in Contemporary Mathematics (submitted).
[6] Atanassov, K., G. Pasi, R. Yager, Intuitionistic fuzzy interpretations of multi-criteria multiperson and multi-measurement tool decision making. International Journal of Systems Science, Vol. 36, 2005, No. 14, 859-868.
[7] Atanassov, K., E. Sotirova, V. Bureva, On index matrices. Part 4: New operations over index matrices. Advanced Studies in Contemporary Mathematics, Vol. 23, 2013, No. 3, 547-552.
[8] Atanassov, K., E. Szmidt, J. Kacprzyk, On intuitionistic fuzzy pairs, Notes on Intuitionistic Fuzzy Sets, Vol. 19, 2013, No. 3, 1-13.

