3rd Int. IFS Conf., 29 Aug – 1 Sep 2016, Mersin, Turkey
Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
Vol. 22, 2016, No. 4, 6–12

Properties of the intuitionistic fuzzy implication $\rightarrow_{186}$

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Received: 10 June 2016 **Accepted:** 29 October 2016

Abstract: In [8], a new Fodor's type of intuitionistic fuzzy implication, numbered as $\rightarrow_{186}$, was defined and some of its properties were studied. The present paper is a continuation of the previous one. New interesting properties of implication $\rightarrow_{186}$ are formulated and checked.
Keywords: Implication, Intuitionistic fuzzy implication, Intuitionistic fuzzy logic.
AMS Classification: 03E72.

1 Introduction

In [7, 15, 16] some Fodor's type of intuitionistic fuzzy implications were introduced. The first five were included in [6], while the latest one, numbered as $\rightarrow_{186}$, was introduced latter – in [8]. There, some of its basic properties were studied, e.g., it was checked which axioms of Klir and Yuan, of Kolmogorov, of Łukasiewicz–Tarski and of intuitionistic logic axioms are valid. Here, we continue this research, studying new properties of intuitionistic fuzzy implication $\rightarrow_{186}$.

In the intuitionistic fuzzy logic (see [1, 2, 4, 5], each proposition, variable or formula is evaluated with two degrees – "truth degree" or "degree of validity" and 'falsity degree" or "degree of non-validity". Thus, to each one of these objects, e.g., p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0,1] \text{ and } \mu(p) + \nu(p) \leq 1.$$

Let

$$\pi(p) = 1 - \mu(p) - \nu(p).$$

This function determines the degree of uncertainty (indeterminacy).

Let an evaluation function V be defined over a set of propositions $\mathcal{S}$, in such a way that for $p \in \mathcal{S}$:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence the function $V : \mathcal{S} \to [0,1] \times [0,1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$ – the set of logical objects that we use (in general case – formulas).

We assume that the evaluation function V assigns to the logical truth T:

$$V(T) = \langle 1, 0 \rangle,$$

and to the logical falsity it assigns the value: F

$$V(F) = \langle 0, 1 \rangle.$$

As it was discussed [3], the first two intuitionistic fuzzy negations are

$$V(\neg_1 p) = \langle \nu(p), \mu(p) \rangle,$$

$$V(\neg_2 p) = \langle \overline{\text{sg}}(\mu(p)), \text{sg}(\mu(p)) \rangle,$$

where here and below

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ \\ 0 & \text{if } x \leq 0 \end{cases},$$

and

$$\overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ \\ 1 & \text{if } x \leq 0 \end{cases}.$$

The second negation does not satisfy the equality $V(\neg_2\neg_2 p) = V(p)$, and, as we see below, it exhibits truly intuitionistic behaviour. Here, we define only the operations "disjunction", "conjunction" and "implication", originally introduced in [1], that have classical logic analogues, as follows:

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle,$$

$$V(p \wedge q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle,$$

$$V(p \to_{186} q) = \langle \max(\nu(p), \mu(q)), \min(\nu(p), \mu(q)) \rangle.$$

For the needs of the discussion below, we define the notions of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [1, 3]) and tautology.

Formula A is an IFT if and only if (iff) for every evaluation function V, if $V(A) = \langle a, b\rangle$, then,

$$a \geq b,$$

while it is a (classical) tautology if and only if for every evaluation function V, if $V(A) = \langle a, b\rangle$, then,

$$a = 1,\ b = 0.$$

Below, when it is clear, we will omit notation "$V(A)$", using directly "A" of the intuitionistic fuzzy evaluation of A. Also, for brevity, in a lot of places, instead of the IFP $\langle \mu(A), \nu(A)\rangle$ we will use the IFP $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$.

It is also suitable, if $\langle a, b\rangle$ and $\langle c, d\rangle$ are IFPs, to have

$$\langle a, b\rangle \leq \langle c, d\rangle \text{ iff } a \leq c \text{ and } b \geq d$$

and

$$\langle a, b\rangle \geq \langle c, d\rangle \text{ iff } a \geq c \text{ and } b \leq d.$$

2 Properties of the intuitionistic fuzzy implication $\rightarrow_{186}$

In [8], the following intuitionistic fuzzy implication was introduced:

$$V(x \rightarrow_{186} y) = \langle \overline{\text{sg}}(d-b) + \text{sg}(d-b)\max(b, c), \text{sg}(d-b)\min(a, d)\rangle$$

and it is shown that it generates the standard (classical) intuitionistic fuzzy negation $\neg_1$, or briefly $\neg$.

G. F. Rose's formula [13, 14] has the form:

$$((\neg\neg A \rightarrow_{186} A) \rightarrow_{186} (\neg\neg A \vee \neg A)) \rightarrow_{186} (\neg\neg A \vee \neg A). \tag{1}$$

For it, the following two theorems are valid.

Theorem 1. Implication $\rightarrow_{186}$ satislies Rose's formula (1) as a tautology.

Proof. Let A be a formula for which $V(A) = \langle a, b\rangle$. Then

$$V(((\neg\neg A \rightarrow_{186} A) \rightarrow_{186} (\neg\neg A \vee \neg A)) \rightarrow_{186} (\neg\neg A \vee \neg A))$$

$$= ((\neg\neg\langle a, b\rangle \rightarrow_{186} \langle a, b\rangle) \rightarrow_{186} (\neg\neg\langle a, b\rangle \vee \neg\langle a, b\rangle)) \rightarrow_{186} (\neg\neg\langle a, b\rangle \vee \neg\langle a, b\rangle)$$

$$= (((\langle a, b\rangle \rightarrow_{186} \langle a, b\rangle) \rightarrow_{186} (\langle a, b\rangle \vee \langle b, a\rangle)) \rightarrow_{186} (\langle a, b\rangle \vee \langle b, a\rangle)$$

$$= ((\langle \overline{\text{sg}}(b-b) + \text{sg}(b-b)\max(b, a), \text{sg}(b-b)\min(a, b)\rangle \rightarrow_{186} \langle \max(a, b), \min(a, b)\rangle)$$

$$\rightarrow_{186} \langle \max(a, b), \min(a, b)\rangle)$$

$$= ((\langle 1, 0\rangle \rightarrow_{186} \langle \max(a, b), \min(a, b)\rangle) \rightarrow_{186} \langle \max(a, b), \min(a, b)\rangle)$$

$$= \langle \overline{\text{sg}}(\min(a, b)-0) + \text{sg}(\min(a, b)-0)\max(0, \max(a, b)), \text{sg}(\min(a, b)-0)\min(1, \min(a, b))\rangle$$

$$\rightarrow_{186} \langle \max(a, b), \min(a, b)\rangle)$$

$= \langle \overline{\text{sg}}(\min(a,b)){+}\text{sg}(\min(a,b))\max(a,b), \text{sg}(\min(a,b))\min(a,b)\rangle \rightarrow_{186} \langle \max(a,b), \min(a,b)\rangle)$

(because for each real number $x \geq 0$: $\text{sg}(x)x = x$.)

$$= \langle \overline{\text{sg}}(\min(a,b)) + \text{sg}(\min(a,b))\max(a,b), \min(a,b)\rangle \rightarrow_{186} \langle \max(a,b), \min(a,b)\rangle)$$

$$= \langle \overline{\text{sg}}(\min(a,b) - \min(a,b)) + \text{sg}(\min(a,b) - \min(a,b))\max(\min(a,b), \max(a,b)),$$

$$\text{sg}(\min(a,b) - \min(a,b))\min(\overline{\text{sg}}(\min(a,b)) + \text{sg}(\min(a,b))\max(a,b), \min(a,b))\rangle$$

$$= \langle \overline{\text{sg}}(0) + \text{sg}(0)\max(\min(a,b), \max(a,b)),$$

$$\text{sg}(0)\min(\overline{\text{sg}}(\min(a,b)) + \text{sg}(\min(a,b))\max(a,b), \min(a,b))\rangle$$

$$= \langle 1, 0\rangle.$$

Therefore, Rose's formula (1) is a tautology. □

Corollary 1. Implication $\rightarrow_{186}$ satislies Rose's formula (1) as an intuitionistic fuzzy tautology.

The next assertions are proved in a similar manner and by this reason, we omit their proofs. Now, we discuss the following formulas, inspired by (1):

$$(A \vee \neg A) \rightarrow_{186} (A \rightarrow_{186} \neg\neg A) \tag{2}$$

$$(\neg\neg A \vee \neg A) \rightarrow_{186} (A \rightarrow_{186} \neg\neg A) \tag{3}$$

$$(A \rightarrow_{186} \neg\neg A) \rightarrow_{186} (A \vee \neg A) \tag{4}$$

$$(A \rightarrow_{186} \neg\neg A) \rightarrow_{186} (\neg\neg A \vee \neg A) \tag{5}$$

Obviously, in the classical propositional calculus, all these four formulas are tautologies. Now, we study their properties in the intuitionistic fuzzy case.

Theorem 2. Implication $\rightarrow_{186}$ satisfies (2) and (3) as tautologies.

Corollary 2. Implication $\rightarrow_{186}$ satislies (2) and (3) as intuitionistic fuzzy tautologies.

Theorem 3. Implication $\rightarrow_{186}$ satisfies (4) and (5) as intuitionistic fuzzy tautologies, but not as tautologies.

Now, following [4], we discuss the well-known Contraposition Law

$$(A \rightarrow_{186} B) \rightarrow_{186} (\neg B \rightarrow_{186} \neg A) \tag{6}$$

and its modified version

$$(\neg\neg A \rightarrow_{186} \neg\neg B) \rightarrow_{186} (\neg B \rightarrow_{186} \neg A). \tag{7}$$

For them, the following assertions are valid.

Theorem 4. Implication $\rightarrow_{186}$ satisfies (6) and (7) as intuitionistic fuzzy tautologies, but not as tautologies.

Proof. Let A and B be a formulas for which $V(A) = \langle a, b\rangle$ and $V(B) = \langle c, d\rangle$. Then for (6) we obtain

$$V((A \rightarrow_{186} B) \rightarrow_{186} (\neg B \rightarrow_{186} \neg A)))$$

$$= (\langle a, b\rangle \rightarrow_{186} \langle c, d\rangle) \rightarrow_{186} (\neg\langle c, d\rangle \rightarrow_{186} \neg\langle a, b\rangle)$$

$$= (\langle a, b\rangle \rightarrow_{186} \langle c, d\rangle) \rightarrow_{186} (\langle d, c\rangle \rightarrow_{186} \langle b, a\rangle)$$

$$= \langle \overline{\mathrm{sg}}(d - b) + \mathrm{sg}(d - b) \max(b, c), \mathrm{sg}(d - b) \min(a, d)\rangle$$

$$\rightarrow_{186} \langle \overline{\mathrm{sg}}(a - c) + \mathrm{sg}(a - c) \max(c, b), \mathrm{sg}(a - c) \min(a, d)\rangle$$

$$= \langle \overline{\mathrm{sg}}(\mathrm{sg}(a - c) \min(a, d) - \mathrm{sg}(d - b) \min(a, d)) + \mathrm{sg}(\mathrm{sg}(a - c) \min(a, d)$$

$$-\mathrm{sg}(d - b) \min(a, d)). \max(\mathrm{sg}(d - b) \min(a, d), \overline{\mathrm{sg}}(a - c) + \mathrm{sg}(a - c) \max(c, b)),$$

$$\mathrm{sg}(\mathrm{sg}(a - c) \min(a, d) - \mathrm{sg}(d - b) \min(a, d)) \min(\overline{\mathrm{sg}}(d - b) + \mathrm{sg}(d - b) \max(b, c),$$

$$\mathrm{sg}(a - c) \min(a, d))\rangle.$$

Let

$$X \equiv \overline{\mathrm{sg}}(\mathrm{sg}(a-c) \min(a, d) - \mathrm{sg}(d-b) \min(a, d)) + \mathrm{sg}(\mathrm{sg}(a-c) \min(a, d) - \mathrm{sg}(d-b) \min(a, d))$$

$$. \max(\mathrm{sg}(d - b) \min(a, d), \overline{\mathrm{sg}}(a - c) + \mathrm{sg}(a - c) \max(c, b)) - \mathrm{sg}(\mathrm{sg}(a - c) \min(a, d)$$

$$-\mathrm{sg}(d - b) \min(a, d)). \min(\overline{\mathrm{sg}}(d - b) + \mathrm{sg}(d - b) \max(b, c), \mathrm{sg}(a - c) \min(a, d)).$$

1. If $a > c$, then

$$X = \overline{\mathrm{sg}}(\min(a, d) - \mathrm{sg}(d - b) \min(a, d)) + \mathrm{sg}(\min(a, d) - \mathrm{sg}(d - b) \min(a, d))$$

$$. \max(\mathrm{sg}(d - b) \min(a, d), \max(c, b)) - \mathrm{sg}(\min(a, d) - \mathrm{sg}(d - b) \min(a, d))$$

$$. \min(\overline{\mathrm{sg}}(d - b) + \mathrm{sg}(d - b) \max(b, c), \min(a, d)).$$

1.1. If $d > b$, then

$$X = \overline{\mathrm{sg}}(\min(a, d) - \min(a, d)) + \mathrm{sg}(\min(a, d) - \min(a, d)). \max(\min(a, d), \max(c, b))$$

$$-\mathrm{sg}(\min(a, d) - \min(a, d)). \min(\max(b, c), \min(a, d))$$

$$= 1 - 0 = 1 > 0.$$

1.2. If $d \leq b$, then

$$X = \overline{\mathrm{sg}}(\min(a, d)) + \mathrm{sg}(\min(a, d)). \max(c, b) - \mathrm{sg}(\min(a, d)). \min(1, \min(a, d))$$

$$= \overline{\mathrm{sg}}(\min(a, d)) + \mathrm{sg}(\min(a, d)). \max(c, b) - \mathrm{sg}(\min(a, d)). \min(a, d).$$

1.2.1. From $a > c$ it follows that $a > 0$. Let $d = 0$. Then

$$X = \overline{\mathrm{sg}}(0) + \mathrm{sg}(0). \max(c, b) - \mathrm{sg}(0). \min(a, d) = 1 > 0.$$

1.2.2. Let $d > 0$. Then

$$X = 0 + \max(c, b) - \min(a, d) \geq b - d \geq 0.$$

2. If $a \leq c$, then

$$X = \overline{\text{sg}}(-\text{sg}(d-b)\min(a,d)) + \text{sg}(-\text{sg}(d-b)\min(a,d))$$

$$.\max(\text{sg}(d-b)\min(a,d),1) - \text{sg}(0-\text{sg}(d-b)\min(a,d)).\min(\overline{\text{sg}}(d-b)+\text{sg}(d-b)\max(b,c),0).$$

$$X = 1 - 0.\min(\overline{\text{sg}}(d-b) + \text{sg}(d-b)\max(b,c),0) = 1 > 0.$$

Therefore, in all cases $X \geq 0$, i.e., (6) is an IFT, but in case 1.2.2, the expression is not 1, as in the rest cases, i.e., (6) is not a tautology. The validity of (7) follows directly from (6). □

3 Conclusion

In next research other properties of the implications will be introduced and studied. All they show that intuitonistic fuzzy sets and logics in the sense, described in [2, 3], correspond to the ideas of Brouwer's intuitionism (see [9, 11, 12]).

Acknowledgements

The first and second authors are thankful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. DFNI-I-02-5 "InterCriteria Analysis: A New Approach to Decision Making".

References

[1] Atanassov, K. (1988) Two variants of intuitonistc fuzzy propositional calculus. *Preprint IM-MFAIS-5-88*, Sofia, 1988.

[2] Atanassov, K. (1999) *Intuitionistic Fuzzy Sets*. Springer, Heidelberg.

[3] Atanassov, K. (2012) *On Intuitionistic Fuzzy Sets*. Springer, Berlin.

[4] Atanassov K. (2014) On Intuitionistic Fuzzy Logics: Results and Problems. In: *Modern Approaches in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics, Volume 1: Foundations*. (Atanassov, K., M. Baczynski, J. Drewniak, J. Kacprzyk, M. Krawczak, E. Szmidt, M. Wygralak, S. Zadrozny, eds.), SRI-PAS, Warsaw, pp. 23–49.

[5] Atanassov, K. (2015) Intuitionistic fuzzy logics as tools for evaluation of data mining processes, *Knowledge-Based Systems*, 80, 122–130.

[6] Atanassov, K. (2017) *Intuitionistic Fuzzy Logics*. Springer Publishing House (in press).

[7] Atanassov, K., E. Szmidt, J. Kacprzyk, On Fodor's type of intuitionistic fuzzy implication and negation, *Notes on Intuitionistic Fuzzy Sets*, 21, 2015, No. 2, 25–34.

[8] Atanassov, K., Szmidt, E. & Kacprzyk, J. (2016) New Fodor's Type of Intuitionistic Fuzzy Implication and Negation, *Notes on Intuitionistic Fuzzy Sets*, 22(3), 1–8.

[9] van Atten, M. (2004) *On Brouwer*, Wadsworth, Behnout.

[10] Baczynski, M. & Jayaram, B. (2008) *Fuzzy Implications*, Springer, Berlin.

[11] Brouwer, L. E. J. (1975) *Collected Works, Vol. 1*, North Holland, Amsterdam.

[12] van Dalen, D. (Ed.) (1981) *Brouwer's Cambridge Lectures on Intuitionism*, Cambridge Univ. Press, Cambridge.

[13] Plisko, V. (2009) A survey of propositional realizability logic. *The Bulleting of Symbolic Logic*, 15(1), 1–42.

[14] Rose, G. F. (1953) Propositional calculus and realizability. *Transactions of the American Mathematical Society*, 75, 1–19.

[15] Szmidt, E., Kacprzyk, J. & Atanassov, K. (2015) Properties of Fodor's intuitionistic fuzzy implication and negation. *Notes on Intuitionistic Fuzzy Sets*, 21(4), 6–12.

[16] Szmidt, E., Kacprzyk, J. & Atanassov, K. (2015) Modal forms of Fodor's type of intuitionistic fuzzy implication. *Notes on Intuitionistic Fuzzy Sets*, 21(5), 1–5.