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Sendograph metric on intuitionistic fuzzy number space

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Abstract: In this paper, we propose a metric based on Hausdorff distance between sendographs of intuitionistic fuzzy numbers. Then we investigate some fundamental properties of this metric and give numerical examples. In section 3.1, it's generalized the well-known Kloeden's theorem on IFN space. In section 3.2, we show that IFN space is not complete with respect to sendograph metric and we construct a completion of IFN space with respect to sendograph metric.

Keywords: Intuitionistic fuzzy number, Hausdorff metric, Sendograph, Endograph, Distance measure.
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1 Introduction

The concept of fuzzy set was introduced by Zadeh in 1965 and has been well understood and used in various aspect of science and technology such as engineering and medicine. The theory of fuzzy set is one of the most important inventions of our time. It should be pointed out that in the study of fuzzy sets various metrics defined on fuzzy sets play a very important role, especially in the case of fuzzy numbers [5]. The most often used metrics are the supremum metric d_∞, the L_p metrics, the sendograph metric and the endograph metric. Among these metrics the sendograph metric was introduced by Kloeden and was used successfully in fuzzy dynamic systems [16]. The sendograph metric is also used in many studies such as [6, 7, 14].

On the other hand, as a natural generalization of fuzzy set, intuitionistic fuzzy set (IFS for short) was introduced by Atanassov in 1983 [1]. His definition was found to be useful to deal with vagueness of knowledge. In the concept of intuitionistic fuzzy set, each element has two degrees named degree of membership and degree of non-membership to IFS respectively [1]. Intuitionistic fuzzy number (IFN for short) was defined by Burillo and Bustince in 1992 [4]. Grzegorzewski redefined the concept of IFN in [13]. IFN is a basic concept for intuitionistic fuzzy theory. Distance between IFS's and IFN's is one of the most fundamental problems in this area. In the literature, there are many definitions of distance measures for IFN [2], [10], [12], [13], [18], [19]. In this paper, we proposed a metric depending on the Hausdorff distance between sendographs of IFN's and investigate some of its properties.

The structure of the paper is as follows: In Section 2, we give some basic definitions about IFS and IFN. In Section 3, we define the sendograph metric for IFN and investigate some properties of this metric. Fundamental properties about this metric are presented. We will first prove that IFNs space is separable with respect to the sendograph metric. In section 3.1, we generalize the well-known Kloeden's theorem on fuzzy number space to the case of IFNs space. In section 3.2. We show that with respect to sendograph metric IFN space is not complete and we construct a completion of IFNs space with respect to the sendograph metric.

2 Preliminaries

Definition 2.1. An intuitionistic fuzzy set in a non-empty set X given by a set $A=\{(x,\mu_A(x),\eta_A(x)): x\in X\}$ where $\mu_A(x):X\to I=[0,1],\eta_A(x):X\to I=[0,1]$ are functions defined such that $0\le\mu(x)+\eta(x)\le 1$ for all $x\in X$. For $x\in X$, $\mu(x)$ and $\eta(x)$ represent the degree of membership and degree of non-membership of x to A respectively.

For each $x\in X$; intuitionistic fuzzy index of x in A can be defined as follows $\pi_A(x)=1-\mu_A(x)-\eta_A(x)$. π_A is called degree of hesitation or indeterminacy. [1]

Definition 2.2. The definition of intuitionistic fuzzy number was defined by Burillo et al. in [4] as follows. An intuitionistic fuzzy number *A* is

i. An intuitionistic fuzzy subset of real line,

ii. Normality i.e. there is an $x_0\in\mathbb{R}$ such that $\mu_A(x)=1$ (so $\eta_A(x)=0$)

iii. Fuzzy convex for the membership function μ_A i.e. for every $\lambda\in[0,1]$ and $x_1,x_2\in\mathbb{R}$

$$\mu_A(\lambda x_1+(1-\lambda)x_2)\ge\min\{\mu_A(x_1),\mu_A(x_2)\}$$

iv. Fuzzy concave for the non-membership function η_A i.e. for every $\lambda\in[0,1]$ and $x_1,x_2\in\mathbb{R}$

$$\eta_A(\lambda x_1+(1-\lambda)x_2)\le\max\{\eta_A(x_1),\eta_A(x_2)\}$$

In [13], Grzegorzewski re-defined intuitionistic fuzzy number (shortly IFN) as follows: An intuitionistic fuzzy subset $A=\{(x,\mu_A(x),\eta_A(x)):x\in\mathbb{R}\}$ of the real line is called an intuitionistic fuzzy number if:

i. A is if-normal (there exist at least two points $x_0,x_1\in\mathbb{R}$ such that $\mu_A(x_0)=1$ and $\eta_A(x_1)=1$)

ii. A is if-convex (i.e. its membership function fuzzy convex and non-membership function fuzzy concave)

iii. μ_A is upper semi continuous and η_A lower semi continuous.

iv. $suppA=cl(\{x;\ \eta_A(x)<1\})$ is bounded subset of real line

From the definition given above we get at once that for any intuitionistic fuzzy number A, there exist eight numbers $a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4$ such that $a_1\le b_1\le a_2\le b_2\le a_3\le b_3\le a_4\le b_4$ and four functions $f_A,g_A,h_A,k_A:\mathbb{R}\to I$, called the sides of IFN, where f_A and k_A non-decreasing and g_A,h_A non-increasing, such that f_A and g_A are upper semi continuous and h_A and k_A are lower semi continuous. Then we can describe a membership function and μ_A non-membership function η_A as follows:

$$\mu_A(x)=\begin{cases}0 & if\ x<a_1\\ f_A(x) & if\ a_1\le x<a_2\\ 1 & if\ a_2\le x\le a_3\\ g_A(x) & if\ a_3<x\le a_4\\ 0 & if\ a_4<x\end{cases}\quad,\quad \eta_A(x)=\begin{cases}1 & if\ x<b_1\\ h_A(x) & if\ b_1\le x<b_2\\ 0 & if\ b_2\le x\le b_3\\ k_A(x) & if\ b_3<x\le b_4\\ 1 & if\ b_4<x\end{cases}$$

Let us denote the set of all IFNs by E_{IFN} .[10]

In [3] , Atanassov generalized these two definitions of IFN in three ways as follows:

1. The maximum condition is replaced with

$$\max_{y\in E}\mu_A(y)=\mu_A(x_0)>0.5>\eta_A(x_0)=\max_{y\in E}\eta_A(y)$$

2. The maximum condition is replaced with

$$\max_{y\in E}\mu_A(y)=\mu_A(x_0)>\eta_A(x_0)=\max_{y\in E}\eta_A(y)$$

3. Finally, in the third generalization of the above three definitions, Atanassov omit the condition for maximal values of μ_A and minimal values for η_A, keeping only the conditions for increasing and decreasing of the two functions. [3]

The concept of triangular fuzzy number can be generalized to define triangular intuit-ionistic fuzzy numbers as follows:

Definition 2.3. A triangular intuitionistic fuzzy number $u=\{(x,\mu_u(x),\eta_u(x));x\in\mathbb{R},\}$, (TIFN for short) is an IFN on $\mathbb{R}$ with its membership and non-membership functions are defined as:

$$\mu_u(x)=\begin{cases}0 & \text{if } x<m-\dot{\alpha}\\ 1-\dfrac{m-x}{\dot{\alpha}} & \text{if } m-\dot{\alpha}\le x<m\\ 1 & \text{if } x=m\\ 1-\dfrac{x-m}{\dot{\beta}} & \text{if } m\le x\le m+\dot{\beta}\\ 0 & \text{if } x\ge m+\dot{\beta}\end{cases}\quad , \qquad \mu_u(x)=\begin{cases}1 & \text{if } x<m-\ddot{\alpha}\\ \dfrac{m-x}{\ddot{\alpha}} & \text{if } m-\ddot{\alpha}\le x<m\\ 0 & \text{if } x=m\\ \dfrac{x-m}{\ddot{\beta}} & \text{if } m\le x\le m+\ddot{\beta}\\ 1 & \text{if } x\ge m+\ddot{\beta}\end{cases}$$

where $\dot{\alpha}$, $\dot{\beta}$, $\ddot{\alpha}$, $\ddot{\beta}$ are positive real numbers such that $\ddot{\alpha}>\dot{\alpha}$, $\ddot{\beta}>\dot{\beta}$. Here, $\dot{\alpha}$, $\dot{\beta}$ are called right and left spreads of the membership function, respectively. $\ddot{\alpha},\ddot{\beta}$ are called right and left spreads of the non-membership function, respectively, [10].

Definition 2.4. The (α,β)**-**cut of an IFS $A=\{(x,\mu_A(x_1),\eta_A(x_1)):x\in\mathbb{R}\}$ is crisp set of $x\in\mathbb{R}$ defined as $A_{\alpha,\beta}=\{x;\mu_A(x)\ge\alpha,\eta_A(x)\le\beta\}$**,** where α,β are fixed non-negative numbers which satisfy the condition $0\le\alpha+\beta\le1$. In other words (α,β)**-**cut of IFS *A* is crisp set of elements x which belong to A at least to degree α and not belong to *A* at most to the degree β.

The following lemma is direct generalization of lemma in [16]:

Lemma 2.1. For a subset $H\subseteq\mathbb{R}\times I\times I$ such that $\alpha,\beta\in I$ and $0\le\alpha+\beta\le1$, let $H_{\alpha,\beta}=\{x;\,(x,\alpha',\beta')\in H,\alpha'\le\alpha,\beta'\le\beta\}$ where $\alpha',\beta'\in I$ and $0\le\alpha'+\beta'\le1$. Then $H_{\alpha,\beta}$ is equal to (α,β)-cut of a unique IFN *u*, if $\{H_{\alpha,\beta};\,\alpha,\beta\in I,\,0\le\alpha+\beta\le1\}$ family satisfies following conditions:

i. For every $\alpha,\beta\in I$ and $0\le\alpha+\beta\le1$, $H_{\alpha,\beta}$ is non-empty closed and convex subset of the real line,
ii. For every $\alpha_i,\beta_i\in I$, $(i=1,2)$ satisfying conditions that $\alpha_1\le\alpha_2$, $\beta_1\ge\beta_2$, $0\le\alpha_i+\beta_i\le1$; $H_{\alpha_1,\beta_1}\supseteq H_{\alpha_2,\beta_2}$,
iii. For every $\{\alpha_n\},\{\beta_n\}\subseteq I$ sequences which satisfy that $\alpha_n\uparrow\alpha$ and $\beta_n\downarrow\beta$, $H_{\alpha,\beta}=\bigcap_{n=1}^{\infty}H_{\alpha_n,\beta_n}$,
iv. For every $\{\alpha_n\},\{\beta_n\}\subseteq I$ sequences which satisfy that $\alpha_n\downarrow0$ and $\beta_n\uparrow1$, $H_{0,1}=\bigcup_{n=1}^{\infty}H_{\alpha_n,\beta_n}$,

Conversely, the (α,β)-cut of an IFN satisfies condition (i)-(iv).

3 Main results

Definition 3.1. Let $A=\{(x,\mu_A(x),\eta_A(x)):x\in\mathbb{R}\}$ be an IFN. Then the endograph of *A*, denoted by *end*(*A*), defined as follows:

$$end(A)=\{(x,r,s);\mu(x)\geq r,\eta(x)\leq s,0\leq r+s\leq 1\}$$

while the supported endograph (shortly sendograph of *A*), denoted by *send*(*A*) is defined as follows:

$$send(A)=\{(x,r,s);x\in suppA;\ 0\leq r\leq\mu(x)\ ,\eta(x)\leq s\leq 1,\ 0\leq r+s\leq 1\}$$

From definition of endograph and sendograph it is clearly seen that:

$$send(A)=end(A)\cap(supp(A)\times I\times I)$$

Since μ_A is upper semi continuous, η_A is lower semi continuous and $supp(A)$ is a bounded and closed subset of $\mathbb{R}$, it is clearly seen that $send(A)$ is a compact subset of $\mathbb{R}^3$.

Definition 3.2. Let d_m be product metric on $\mathbb{R}\times I\times I$ defined as follows:

$$d_m((x_1,r_1,s_1),(x_2,r_2,s_2))=\max\{|x_1-x_2|,|r_1-r_2|,|s_1-s_2|\}.$$

Then for all $A,B\in E_{IFN}$, Let $D_{send}(A,B)=H(send(A),send(B))$, where H is the Hausdorff metric defined as:

$$H(send(A),send(B))=\max\{H^*(send(A),send(B)),H^*(send(B),send(A))\}$$

where

$$H^*(send(A),send(B))=\max\{d(x,send(B));x\in send(A)\}$$

and

$$d(x,send(B))=\min\{d_p(x,y);y\in send(B)\}.$$

Similar to case of fuzzy numbers, it can be easily seen that $D_{send}(A,B)$ is a metric for all IFNs $D_{send}(A,B)$ is called the sendograph metric of IFNs.

Theorem 3.1. (E_{IFN},D_{send}) is separable.

Proof: Let $0=\alpha_1\leq\alpha_2\leq...\leq\alpha_n=1$ and $0=\beta_n\leq\beta_{n-1}\leq...\leq\beta_1=1$ be two partitions of [0, 1], which satisfy the following conditions:

i. For each $i=1,2,...,n$; α_i and β_i are rational numbers,

ii. $|\alpha_i-\alpha_{i-1}|<\varepsilon$ and $|\beta_i-\beta_{i-1}|<\varepsilon$ for each $i=1,2,...,n$ and $\varepsilon>0$

iii. $0\leq\alpha_i+\beta_i\leq 1$ for $i=1,2,...,n$

Let be $A \in E_{IFN}$. Then we define IFS ϕ whose membership functions and non-membership functions are defined respectively as:

$$\mu_{\phi} = \begin{cases} \alpha_i & , \ \alpha_{i-1} < \mu_A(x) < \alpha_i \\ 0 & , \quad otherwise \end{cases}, \ \eta_{\phi}(x) = \begin{cases} \beta_i & , \ \beta_i < \mu_A(x) < \beta_{i-1} \\ 1 & , \quad otherwise \end{cases}$$

From the definition of ϕ, it can be clearly seen that $0 \le \mu_{\phi}(x) + \eta_{\phi}(x) \le 1$ for all $x \in \mathbb{R}$. Since $A \in E_{IFN}$, ϕ satisfies the conditions of IFN. Obviously, $supp(A) = supp(\phi)$ and $D_{send}(A,\phi) \le \varepsilon$. Let denote the set of all ϕ by $\tilde{E}_{IFN}$. $\tilde{E}_{IFN}$ is countable and dense subset of E_{IFN}. Therefore (E_{IFN}, D_{send}) is separable.

Definition 3.3. A sequence $\{A_n\}$ of intuitionistic fuzzy numbers is called support bounded, if there exists an interval $(a,b) \subset R$ such that $supp(A_n) \subset (a,b)$ for every $n \in \mathbb{N}$.

3.1 Generalization of Kloeden's theorem on E_{IFN}

In this section, we generalized Kloeden's well-known theorem on E_{IFN}. Kloeden's theorem (in [16]) is a useful tool to determining convergence of sequences of IFN with respect to sendograph metric.

Theorem 3.2. Suppose that A and sequence $\{A_n\}$ belong to E_{IFN}. Then $D_{send}(A_n, A) \to 0$ if only if

1. $h(supp(A_n), supp(A)) \to 0$, where h is Hausdorff metric defined on $\mathbb{R}$
2. For every $\varepsilon > 0$, there exist an integer $n(\varepsilon)$ satisfied both of the following conditions.
   a. For all $x \in \mathbb{R}$ there can be found a sequence $x_n = x_n(x,\varepsilon) \subset \mathbb{R}$ with $\mu_{A_n}(x) < \mu_A(x_n) + \varepsilon$, $\eta_{A_n}(x) > \eta_A(x_n) - \varepsilon$ and $|x_n - x| < \varepsilon$ for all $n \ge n(\varepsilon)$;
   b. For all $x \in supp(A)$ there can be found a sequence $(x_n) = (x_n)(x,\varepsilon) \subset supp(A_n)$ with $\mu_A(x) < \mu_{A_n}(x_n) + \varepsilon$, $\eta_A(x) > \eta_{A_n}(x_n) - \varepsilon$ and $|x_n - x| < \varepsilon$ for all $n \ge n(\varepsilon)$.

Proof: It can be proven as Kloeden's proof in the case of fuzzy numbers [11]. In our proof, there are minor modifications to Kloeden's proof.

$\Rightarrow$: Suppose that $D_{send}(A_n, A) \to 0$. We must show that (1.), (2a.) and (2b.) are true.

1. Suppose that $D_{send}(A,B) < \varepsilon$ for A, $B \in E_{IFN}$ and every $\varepsilon > 0$.

Then $H^*(send(A), send(B)) < \varepsilon$. Since $supp(A)$ is compact, There exists $x_a \in supp(A)$ which satisfies that $h^*(supp(A), supp(B)) = d(x_a, supp(A))$. Then $(x_a, 0, 1) \in send(A)$. So,

$$d((x_a,0,1), send(B)) \le H^*(send(A), send(B)) < \varepsilon.$$

Since $send(B)$ is compact, There exists $(x_b, r_b, s_b) \in send(B)$ which satisfies that

$$d_m((x_a,0,1),(x_b,r_b,s_b)) = d((x_a,0,1), send(B)) < \varepsilon.$$

Therefore, we have: $|x_a - x_b| < \varepsilon$, $|0 - r_b| < \varepsilon$, $|1 - s_b| < \varepsilon$. Since $x_b \in supp(B)$,

$$h^*(supp(A), supp(B)) = d(x_a, supp(A)) \leq d(x_a, x_b) < \varepsilon.$$

The inequality $h^*(supp(B), supp(A)) < \varepsilon$ can be shown by reversing roles of *A* and *B*. Therefore, by the combination of the last two inequalities, we obtain that $h(supp(A), supp(B)) < \varepsilon$.

2a. Since $D_{send}(A_n, A) \to 0$, for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that

$$H^*(send(A_n), send(A)) \leq D_{send}(A_n, A) < \varepsilon$$

for all $n \geq n_0(\varepsilon)$. Let be $x \in \mathbb{R}$. Then, define $x_n = x$ for $n < n_0(\varepsilon)$. If $n \geq n_0(\varepsilon)$ and $x \notin supp(A_n)$. $\mu_{A_n}(x) = 0 < \mu_A(x) + \varepsilon$ and $\eta_{A_n}(x) = 1 > \eta_A(x) - \varepsilon$, since $\mu_A(x) \geq 0$ and $\eta_A(x) \leq 1$. Therefore the sequence can be chosen $x_n = x$ for $n \in \mathbb{N}$. On the other hand, if $n \geq n_0(\varepsilon)$ and $x \in supp(A_n)$. Then, $(x, \mu_{A_n}(x), \eta_{A_n}(x)) \in send(A_n)$ and

$$d((x, \mu_{A_n}(x), \eta_{A_n}(x)),\ send(A)) \leq H^*(send(A_n), send(A)) < \varepsilon.$$

Since $send(A)$ is compact, there exists $(x_n, r_n, s_n) \in send(A)$ such that

$$d_m((x, \mu_{A_n}(x), \eta_{A_n}(x)),\ (x_n, r_n, s_n)) = d((x, \mu_{A_n}(x), \eta_{A_n}(x)),\ send(A)) < \varepsilon\ .$$

Therefore, we have: $|x_n - x| < \varepsilon$, $|\mu_{A_n}(x) - r_n| < \varepsilon$, $|\eta_{A_n}(x) - s_n| < \varepsilon$. Since $(x_n, r_n, s_n) \in send(A)$, $\mu_A(x_n) \geq r_n$ and $\eta_A(x_n) \leq s_n$. Thus, $|x_n - x| < \varepsilon, \mu_{A_n}(x) < r_n + \varepsilon \leq \mu_A(x_n) + \varepsilon$ and $\eta_{A_n}(x) > s_n - \varepsilon \leq \eta_A(x_n) - \varepsilon$ for all $n \in \mathbb{N}$. Therefore (2a) holds.

2b. Since $D_{send}(A_n, A) \to 0$, for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that

$$H^*(send(A), send(B)) \leq D_{send}(A_n, A) < \varepsilon$$

for all $n \geq n_0(\varepsilon)$. Let be $x \in supp(A)$. Then $(x, \mu_A(x), \eta_A(x)) \in send(A)$. So,

$$d((x, \mu_{A_n}(x), \eta_{A_n}(x)),\ send(A)) \leq H^*(send(A_n), send(A)) < \varepsilon$$

for all $n \geq n(\varepsilon)$. Since $send(A_n)$ is compact, there exists a $(x_n, r_n, s_n) \in send(A_n)$ such that $d_m((x, \mu_A(x), \eta_A(x)),\ (x_n, r_n, s_n)) = d((x, \mu_A(x), \eta_A(x)),\ send(A_n)) < \varepsilon$ i.e. $|x_n - x| < \varepsilon$, $|\mu_A(x) - r_n| < \varepsilon$, $|\eta_A(x) - s_n| < \varepsilon$. Therefore, $\mu_A(x) < r_n + \varepsilon \leq \mu_{A_n}(x_n) + \varepsilon$ and $\eta_A(x) > s_n - \varepsilon \leq \eta_{A_n}(x_n) - \varepsilon$ for all $n \geq n(\varepsilon)$.

$\Leftarrow$: Suppose that conditions (1), (2a), (2b) are satisfied for some A and some sequence $\{A_n\}$ in E_{IFN}. We must show that $D_{send}(A_n, A) \to 0$

A. From condition (1), for any $\delta > 0$ there exist $n' = n'(\delta) \in \mathbb{N}$ such that for $n \geq n_0$,

$$h^*(supp(A_n), supp(A)) \leq h(supp(A_n), supp(A)) < \delta$$

i.e. $supp(A_n) \subset U_\delta(supp(A))$ where δ-neighbourhood of $supp(A)$ is denoted by $U_\delta(supp(A))$. So; $supp(A_n \times I \times I) \subset U_\delta(supp(A \times I \times I))$ for all $n \geq n'$. Also, from (2a), there exist $n = n(\delta)$ such that for each $x \in \mathbb{R}$ and $\delta > 0$ a sequence $(x_n) = x_n(x, \delta)$ can be found with $|x_n - x| < \delta$, $\mu_{A_n}(x) < \mu_A(x_n) + \delta$ and $\eta_{A_n}(x) > \eta_A(x_n) - \delta$. i.e. for all $x \in U_\delta(x_n)$, $[0, \mu_{A_n}(x)] \subset [0, \mu_A(x_n) + \delta]$ and $[\eta_{A_n}(x), 1] \subset [\eta_A(x_n) - \delta, 1]$ for all $n \geq n(\delta)$. So

$$\begin{aligned}\{x\} \times [0, \mu_A(x_n)] \times [\eta_A(x_n), 1] &\subset U_\delta(x_n) \times [0, \mu_A(x_n) + \delta] \times [\eta_A(x_n) - \delta, 1] \\ \subset U_\delta(\{x_n\} \times [0, \mu_A(x_n)] \times [\eta_A(x_n), 1]) &\subset U_\delta\left(\bigcup_{y \in \mathbb{R}} \{y\} \times [0, \mu_{A_n}(x)] \times [\eta_A(x_n), 1]\right) \\ &= U_\delta(end(A))\end{aligned}$$

for all $n \geq n(\delta)$ and $x \in \mathbb{R}$.

For $\varepsilon > 0$, since $end(A)$ is closed subset and $supp(A) \times I \times I$ is compact subset of $\mathbb{R} \times I \times I$ such that $(supp(A) \times I \times I) \cap (\mathbb{R} \times I \times I) \neq \varnothing$ there exists a $\delta = \delta(end(A_n),\ supp(A) \times I \times I,\ \varepsilon) > 0$ with $U_\delta(end(A_n)) \cap U_\delta(supp(A) \times I \times I) \neq \varnothing$ and $U_\delta(end(A_n)) \cap U_\delta(supp(A) \times I \times I) \subset U_\varepsilon(end(A_n) \cap (supp(A) \times I \times I))$. Then for all $n \geq n_2(\varepsilon) = \max\{n(\delta), n'(\delta)\}$;

$$\begin{aligned}send(A_n) = end(A_n) \cap (supp(A_n) \times I \times I) &\subset U_\delta(end(A_n)) \cap U_\delta(supp(A_n) \times I \times I) \\ &\subset U_\varepsilon(end(A_n) \cap (supp(A_n) \times I \times I)) = U_\varepsilon(send(A))\end{aligned}$$

Therefore $H^*(send(A_n), send(A)) < \varepsilon$ for all $n \geq n_2(\varepsilon)$.

B. From (2b). We can find a $n(\varepsilon)$ such that for all $x \in supp(A)$ and $\varepsilon > 0$ there exists a sequence $\{x_n\} \subset supp(A)$ which satisfies following condition: $\mu_A(x) < \mu_{A_n}(x_n) + \varepsilon$, $\eta_A(x) > \eta_{A_n}(x_n) - \varepsilon$ and $|x_n - x| < \varepsilon$ for all $n \geq n(\varepsilon)$. Since $\{x_n, \mu_{A_n}(x_n), \eta_{A_n}(x_n)\} \in send(A)$, thus

$$\begin{aligned}\{x\} \times [0, \mu_A(x)] \times [\eta_A(x), 1] &\subset U_\varepsilon(x_n) \times [0, \mu_{A_n}(x_n) + \varepsilon] \times [\eta_{A_n}(x_n) - \varepsilon, 1] \\ &\subset U_\varepsilon(\{x_n\} \times [0, \mu_{A_n}(x_n)] \times [\eta_{A_n}(x_n), 1]) \subset U_\varepsilon(send(A))\end{aligned}$$

for all $n \geq n(\varepsilon)$.

Hence

$$send(A) = \bigcup_{x \in supp(A)} \{x\} \times [0, \mu_{A_n}(x)] \times [\eta_{A_n}(x), 1] \subset U_\varepsilon(send(A_n)),$$

i.e. $H^*(send(A), send(A_n)) < \varepsilon$ for all $n \geq n(\varepsilon)$. By combining results of (A) and (B) we obtain $D_{send}(A_n, A) \to 0$. □

Corollary 3.3. Every convergent sequence in E_{IFN} is support bounded.

Proof: Let $\{A_n\}$ be a sequence of IFNs, which converges to A, i.e. $\lim_{n\to\infty} D_{send}(A_n, A) = 0$ for every $n \geq n_0(\varepsilon)$. From Kloeden's Theorem, it is obtained that $\lim_{n\to\infty}\left(h\left(supp(A_n), supp(A)\right)\right) = 0$. Hence $supp(A_n)$ converges to $supp(A)$. Therefore $supp(A_n)$ is bounded for all $n \in \mathbb{N}$. From Definition 3.3, $\{A_n\}$ is support bounded. □

3.2 Completion of E_{IFN} with respect to D_{send}

Space E_{IFN} is not complete with respect to sendograph metric. To show that fact, we give an example.

Example 3.2 Let A_n a sequence of IFNs and A as below: For each $n \in \mathbb{N}$, membership function μ_{A_n} and non-membership function η_{A_n} are defined as follows, respectively as follows

$$\mu_{A_n}(x) = \begin{cases} 0 & , \ x < 0 \\ \dfrac{1}{2n} & , \ 0 \leq x < 1 \\ 1 & , \ x = 1 \\ \dfrac{1}{2n} & , \ 1 < x \leq 2 \\ 0 & , \ 2 < x \end{cases} \quad , \ \mu_{A_n}(x) = \begin{cases} 1 & , \ x < 0 \\ \dfrac{1}{4n} & , \ 0 \leq x < 1 \\ 0 & , \ x = 1 \\ \dfrac{1}{4n} & , \ 1 < x \leq 2 \\ 1 & , \ 2 < x \end{cases}$$

Similar manner; μ_A and η_A are defined as membership and non-membership function of A, respectively:

$$\mu_A(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases} \text{ and } \eta_A(x) = \begin{cases} 1, & x \neq 1 \\ 0, & x = 1 \end{cases}.$$

It can be easily seen that A_n is a Cauchy sequence with respect to sendograph metric. But it does not convergence to A with sendograph metric since it convergences to A with endograph metric and $\lim_{n\to\infty} h\left(supp(A_n), supp(A)\right) \neq 0$ for all $n \in \mathbb{N}$.

Therefore, it is clearly understood that E_{IFN} with respect to sendograph metric is not complete.

Theorem 3.4. Let $E_{IFN}^{*} = \{[a,b]\times\{0\}\times\{1\} \cup send(A); A \in E_{IFN}, \mathrm{A}_{0,1} \subseteq [a,b]\}$. Then $\left(E_{IFN}^{*}, D_{send}\right)$ is completion of $\left(E_{IFN}, D_{send}\right)$.

Proof: To prove our claim we must show that $cl\{E_{IFN}\} = E_{IFN}^{*}$ and E_{IFN}^{*} is complete.

I. It can be clearly seen that each member of E_{IFN}^{*} satisfies the (i), (ii), (iii), conditions of Lemma 2.1. But the condition (iv) does not hold in generally. General case is $A_{0,1} \subseteq A_{0,1}^{*}$ for $A \in E_{IFN}$ and $A^{*} \in E_{IFN}^{*}$. Since the conditions (i), (ii), (iii) are satisfied in E_{IFN}^{*}, We can say that $A^{*} \in E_{IFN}^{*}$ differs with an unique $A \in E_{IFN}$ only at $(0,1)$-cut. Then, if $A^{*} \neq send(A)$ for each $A \in E_{IFN}$, we can define $A' = A \cup \left(A_{0,1} \times [0,\varepsilon] \times [\varepsilon'', 1]\right)$ for all $\varepsilon', \varepsilon'' > 0$ wit h $0 \leq \varepsilon' + \varepsilon'' \leq 1$.

Thus A' corresponds to an unique IFN $B \in E_{IFN}$ such that $A' = send(B)$. Therefore $D_{send}(A, A') \le \min\{\varepsilon', \varepsilon''\}$. Thus $cl\{E_{IFN}\} = E^*_{IFN}$.

II. Let $\{A^*_n\}$ be an Cauchy sequence in E^*_{IFN}. Then for every $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $\{A^*_n\}$ for $n, m \ge n_0$ $D_{send}(A^*_n, A^*_m) \le \varepsilon$ .i.e. $H(send(A^*_n), send(A^*_m)) \le \varepsilon$. Since the membership and non-membership function of IFN which corresponds to A^*_n for each $n \in \mathbb{N}$ are upper semi continuous and lower semi continuous, $send(A^*_n)$ is closed. Thus $\{send(A^*_n)\}$ is a Cauchy sequence in C^3 with respect to Hausdorff metric. Since (C^3, H) is complete, $A^* = \bigcap_{n=1}^{\infty} cl\left\{\bigcup_{m \ge n}^{\infty} send(A^*_m)\right\}$ is member of C^3 and for each $n \in \mathbb{N}$, there exist $n_0 \in \mathbb{N}$ such that $H(send(A^*_n), A^*) < \varepsilon$. Now, we must prove that $A^* \in E^*_{IFN}$. To prove our claim we must show that A^* satisfies the conditions (i), (ii) and (iii) of Lemma 2.1.

i. Since for each $n \in \mathbb{N}$, $(A^*_n)_{\alpha,\beta}$ is a non-empty, closed and convex set, it is clearly seen that $(A^*)_{a,\beta} = \bigcap_{n=1}^{\infty} cl\left\{\bigcup_{m \ge n}^{\infty} (A^*_m)_{a,\beta}\right\}$ is a non-empty, closed and convex set.

ii. From definition of A^*, it is easily seen that for every $\alpha_i, \beta_i \in I$, $(i = 1, 2)$ satisfying the condition that $\alpha_1 < \alpha_2$, $\beta_1 > \beta_2$, $0 \le \alpha_i + \beta_i \le 1$ $A^*_{\alpha_1,\beta_1} \supseteq A^*_{\alpha_2,\beta_2}$.

iii. Since for every pair of sequences $\{\alpha_m\}, \{\beta_m\} \subseteq I$ such that $\alpha_m \uparrow \alpha$ and $\beta_m \downarrow \beta$, $(A^*_n)_{a,\beta} = \bigcap_{n=1}^{\infty} (A^*_n)_{\alpha_m,\beta_m}$ and $D_{send}(A^*_n, A_n) \to 0$, it is obtained that

$$(A^*)_{a,\beta} = \bigcap_{n=1}^{\infty} cl\left\{\bigcup_{m \ge n}^{\infty} (A^*_m)_{a,\beta}\right\} = \bigcap_{n=1}^{\infty} cl\left\{\bigcup_{m \ge n}^{\infty}\left\{\bigcap_{n=1}^{\infty} (A^*_m)_{\alpha_k,\beta_k}\right\}\right\} = \bigcap_{n=1}^{\infty} (A^*)_{\alpha_k,\beta_k}$$

This completes the proof. □

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