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Some properties of operations conjunction and disjunction from Łukasiewicz type over intuitionistic fuzzy sets. Part 2

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Abstract: Some properties of two operations – conjunction and disjunction from Łukasiewicz type – over intuitionistic fuzzy sets are studied. An open problem is formulated.
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1 Introduction

The paper is a continuation of our previous research [5]. Initially, we give some necessary definitions.

Let a set E be fixed. The Intuitionistic Fuzzy Set (IFS) A in E is defined by (see, e.g., [1]):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

In [5] we studied some properties of two operators over Intuitionistic Fuzzy Sets (IFSs), introduced in [4]. Here, we continue this research, studying the relations between these operations and extended modal and level operators, defined over IFSs.

Different relations and operations have been introduced over the IFSs. Some of them (see, e.g. [1,2]) are the following

$$\begin{array}{llll}
A \subset_{\Box} B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x)), & \\
A \subset_{\Diamond} B & \text{iff} & (\forall x \in E)(\nu_A(x) \geq \nu_B(x)) & \\
A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \ \&\ \nu_A(x) \geq \nu_B(x)) & \text{iff} \quad A \subset_{\Box} B \ \&\ A \subset_{\Diamond} B, \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \ \&\ \nu_A(x) = \nu_B(x)) & \text{iff} \quad A \subset B \ \&\ B \subset A.
\end{array}$$

In [1,2], series of extended modal and local operators are introduced. Two of the most notable of them are $F_{\alpha,\beta}(A)$ and $G_{\alpha,\beta}(A)$.

Let $\alpha, \beta \in [0, 1]$ and let:

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}, \text{ where } \alpha + \beta \leq 1$$
$$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\}.$$

In [4], the first author introduced the following two operations:

$$\begin{array}{rcl}
A \oplus B & = & \{\langle x, \min(1, \mu_A(x) + \mu_B(x)), \max(0, \nu_A(x) + \nu_B(x) - 1)\rangle | x \in E\}, \\
A \odot B & = & \{\langle x, \max(0, \mu_A(x) + \mu_B(x) - 1), \min(1, \nu_A(x) + \nu_B(x))\rangle | x \in E\},
\end{array}$$

The same operations have been discussed in [3].

In this paper we continue to study some of the basic properties of these operations.

2 Main results

Theorem 1. For every two IFSs A, B and for every two real numbers $\alpha, \beta \in [0, 1]$:

(a) $F_{\alpha,\beta}(A) \oplus F_{\alpha,\beta}(B) \subset_{\Box} F_{\alpha,\beta}(A \oplus B)$, (1)

(b) $F_{\alpha,\beta}(A \odot B) \subset_{\Diamond} F_{\alpha,\beta}(A) \odot F_{\alpha,\beta}(B)$. (2)

Proof. (a) Let A, B be two IFSs and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. Then

$$F_{\alpha,\beta}(A \oplus B)$$

$$= F_{\alpha,\beta}(\{\langle x, \min(1, \mu_A(x) + \mu_B(x)), \max(0, \nu_A(x) + \nu_B(x) - 1)\rangle | x \in E\})$$

$$= \{\langle x, \min(1, \mu_A(x) + \mu_B(x)) + \alpha.(1 - \min(1, \mu_A(x) + \mu_B(x))$$

$$- \max(0, \nu_A(x) + \nu_B(x) - 1)), \max(0, \nu_A(x) + \nu_B(x) - 1)$$

$$+\beta.(1 - \min(1, \mu_A(x) + \mu_B(x)) - \max(0, \nu_A(x) + \nu_B(x) - 1))\rangle | x \in E\}.$$

$$F_{\alpha,\beta}(A) \oplus F_{\alpha,\beta}(B)$$

$$= \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}$$

$$\oplus\{\langle x, \mu_B(x) + \alpha.\pi_B(x), \nu_B(x) + \beta.\pi_B(x)\rangle | x \in E\}$$

$$= \{\langle x, \min(1, \mu_A(x) + \alpha.\pi_A(x) + \mu_B(x) + \alpha.\pi_B(x)),$$

$$\max(0, \nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) - 1)\rangle | x \in E\}.$$

Now, we check the values of the following two expressions.

$$X \equiv \min(1, \mu_A(x) + \mu_B(x)) + \alpha.(1 - \min(1, \mu_A(x) + \mu_B(x))$$

$$- \max(0, \nu_A(x) + \nu_B(x) - 1)) - \min(1, \mu_A(x) + \alpha.\pi_A(x) + \mu_B(x) + \alpha.\pi_B(x)).$$

Let $\mu_A(x) + \mu_B(x) \geq 1$. Then

$$\nu_A(x) + \nu_B(x) - 1 \leq 2 - (\mu_A(x) + \mu_B(x)) - 1 = 1 - (\mu_A(x) + \mu_B(x)) \leq 0$$

and

$$X = 1 + \alpha.(1 - \min(1, \mu_A(x) + \mu_B(x)) - 0) - \min(1, \mu_A(x) + \alpha.\pi_A(x) + \mu_B(x) + \alpha.\pi_B(x))$$

$$X = 1 + \alpha.(1 - 1) - \min(1, \mu_A(x) + \alpha.\pi_A(x) + \mu_B(x) + \alpha.\pi_B(x))$$

$$X = 1 - \min(1, \mu_A(x) + \alpha.\pi_A(x) + \mu_B(x) + \alpha.\pi_B(x)) \geq 0.$$

Let $\mu_A(x) + \mu_B(x) < 1$. Then

$$X = \mu_A(x) + \mu_B(x) + \alpha.(1 - \mu_A(x) - \mu_B(x))$$

$$- \max(0, \nu_A(x)+\nu_B(x)-1)) - \min(1, \mu_A(x)+\mu_B(x)+\alpha.(2-\mu_A(x)-\mu_B(x)-\nu_A(x)-\nu_B(x))).$$

Let $\nu_A(x) + \nu_B(x) \geq 1$. Then

$$X = \mu_A(x) + \mu_B(x) + \alpha.(1 - \mu_A(x) - \mu_B(x)$$

$$-\nu_A(x) - \nu_B(x) + 1) - \min(1, \mu_A(x) + \mu_B(x) + \alpha.(2 - \mu_A(x) - \mu_B(x) - \nu_A(x) - \nu_B(x))) \geq 0.$$

Let $\nu_A(x) + \nu_B(x) < 1$. Then

$$X = \mu_A(x) + \mu_B(x) + \alpha.(1 - \mu_A(x) - \mu_B(x)) - 0$$

$$-\min(1, \mu_A(x) + \mu_B(x) + \alpha.(2 - \mu_A(x) - \mu_B(x) - \nu_A(x) - \nu_B(x)))$$
$$= \mu_A(x) + \mu_B(x) + \alpha.(1 - \mu_A(x) - \mu_B(x)) - 0$$
$$-\min(1, \mu_A(x) + \mu_B(x) + \alpha.(1 - \mu_A(x) - \mu_B(x)) - \alpha.(\nu_A(x) + \nu_B(x) - 1))$$
$$\geq \mu_A(x) + \mu_B(x) + \alpha.(1 - \mu_A(x) - \mu_B(x)) - 0$$
$$-\min(1, \mu_A(x) + \mu_B(x) + \alpha.(1 - \mu_A(x) - \mu_B(x))) \geq 0.$$

Therefore, always $X \geq 0$, i.e., (1) is valid.

It is interesting to comment why in (1) there is not relation $\subset$. The reason is the following.

Let
$$Y \equiv \max(0, \nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) - 1)$$
$$-\max(0, \nu_A(x) + \nu_B(x) - 1) - \beta.(1 - \min(1, \mu_A(x) + \mu_B(x)) - \max(0, \nu_A(x) + \nu_B(x) - 1)).$$

Let $\nu_A(x) + \nu_B(x) \geq 1$. Then
$$\mu_A(x) + \mu_B(x) \leq 1 - \nu_A(x) + 1 - \nu_B(x) \leq 1$$
and
$$Y = \nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) - 1$$
$$-(\nu_A(x) + \nu_B(x) - 1) - \beta.(1 - (\mu_A(x) + \mu_B(x)) - \nu_A(x) - \nu_B(x) + 1)$$
$$= \beta.\pi_A(x) + \beta.\pi_B(x) - \beta.(2 - \mu_A(x) - \mu_B(x)) - \nu_A(x) - \nu_B(x)) = 0.$$

Let $\nu_A(x) + \nu_B(x) < 1$. Then
$$Y = \max(0, \nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) - 1) - 0 - \beta.(1 - \min(1, \mu_A(x) + \mu_B(x)) - 0)$$
$$= \max(0, \nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) - 1) - \beta.(1 - \min(1, \mu_A(x) + \mu_B(x))).$$

If $\mu_A(x) + \mu_B(x) \geq 1$, then
$$Y = \max(0, \nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) - 1) - \beta.(1 - 1) \geq 0.$$

If $\mu_A(x) + \mu_B(x) < 1$, then
$$Y = \max(0, \nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) - 1) - \beta.(1 - \mu_A(x) - \mu_B(x)).$$

Let $\nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) \geq 1$. Then
$$Y = \nu_A(x) + \beta.(1 - \mu_A(x) - \nu_A(x)) + \nu_B(x) + \beta.(1 - \mu_B(x) - \nu_B(x)) - 1 - \beta.(1 - \mu_A(x) - \mu_B(x))$$
$$= \nu_A(x) + \nu_B(x) + \beta.(2 - \nu_A(x) - \nu_B(x)) - \beta$$
$$= (\beta - 1).(1 - \nu_A(x) - \nu_B(x)) < 0$$
for $\beta < 1$, because $1 - \nu_A(x) - \nu_B(x) > 0$.

If $\nu_A(x) + \beta.\pi_A(x) + \nu_B(x) + \beta.\pi_B(x) < 1$. Then
$$Y = 0 - \beta.(1 - \mu_A(x) - \mu_B(x)) < 0$$
for $\beta > 0$, because $1 - \nu_A(x) - \nu_B(x) > 0$.

Therefore, Y can be either a positive, or a negative number. By the same reason, there is also not a relation from the type of $\subset_\diamond$.

(b) is proved analogously. $\square$

Theorem 2. For every two IFSs A, B and for every two real numbers $\alpha, \beta \in [0, 1]$:

(a) $G_{\alpha,\beta}(A \oplus B) \subset G_{\alpha,\beta}(A) \oplus G_{\alpha,\beta}(B)$, (3)

(b) $G_{\alpha,\beta}(A) \oplus G_{\alpha,\beta}(B) \subset G_{\alpha,\beta}(A \oplus B)$, (4)

Proof. (a) Let A, B be two IFSs and $\alpha, \beta \in [0, 1]$. Then

$$G_{\alpha,\beta}(A \oplus B)$$

$$= G_{\alpha,\beta}(\{\langle x, \min(1, \mu_A(x) + \mu_B(x)), \max(0, \nu_A(x) + \nu_B(x) - 1)\rangle | x \in E\})$$

$$= \{\langle x, \alpha. \min(1, \mu_A(x) + \mu_B(x)), \beta. \max(0, \nu_A(x) + \nu_B(x) - 1)\rangle | x \in E\}.$$

$$G_{\alpha,\beta}(A) \oplus G_{\alpha,\beta}(B)$$

$$= \{\langle x, \alpha.\mu_A(x), \beta.\nu_A\rangle | x \in E\} \oplus \{\langle x, \alpha.\mu_B(x), \beta.\nu_B(x)\rangle | x \in E\}$$

$$= \{\langle x, \min(1, \alpha.\mu_A(x) + \alpha.\mu_B(x)), \max(0, \beta.\nu_A(x) + \beta.\nu_B(x) - 1)\rangle | x \in E\}.$$

Then, we check that

$$\min(1, \alpha.\mu_A(x) + \alpha.\mu_B(x)) - \alpha. \min(1, \mu_A(x) + \mu_B(x))$$

$$= \min(1, \alpha.\mu_A(x) + \alpha.\mu_B(x)) - \min(\alpha, \alpha.\mu_A(x) + \alpha.\mu_B(x)) \geq 0.$$

and

$$\beta. \max(0, \nu_A(x) + \nu_B(x) - 1) - \beta. \max(0, \nu_A(x) + \nu_B(x) - 1)$$

$$= \max(0, \beta.\nu_A(x) + \beta.\nu_B(x) - \beta) - \beta. \max(0, \nu_A(x) + \nu_B(x) - 1) \geq 0.$$

Therefore, (3) is valid.

(b) is proved analogously. □

3 Conclusion

In the next part of our research, we will study the relations between the operations $\oplus$ and $\odot$ with the other extended modal operators (see [1, 2]), with the level operators (see [1]), and with the second type of the modal operators (see [2]).

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