Notes on Intuitionistic Fuzzy Sets
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Exact sequence of intuitionistic fuzzy G-modules

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Abstract: The concept of intuitionistic fuzzy G-modules and their properties are defined and discussed by the author et al. in [16]. In this paper we develop the notion of exact sequence of intuitionistic fuzzy G-modules and study their properties.
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1 Introduction

The concept of intuitionistic fuzzy sets was introduced by K. T. Atanassov [1–3] as a generalization to the notion of fuzzy sets by L. A. Zadeh [25]. R. Biswas was the first to introduce the intuitionistic fuzzification of Algebraic structure and developed the concept of intuitionistic fuzzy subgroup of a group in [5]. Later on many mathematicians worked on it and introduced the notion of intuitionistic fuzzy subring, intuitionistic fuzzy submodule etc. (see [4, 8–15]). The notion of intuitionistic fuzzy G-modules was introduced by the author et al. in [16]. Many properties like representation, reducibility, complete reducibility and injectivity of intuitionistic fuzzy G-modules have been discussed in [17–24].

2 Preliminaries

In this section, we list some basic concepts and well known results on G-modules, exact sequence of G-modules which are mainly taken from [6, 7]. The concepts about intuitionistic fuzzy set theory and results about intuitionistic fuzzy G-modules are mainly taken from [3,4,14–16,18,19, 23].

Let G be a group and M be a vector space over a field K. Then M is called a G-module if for every $g \in G$ and $m \in M$, if there exists a product (called the action of G on M), $gm \in M$ which satisfies the following axioms

(i) $1_G.m = m, \forall\, m \in M$ (1_G being the identity of G)

(ii) $(g \cdot h) \cdot m = g \cdot (h \cdot m), \forall\, m \in M, g, h \in G$

(iii) $g.(k_1m_1 + k_2m_2) = k_1(g.m_1) + k_2(g.m_2), \forall\, k_1, k_2 \in K; m_1, m_2 \in M$ and $g \in G$.

A subspace of M, which itself is a G-module with the same action is called G-submodule of M. It can be seen that intersection of G-modules is again a G-submodule. A non-zero G-module M is irreducible if the only G-submodules of M are $\{0\}$ and M. Otherwise it is reducible. A non-zero G-module M is completely reducible if for every G-submodule N of M, there exists a G-submodule N^* of M such that $M = N \oplus N^*$. A G-module M is semi-simple if there exists a family of irreducible G-submodules M_i such that $M = \oplus_{i=1}^{n} M_i$. It is evident that completely reducible G-modules are semi-simple.

Definition 2.1. Let M and M^* be G-modules. A mapping $f : M \to M^*$ is a G-module homomorphism if

(i) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$

(ii) $f(gm) = gf(m), \forall\, k_1, k_2 \in K; m, m_1, m_2 \in M$ and $g \in G$.

Definition 2.2. Let $f : M \to M^*$ is a G-module homomorphism. Then
$\ker f = \{m \in M : f(m) = 0^*\}$ is a G-submodule of M and $Imf = \{f(m) : m \in M\}$ is a G-submodule of M^*.

Proposition 2.3. *If M is a G-module and N is a G-submodule of M, then M/N is a G-module which is called Quotient G-modules.*

Definition 2.4. A pair of module homomorphisms $M \xrightarrow{f} N \xrightarrow{g} P$ is said to be exact at N if $Imf = \ker g$.

A sequence of module homomorphisms

$$\ldots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \ldots$$

is exact provided that $Imf_i = \ker f_{i+1}$ for all indices i.

Remark 2.5. (i) For any module M, there are unique trivial module homomorphisms $0 \to M, 0 \mapsto 0$, is a monomorphism and $M \to 0, m \mapsto 0$ is an epimorphism.

(ii) $0 \to M \xrightarrow{f} N$ is exact if and only if $\ker f = 0$, i.e., f is a monomorphism.

(iii) $N \xrightarrow{g} P \to 0$ is exact if and only if $Img = P$, i.e., g is a epimorphism.

(iv) If $M \xrightarrow{f} N \xrightarrow{g} P$ is exact, then $g \circ f = 0$.

(v) An exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is called a short exact sequence, in which $M \cong Imf = \ker g, N/\ker g \cong Img = P$.
Whenever $M \leq N$, there is a short exact sequence $0 \to M \xrightarrow{i} N \xrightarrow{\pi} N/M \to 0$.

(vi) A short exact sequence in (v) is said to be split if there exist homomorphisms $\bar{f} : N \to M$ and $\bar{g} : P \to N$ such that $\bar{f}f = i_M$ and $g\bar{g} = i_P$.

(vii) Let $M \oplus N$ be the external direct sum of modules M and N. Then there exist the following canonical embeddings

$$i_M : M \to M \oplus N \text{ given by } m \mapsto (m, 0)$$
$$i_N : N \to M \oplus N \text{ given by } n \mapsto (0, n)$$

and the canonical projections

$$\pi_M : M \oplus N \to M \text{ given by } (m, n) \to m$$
$$\pi_N : M \oplus N \to N \text{ given by } (m, n) \to n$$

Clearly, $\pi_M i_M = I_M$ (identity map on M) and $\pi_N i_N = I_N$ (identity map on N).
Therefore $i_M\pi_M + i_N\pi_N = I_{M\oplus N}$ is the identity map on $M \oplus N$.

(viii) Consider an exact sequence of the form $0 \to M \xrightarrow{i_M} M \oplus N \xrightarrow{\pi_N} N \to 0$ where i_M and π_N are the canonical maps. Then following is commutative diagram $0 \to M \xrightarrow{i_M}\xleftarrow{\pi_M} M \oplus N \xrightarrow{\pi_N} N/M \to 0$. i.e., $\pi_M i_M = I_M$ and $\pi_N i_N = I_N$.
Then the above sequence split.

Theorem 2.6. *(The Short Five Lemma) Let R be a ring and*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\varphi"] & N \arrow[r, "g"] \arrow[d, "\psi"] & P \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' \arrow[r] & 0
\end{tikzcd}$$

a commutative diagram of R-module homomorphisms such that each row is a short exact sequence. Then

(i) φ and ξ are monomorphisms $\Rightarrow \psi$ is a monomorphism;

(ii) φ and ξ are epimorphisms $\Rightarrow \psi$ is a epimorphism;

(iii) φ and ξ are isomorphisms $\Rightarrow \psi$ is a isomorphism.

In such a case, the row short exact sequences are said to be isomorphic.

Theorem 2.7. *Let R be a ring and $0 \to M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \to 0$ a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent:*

(i) There is an R-module homomorphism $h : M_2 \to N$ with $gh = I_{M_2}$;

(ii) There is an R-module homomorphism $k : N \to M_1$ with $kf = I_{M_1}$;

(iii) The given sequence is isomorphic to the direct sum short exact sequence

(iv) $0 \to M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \to 0$; in particular $N \cong M_1 \oplus M_2$; such a sequence is called a split exact sequence.

Definition 2.8. Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and degree of non-membership of the element $x \in X$ respectively and for any $x \in X$, we have $\mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.9. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$ be any two IFSs of X, then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

(ii) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$

(iii) $A^c = \{\langle x, \mu_{A^c}(x), \nu_{A^c}(x)\rangle : x \in X\}$, where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for all $x \in X$

(iv) $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle : x \in X\}$, where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_{A\cap B}(x) = \nu_A(x) \vee \nu_B(x)$

(v) $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle : x \in X\}$, where $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ and $\nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x)$.

Remark 2.10. For convenience, we write the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ by $A = (\mu_A, \nu_A)$.

Definition 2.11. Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Let A and B be IFSs of X and Y, respectively. Then the image of A under the map f is denoted by $f(A)$ and is defined as $f(A)(y) = (\mu_{f(A)}(y), \nu_{f(A)}(y))$, where

$$\mu_{f(A)}(y) = \begin{cases} \vee\{\mu_A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \oslash \\ 0, & \text{otherwise} \end{cases}$$

$$\nu_{f(A)}(y) = \begin{cases} \wedge\{\nu_A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \oslash \\ 1, & \text{otherwise,} \end{cases} \forall y \in Y.$$

Also the pre-image of B under f is denoted by $f^{-1}(B)$ and is defined as

$$f^{-1}(B)(x) = \{\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)\},$$

where, $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$; $\forall\, x \in X$.

Remark 2.12. In general, $\mu_{f(A)}(f(x)) \geq \mu_A(x)$ and $\nu_{f(A)}(f(x)) \leq \nu_A(x)$ and equality hold if f is one-one.

Definition 2.13. Let $(X, .)$ be a groupoid and A, B be two IFSs of X. Then the intuitionistic fuzzy sum of A and B is denoted by $A + B$ and is defined as:
$(A + B)(x) = (\mu_{A+B}(x), \nu_{A+B}(x))$, where

$$\mu_{A+B}(x) = \begin{cases} (\vee_{x=a+b}\{\mu_A(a) \wedge \mu_B(b)\}, \wedge_{x=a+b}\{\nu_A(a) \vee \nu_B(b)\}, & \text{if } x = a + b \\ (0, 1), & \text{otherwise} \end{cases}; \forall x \in X.$$

Definition 2.14. For any IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ of set X. We denote the support of the IFS set A by A^* and is defined as

$$A^* = \{x \in X : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}.$$

Proposition 2.15. *Let $f : X \to Y$ be a mapping and A, B are IFS of X and Y, respectively. Then the following result holds:*

(i) $f(A^*) \subseteq (f(A))^*$ *and equality hold when the map f is bijective*

(ii) $f^{-1}(B^*) = (f^{-1}(B))^*$

Definition 2.16. Let G be a group and M be a G-module over K, which is a subfield of C. Then an intuitionistic fuzzy G-module (IFGM) on M is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of M such that following conditions are satisfied:

(i) $\mu_A(ax + by) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(ax + by) \leq \max\{\nu_A(x), \nu_A(y)\}$, $\forall a, b \in K$ and $x, y \in M$ and

(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, $\forall g \in G; m \in M$.

Definition 2.17. Let $A \in G^M$ (where G^M denotes the intuitionistic fuzzy power set of G-module M). Then A is called an intuitionistic fuzzy submodule of G-module M, if it satisfies the following:

(i) $\mu_A(0) = 1$ and $\nu_A(0) = 0$;

(ii) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, $\forall g \in G; m \in M$;

(iii) $\mu_A(m_1 + m_2) \geq \min\{\mu_A(m_1), \mu_A(m_2)\}$ and $\nu_A(m_1 + m_2) \leq \max\{\nu_A(m_1), \nu_A(m_2)\}$, $m_1, m_2 \in M$.

We denote the set of all intuitionistic fuzzy submodules of G-module M by $G(M)$.

Theorem 2.18. *Let $A \in G(M)$. Then A^* is a G-submodule of M.*

Theorem 2.19. *For any $A, B \in G(M)$, we have $(A + B)^* = A^* + B^*$ and $(A \cap B)^* = A^* \cap B^*$.*

Theorem 2.20. *Let $A \in G(M)$ and let N be a G-submodule of M. Define $A|_N \in G^N$ (where G^N is the intuitionistic fuzzy power of G-module N) as follows: $\mu_{A|_N}(x) = \mu_A(x)$ and $\nu_{A|_N}(x) = \nu_A(x)$. Then $A|_N \in G(N)$.*

Theorem 2.21. *Let $A \in G(M)$ and let N be a G-submodule of M. Define $A_N \in G^{M/N}$ as follows: $\mu_{A_N}(x+N) = \vee\{\mu_A(x+n) : n \in N\}$ and $\nu_{A_N}(x+N) = \wedge\{\nu_A(x+n) : n \in N\}$, $\forall x \in M$, where M/N denote the quotient module of M with respect to N. Then $A_N \in G(M/N)$.*

Definition 2.22. Let $A, B \in G(M)$ be such that $A \subseteq B$. Then $B/A \in G(B^*/A^*)$ is called the quotient of B with respect to A and is defined as

$$B/A(x + A^*) = (\mu_{B/A}(x + A^*), \nu_{B/A}(x + A^*)),$$

where $\mu_{B/A}(x + A^*) = \vee\{\mu_B(x + y); y \in A^*\}$ and $\nu_{B/A}(x + A^*) = \wedge\{\nu_B(x + y); y \in A^*\}$, where $x \in B^*$.

Definition 2.23. We define two IFS Ω and $\Omega(M)$ of M as

$$\Omega(x) = \begin{cases} (1,0), & \text{if } x = 0 \\ (0,1), & \text{if } x \neq 0 \end{cases}; \quad \Omega(M) = (1,0), \forall x \in M.$$

Then the IFS Ω and $\Omega(M)$ are IFSMs of M which are actually equivalent of $\{0\}$ and M in module theory.

Lemma 2.24. *For any IFS $A = (\mu_A, \nu_A)$ of a module M, $A^* = \{0\}$ if and only if $A = \Omega$.*

Definition 2.25. If $A, B \in G(M)$, then the sum $A + B$ is called the direct sum of A and B if $A \cap B = \Omega$ and we write it as $A \oplus B$.

Theorem 2.26. *Let $A, B, C \in G(M)$ such that $A = B \oplus C$, then $A^* = B^* \oplus C^*$.*

Definition 2.27. Let M and N be G-modules; $A \in G(M), B \in G(N)$. Consider the direct sum $M \oplus N$. We extend A and B on $M \oplus N$ to $A^{'}$ and $B^{'}$ as follows

$$\mu_{A'}(m,n) = \begin{cases} \mu_A(m), & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}; \quad \nu_{A'}(m,n) = \begin{cases} \nu_A(m), & \text{if } n = 0 \\ 1, & \text{if } n \neq 0 \end{cases}; \quad \forall (m,n) \in M \oplus N$$

$$\mu_{B'}(m,n) = \begin{cases} \mu_B(n), & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}; \quad \nu_{B'}(m,n) = \begin{cases} \nu_B(n), & \text{if } m = 0 \\ 1, & \text{if } m \neq 0 \end{cases}; \quad \forall (m,n) \in M \oplus N.$$

Then $A^{'}, B^{'} \in G(M \oplus N)$

$$\mu_{A' \cap B'}(m,n) = \begin{cases} 1, & \text{if } (m,n) = 0 \\ 0, & \text{if } (m,n) \neq 0 \end{cases}; \quad \nu_{A' \cap B'}(m,n) = \begin{cases} 0, & \text{if } (m,n) = 0 \\ 1, & \text{if } (m,n) \neq 0 \end{cases}; i.e., A^{'} \cap B^{'} = \Omega.$$

Therefore $A^{'} + B^{'}$ is infact a direct sum and we denote it by $A \oplus B$.

Remark 2.28. Note that

$$\begin{aligned}\mu_{A\oplus B}(m,n) &= \mu_{A'\cap B'}(m,n)\\ &= \vee_{(m,n)=(m_1,n_1)+(m_2,n_2)}\{\mu_{A'}(m_1,n_1)\wedge\mu_{B'}(m_2,n_2)\}, \forall (m,n)\in M\oplus N\\ &= \mu_{A'}(m,0)\wedge\mu_{B'}(m,0)\\ &= \mu_A(m)\wedge\mu_B(n).\end{aligned}$$

Similarly $\nu_{A\oplus B}(m,n)=\nu_A(m)\vee\nu_B(n)$.

Definition 2.29. Let $A_i\in G(M), i\in J$, then we say that A is the direct sum $\{A_i : i\in J\}$ denoted by $\oplus_{i\in J}A_i$ if

(i) $A=\Sigma_{i\in J}A_i$

(ii) $A_j\cap\Sigma_{i\in J\setminus\{j\}}A_i=\Omega, \forall j\in J$.

Example 2.30. Let $G=\{1,-1\}, M=R^2=\{(p,q):p,q\in R\}$ is a vector space over the field R. Then M is a G-module. Define IFSs $A=(\mu_A,\nu_A), B=(\mu_B,\nu_A), C=(\mu_C,\nu_C)$ of M by

$$\mu_A(x)=\begin{cases}1, & \text{if } x=(0,0)\\ 0.25, & \text{if } x=(p,0), p\notin 0\,;\\ 0.25, & \text{if } x=(p,q), q\notin 0\end{cases}\quad \nu_A(x)=\begin{cases}1, & \text{if } x=(0,0)\\ 0.5, & \text{if } x=(p,0), p\notin 0\\ 0.5, & \text{if } x=(p,q), q\notin 0;\end{cases}$$

$$\mu_B(x)=\begin{cases}1, & \text{if } x=(0,0)\\ 0.25, & \text{if } x=(p,0), p\notin 0\,;\\ 0, & \text{if } x=(p,q), q\notin 0\end{cases}\quad \nu_B(x)=\begin{cases}0, & \text{if } x=(0,0)\\ 0.5, & \text{if } x=(p,0), p\notin 0\\ 0, & \text{if } x=(p,q), q\notin 0;\end{cases}$$

$$\mu_C(x)=\begin{cases}1, & \text{if } x=(0,0)\\ 0.25, & \text{if } x=(p,0), p\notin 0\,;\\ 0, & \text{if } x=(p,q), q\notin 0\end{cases}\quad \nu_C(x)=\begin{cases}0, & \text{if } x=(0,0)\\ 0.5, & \text{if } x=(0,q), q\notin 0\\ 1, & \text{if } x=(p,q), p\notin 0.\end{cases}$$

Then, $A,B,C\in G(M)$ such that $A=B\oplus C$.

Definition 2.31. Let $A\in G(M)$. Then A is said to be a semi-simple G-module if whenever B is strictly proper G-submodule of A (i.e., $B\subset A$), there exists a strictly proper G-submodule C of A such that $A=B\oplus C$. That is if B is a proper G-submodule of A such that $B(x)=A(x)$ $\forall x\in B^*$ then there exists a proper G-submodule C of A satisfying $C(x)=A(x)\ \forall\ x\in C^*$ such that $A=B\oplus C$.

Definition 2.32. Let M and M^* be G-modules and let A,B be two intuitionistic fuzzy G-submodules on M and M^* respectively. Let $f:M\to M^*$ be a G-module homomorphism. Then f is called a weak intuitionistic fuzzy G-homomorphism of A onto B if $f(A)\subseteq B$. The homomorphism f is an intuitionistic fuzzy G-homomorphism of A onto B if $f(A)=B$. We say that A is an intuitionistic fuzzy G-homomorphic onto B and we write as $A\approx B$.

Let $f:M\to M^*$ be a G-module isomorphism. Then f is called a weak intuitionistic fuzzy G-isomorphism if $f(A)\subseteq B$ and f is an intuitionistic fuzzy G-isomorphism if $f(A)=B$ and we write it as $A\cong B$.

3 Exact sequence of intuitionistic fuzzy G-modules

From the theory of G-modules recall that a sequence of G-modules and G-module homomorphisms

$$\ldots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \ldots \tag{3.1}$$

is said to be exact at M_i if $Im(f_i) = \ker(f_{i+1})$; and the sequence is said to be exact if it is exact at each M_i. In this section, we extend this notion to intuitionistic fuzzy G-modules and prove some results.

Definition 3.1. Let $M_i, i \in Z$ be G-modules and let $A_i \in G(M_i), i \in Z$. Suppose that (1) is exact sequence of G-modules. Then the sequence

$$\ldots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \ldots \tag{3.2}$$

of intuitionistic fuzzy G-modules is said to be exact if, for all $i \in Z$,

(i) $f_{i+1}(A_i) \subseteq A_{i+1}$ and

(ii) $(f_i(A_{i-1}))^* = \ker(f_{i+1})$.

Theorem 3.2. *Let $A, B \in G(M)$ be such that $A \oplus B$ is a direct sum of intuitionistic fuzzy submodules of G-module M so that $A^* \oplus B^*$ is a direct sum of G-modules. Then the sequence $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ is exact, considering $A \in G(A^*)$ and $B \in G(B^*)$.*

Proof. Note that the sequence $0 \to A^* \xrightarrow{i} A^* \oplus B^* \xrightarrow{\pi} B^* \to 0$ is an exact sequence of G-modules where "i" and "π" are respectively the canonical injection and projection. We have to prove that the sequence $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ is an exact sequence of intuitionistic fuzzy G-modules.

Let $x \in A^* + B^*$. Then $i(A)(x) = (\mu_{i(A)}(x), \nu_{i(A)}(x))$, where

$$\mu_{i(A)}(x) = \begin{cases} \vee\{\mu_A(t) : t \in A^*, i(t) = x\}, & \text{if } i^{-1}(x) \neq \oslash \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \mu_A(x), & \text{if } x \in A^* \\ 0, & \text{if } x \notin A^* \end{cases} \textit{and}$$

$$\nu_{i(A)}(x) = \begin{cases} \wedge\{\nu_A(t) : t \in A^*, i(t) = x\}, & \text{if } i^{-1}(x) \neq \oslash \\ 1, & \text{otherwise} \end{cases} = \begin{cases} \nu_A(x), & \text{if } x \in A^* \\ 1, & \text{if } x \notin A^*. \end{cases}$$

$$\text{Thus, } i(A) = A \forall x \in A^* \tag{3.3}$$

Also, $(A + B)(x) = (\mu_{A+B}(x), \nu_{A+B}(x))$, where

$$\begin{aligned} \mu_{A+B}(x) &= \begin{cases} \vee\{\mu_A(y) \wedge \mu_B(z) : y, z \in M, y + z = x\}, & \text{if } x = y + z \\ 0, & \text{if } x \neq y + z \end{cases} \\ &= \begin{cases} \mu_A(x), & \text{if } x \in A^* \\ 0, & \text{if } x \notin A^* \end{cases} \textit{and} \end{aligned}$$

$$\nu_{A+B}(x) = \begin{cases} \wedge\{\nu_A(y) \vee \nu_B(z) : y, z \in M, y+z = x\}, & \text{if } x = y+z \\ 1, & \text{if } x \neq y+z \end{cases}$$
$$= \begin{cases} \nu_A(x), & \text{if } x \in A^* \\ 1, & \text{if } x \notin A^*. \end{cases}$$

[Note that $A \oplus B$ is a direct sum, so $A \cap B = \Omega$. If $x = y + z$ with $x \in A^*$, then the only possibility is $x = x + 0$ or $x = y + z$; $y, z \in A^*$. But in the second case $\mu_B(z) = 0, \nu_B(z) = 1$].

$$\text{Thus, } A + B = A \text{ if } x \in A^* \tag{3.4}$$

It follows from (3.3) and (3.4) that $i(A) \subseteq A + B$.
For $x \in B^*, (\pi(A+B))(x) = (\mu_{\pi(A+B)}(x), \nu_{\pi(A+B)}(x))$, where

$$\begin{aligned} \mu_{\pi(A+B)}(x) &= \vee\{\mu_{A+B}(t) : t \in A^* + B^*; \pi(t) = x\} \\ &= \vee\{\mu_{A+B}(r+x) : r \in A^*\}[\text{ Since } \pi : A^* + B^* \to B^* \text{is the projection}] \\ &= \vee\{\mu_A(r) \wedge \mu_B(x) : r \in A^*\} \\ &= \mu_B(x).[\text{ Since } \mu_A(r) = 1 \text{ with } r = 0] \end{aligned}$$

Similarly, we have $\nu_{\pi(A+B)}(x) = \nu_B(x)$. Hence $\pi(A + B) = B$.
Now by (1), we have

$$i(A)(x) = \begin{cases} (\mu_A(x), \nu_A(x)), & \text{if } x \in A^* = \ker(\pi) \\ (0, 1), & \text{if } x \notin A^* = \ker(\pi). \end{cases} \quad i.e., (i(A))^* = \ker(\pi).$$

Therefore, $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ is an exact sequence of intuitionistic fuzzy G-modules. □

Remark 3.3. Note that in the above theorem, for convenience, we have denoted the intuitionistic fuzzy G-module $\Omega \in G(M)$ by 0.

Also if $0 \to M \xrightarrow{f} N$ is a sequence of G-modules and $A \in G(M), B \in G(N)$, then it is easy to see that $0 \to A \xrightarrow{f} B$ is an exact sequence of intuitionistic fuzzy G-modules if and only if f is injective.

Definition 3.4. Let $A, B \in G(M)$ be such that $A \oplus B$ is a direct sum of intuitionistic fuzzy submodules of G-module M. Then the sequence $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ of intuitionistic fuzzy G-modules is called a split exact sequence of intuitionistic fuzzy G-modules.

Now we obtain a necessary condition for a given sequence $A \xrightarrow{f} B \xrightarrow{g} C$ to be exact at B.

Theorem 3.5. *Let $M \xrightarrow{f} N \xrightarrow{g} P$ be a sequence of G-modules exact at N and let $A \in G(M), B \in G(N), C \in G(P)$. Then the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of intuitionistic fuzzy G-modules is exact at B only if $A^* \xrightarrow{f'} B^* \xrightarrow{g'} C$ is a sequence of G-modules exact at B^*, where f' and g' are restriction of f and g to A^* and B^* respectively.*

Proof. Suppose that $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B. Then by definition $f(A) \subseteq B, g(B) \subseteq C$ and $(f(A))^* = \ker(g)$.
Now, consider the sequence $A^* \xrightarrow{f'} B^* \xrightarrow{g'} C^*$.
We claim that this sequence is exact at B^*.
For $x \in (f(A))^*$:
$\Leftrightarrow \mu_{f(A)}(x) > 0$ and $\nu_{f(A)}(x) < 1$
$\Leftrightarrow \vee\{\mu_A(t) : f(t) = x, t \in M\} > 0$ and $\wedge\{\nu_A(t) : f(t) = x, t \in M\} < 1$
$\Leftrightarrow \exists' s\, t_1, t_2 \in M$ such that $x = f(t_1) = f(t_2), \mu_A(t_1) > 0, \nu_A(t_2) < 1$
(As $\mu_A(t_1) + \nu_A(t_1) \leq 1$ always, so if $\mu_A(t_1) > 0$ then $\nu_A(t_2) < 1$)
$\Leftrightarrow \exists' s\, t_1 \in M$ such that $x = f(t_1), \mu_A(t_1) > 0$ and $\nu_A(t_1) < 1$, i.e., $t_1 \in A^*$
$\Leftrightarrow x = f(t_1) \in f(A^*)$.
Thus, we get $(f(A))^* = f(A^*)$. Similarly, we get $(g(A))^* = g(A^*)$.
Therefore, $f'(A^*) = f(A^*) = (f(A))^* \subseteq B^*$ as $f(A) \subseteq B$.
Similarly, $g'(B^*) = (g(B))^* \subseteq C^*$.
Now, Since $(f(A))^* = \ker(g)$ it follows that $f'(A^*) = \ker(g')$.
Thus, the sequence $A^* \xrightarrow{f'} B^* \xrightarrow{g'} C^*$ is exact at B^*.
This completes the proof of the theorem. □

Remark 3.6. The converse of the above theorem is not true. That is the sequence $A^* \to B^* \to C^*$ is exact at B^* does not implies that the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B.

Example 3.7. Let M be a G-module, N and P are submodules of M such that $N \oplus P$ is a direct sum. Define $A \in G(N), B \in G(P)$ and $C \in G(N \oplus P)$ as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.8, & \text{if } x \in N - \{0\} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.1, & \text{if } x \in N - \{0\} \end{cases};$$

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x \in P - \{0\} \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.3, & \text{if } x \in P - \{0\} \end{cases};$$

$$\mu_C(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.3, & \text{if } x \in N \oplus P - \{0\} \end{cases}; \quad \nu_C(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.5, & \text{if } x \in N \oplus P - \{0\}. \end{cases}$$

Clearly, $A^* = N, B^* = P$ and $C^* = N \oplus P$. Obviously, $N \xrightarrow{i} N \oplus P \xrightarrow{\pi} P$ is exact at $N \oplus P$. That is $A^* \xrightarrow{i} C^* \xrightarrow{\pi} B^*$ is exact at C^*.
Now, $\mu_{i(A)}(x) = \vee\{\mu_A(t) : t \in N, i(t) = x\} = \mu_A(x)$(with $t = x \in N$)
and $\nu_{i(A)}(x) = \wedge\{\nu_A(t) : t \in N, i(t) = x\} = \nu_A(x)$(with $t = x \in N$).
That is $i(A) = A$ and clearly $A \not\subset B$. Therefore, the sequence $A \xrightarrow{i} C \xrightarrow{\pi} B$ is not an exact sequence of intuitionistic fuzzy G-modules.

Theorem 3.8. *Let $M \xrightarrow{f} N \xrightarrow{g} P$ be a sequence of G-modules exact at N and let $A \in G(M), B \in G(N)$ and $C \in G(P)$ be such that $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of intuitionistic fuzzy G-modules exact at B. Then $f(C_{(\alpha,\beta)}(A)) \subseteq \ker(g)\ \forall \alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$.*

Proof. Since $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B.
Therefore, $f(A) \subseteq B, g(B) \subseteq C$ and $(f(A))^* = \ker(g)$.
We know that $f(C_{(\alpha,\beta)}(A)) \subseteq C_{(\alpha,\beta)}(f(A))$ and $g(C_{(\alpha,\beta)}(A)) \subseteq C_{(\alpha,\beta)}(g(A))$.

Thus, if $x \in f(C_{(\alpha,\beta)}(A))$ be any element, then $x \in C_{(\alpha,\beta)}(f(A))$
$\Rightarrow \mu_{f(A)}(x) \geq \alpha > 0$ and $\nu_{f(A)}(x) \leq \beta < 1$ [$\because \alpha, \beta \in$ (0,1)]
$\Rightarrow x \in (f(A))^*$. But $(f(A))^* = \ker(g)$. Therefore, $x \in \ker(g)$.
Hence $f(C_{(\alpha,\beta)}(A)) \subseteq \ker(g)$. □

4 Isomorphism of short exact sequences of intuitionistic fuzzy G-modules

Definition 4.1. Let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be a short exaxt sequence of G-modules. Let $A \in G(M), B \in G(N)$ and $C \in G(P)$. Then an exact sequence of intuitionistic fuzzy G-modules of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a short exact sequence of intuitionistic fuzzy G-modules.

Extending the concept of isomorphism between short exact sequences of G-modules in classical module theory to the intuitionistic fuzzy setting. We define isomophism and weak isomorphism between short exact sequences of intuitionistic fuzzy G-modules and obtain some sufficient conditions under which the exact sequence $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ is weakly isomorphism to the exact sequence $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$. Also we get another set of sufficient conditions under which the exact sequence $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ is weakly isomorphic to the exact sequence $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$.

Recall that two short exact sequences of G-modules are said to be isomorphic if there is a commutative diagram of G-modules homomorphism

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\varphi"] & N \arrow[r, "g"] \arrow[d, "\psi"] & P \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' \arrow[r] & 0
\end{tikzcd}$$

such that φ, ψ, ξ are G-isomorphism.

Definition 4.2. Let

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\varphi"] & N \arrow[r, "g"] \arrow[d, "\psi"] & P \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & P' \arrow[r] & 0
\end{tikzcd}$$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.1}$$

and

$$0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow 0 \tag{4.2}$$

are two exact sequences of intuitionistic fuzzy G-modules. Then the sequence (4.1) is said to be weakly isomorphic to the sequence (4.2) if $\varphi(A) \subseteq A', \psi(B) \subseteq B'$ and $\xi(C) \subseteq C'$. The sequence (4.1) is said to be isomorphic to the sequence (4.2) if $\varphi(A) = A', \psi(B) = B'$ and $\xi(C) = C'$.

Related to (2.26) we have the following theorems in intuitionistic fuzzy module theory.

Theorem 4.3. *Let $0 \to M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \to 0$ be a short exact sequence of G-modules and let $A_1 \in G(M_1), A_2 \in G(M_2), B \in G(N)$ be such that $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ is a short exact sequence of intuitonistic fuzzy G-modules. If there is a G-module homomorphism $h : M_2 \to N$ with $goh = I_{M_2}$ such that $h(A_2) \subseteq B$, then the short exact sequence $0 \to A_1 \xrightarrow{i} A_1 \oplus A_2 \xrightarrow{\pi} A_2 \to 0$ is weakly isomorphic to a given short sequence $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$. In particular $A_1 \oplus A_2 \cong B$.*

Proof. We have by definition, $f(A_1) \subseteq B, g(B) \subseteq A_2$ and $(f(A_1))^* = \ker(g)$.
Also it is given that $h(A_2) \subseteq B$. Now consider the diagram

$$0 \longrightarrow M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d, "I_{M_1}"] & A_1 \oplus A_2 \arrow[r, "g"] \arrow[d, "\phi"] & A_2 \arrow[r] \arrow[d, "I_{M_2}"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "f"] & B \arrow[r, "g"] & A_2 \arrow[r] & 0
\end{tikzcd}$$

$$0 \longrightarrow M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \longrightarrow 0$$

where $\phi : M_1 \oplus M_2 \to N$ is defined by $\phi(m_1, m_2) = f(m_1) + h(m_2)$. Then ϕ is a module G-homomorphism. Moreover $\phi \circ i = f \circ I_{M_1}$ and $g \circ \phi = I_{M_2} \circ \pi$.

Since I_{M_1} and I_{M_2} (identity maps) are isomorphisms ϕ is also an isomorphism (by short five lemma of exact sequences of G-modules) and so N is isomorphic to $M_1 \oplus M_2$ and the sequences $0 \to M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \to 0$ and $0 \to M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \to 0$ are isomorphic short exact sequences of G-modules.

Obviously $I_{M_1}(A_1) = A_1, I_{M_2}(A_2) = A_2$.

Now let $x = \phi(m_1', m_2') \in N$ be arbitrary, where $m_1' \in M_1, m_2' \in M_2$. Then we get

$$\begin{aligned}
\mu_{\phi(A_1 \oplus A_2)}(x) &= \vee\{\mu_{A_1 \oplus A_2}(t_1, t_2) : (t_1, t_2) \in M_1 \oplus M_2; \phi(t_1, t_2) = x\} \\
&= \vee\{\mu_{A_1}(t_1) \wedge \mu_{A_2}(t_2) : t_1 \in M_1, t_2 \in M_2; f(t_1) + h(t_2) = \phi(m_1', m_2')\} \\
&= \vee\{\mu_{A_1}(t_1) \wedge \mu_{A_2}(t_2) : t_1 \in M_1, t_2 \in M_2; f(t_1) + h(t_2) = f(m_1') + h(m_2')\} \\
&= \vee\{\mu_{A_1}(t_1) \wedge \mu_{A_2}(t_2) : t_1 \in M_1, t_2 \in M_2; f(t_1) = f(m_1'), h(t_2) = h(m_2')\}
\end{aligned}$$

Thus,

$$\mu_{\phi(A_1\oplus A_2)}(x) = \vee\{\mu_{A_1}(t_1)\wedge\mu_{A_2}(t_2) : t_1 \in M_1, t_2 \in M_2; f(t_1) = f(m_1^{'}), h(t_2) = h(m_2^{'})\} \quad (4.3)$$

[Since $M_1 \simeq \phi(M_1) = f(M_1); M_2 \simeq \phi(M_1) = f(M_1)$, we get $N \simeq M_1 \oplus M_2 \simeq f(M_1) \oplus h(M_2)$].
Since we have $f(A_1) \subseteq B$ and $h(A_2) \subseteq B$ it follows that

Since we have $f(A_1) \subseteq B$ and $h(A_2) \subseteq B$ it follows that

$$\vee\{\mu_{A_1}(t_1) : t_1 \in M_1; f(t_1) = f(m_1^{'})\} \leq \mu_B(f(m_1^{'})) \quad (4.4)$$

and

$$\vee\{\mu_{A_2}(t_2) : t_2 \in M_2; h(t_2) = h(m_2^{'})\} \leq \mu_B(h(m_2^{'})) \quad (4.5)$$

Since B is an intuitionistic fuzzy G-module, so from (4.4) and (4.5) we get we get
$[\vee\{\mu_{A_1}(t_1) : t_1 \in M_1; f(t_1) = f(m_1^{'})\}] \wedge [\vee\{\mu_{A_2}(t_2) : t_2 \in M_2; h(t_2) = h(m_2^{'})\}]$
$\leq \mu_B(f(m_1^{'}) + h(m_2^{'})) = \mu_B(\phi(m_1^{'}, m_2^{'})) = \mu_B(x)$.
Using the complete distribuitity, we get $\vee\{\mu_{A_1}(t_1) \wedge \mu_{A_2}(t_2) : t_1 \in M_1, t_2 \in M_2; f(t_1) = f(m_1^{'}), h(t_2) = h(m_1^{'})\} \leq \mu_B(x)$.

Therefore from (1), we get $\mu_{\phi(A_1\oplus A_2)}(x) \leq \mu_B(x)\ \forall x \in N$.

Similarly, we can show that $\nu_{\phi(A_1\oplus A_2)}(x) \geq \nu_B(x)\ \forall x \in N$. Hence $\phi(A_1 \oplus A_2) \subseteq B$.
Hence by definition, short exact sequence $0 \to A_1 \xrightarrow{i} A_1 \oplus A_2 \xrightarrow{\pi} A_2 \to 0$ is weakly isomorphic (with identity map on A_1 and A_2) to the given short sequence $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ and hence $A_1 \oplus A_2 \cong B$. □

Theorem 4.4. *Let $0 \to M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \to 0$ be a short exact sequence of G-modules and let $A_1 \in G(M_1), A_2 \in G(M_2), B \in G(N)$ be such that $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ is a short exact sequence of intuitionistic fuzzy G-modules. If there is a G-module homomorphism $k : M_2 \to N$ with $kof = I_{M_1}$ such that $k(B) \subseteq A_1$, then the short exact sequence $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ is weakly isomorphic to the given short sequence $0 \to A_1 \xrightarrow{i} A_1 \oplus A_2 \xrightarrow{\pi} A_2 \to 0$. In particular $B \cong A_1 \oplus A_2$.*

Proof. We have $f(A_1) \subseteq B, g(B) \subseteq A_2$ and $k(B) \subseteq A_1$ and we have the diagram

$$0 \longrightarrow M_1 \xrightarrow{f} N \xrightarrow{\mathrm{g}} M_2 \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d, "I_{M_1}"] & B \arrow[r, "\pi"] \arrow[d, "\psi"] & A_2 \arrow[r] \arrow[d, "I_{M_2}"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "f"] & A_1 \oplus A_2 \arrow[r, "g"] & A_2 \arrow[r] & 0
\end{tikzcd}$$

$$0 \longrightarrow M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \longrightarrow 0$$

where $\psi : N \to M_1 \oplus M_2$ is defined by $\psi(n) = (k(n), g(n))$.

Then ψ is a G-module homomorphism. Moreover $ioI_{M_1} = \psi of$ and $\pi o\psi = I_{M_2}og$.

Since I_{M_1} and I_{M_2} (identity maps) are isomorphisms ψ is also an isomorphism (by short five lemma for exact sequences of G-modules) and so N is isomorphic to $M_1 \oplus M_2$ and the sequences $0 \to M_1 \xrightarrow{f} N \xrightarrow{g} M_2 \to 0$ and $0 \to M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \to 0$ are isomorphic short exact sequences of G-modules.

Obviously $I_{M_1}(A_1) = A_1, I_{M_2}(A_2) = A_2$.

Now, for $(m_1, m_2) \in M_1 \oplus M_2$, we get

$$\mu_{\psi(B)}(m_1, m_2) = \vee\{\mu_B(n) : n \in N; k(n) = m_1 and g(n) = m_2\} \tag{4.6}$$

Also, since $k(B) \subseteq A_1$ and $g(B) \subseteq A_2$, we get

$$\vee\{\mu_B(n) : n \in N; k(n) = m_1\} \leq \mu_{A_1}(m_1) \tag{4.7}$$

$$\vee\{\mu_B(n) : n \in N; g(n) = m_2\} \leq \mu_{A_1}(m_2) \tag{4.8}$$

From (4.7) and (4.8) we deduce that

$$\vee\{\mu_B(n) : n \in N; k(n) = m_1 and g(n) = m_2\} \leq \mu_{A_1}(m_1) \wedge \mu_{A_1}(m_2) \tag{4.9}$$

Similarly we can show that

$$\wedge\{\nu_B(n) : n \in N; k(n) = m_1 and g(n) = m_2\} \geq \nu_{A_1}(m_1) \vee \nu_{A_1}(m_2) \tag{4.10}$$

Also we get
$\mu_{A_1 \oplus A_2}(m_1, m_2) = \vee\{\mu_{A_1}(x_1, x_2) \wedge \mu_{A_2}(y_1, y_2) : (x_1, x_2), (y_1, y_2) \in$
$M_1 \oplus M_2; (x_1, x_2) + (y_1, y_2) = (m_1, m_2)\}$
$= \mu_{A_1}(m_1) \wedge \mu_{A_1}(m_2)$.
Thus,

$$\mu_{A_1 \oplus A_2}(m_1, m_2) = \mu_{A_1}(m_1) \wedge \mu_{A_1}(m_2). \tag{4.11}$$

Similarly, we have

$$\mu_{A_1 \oplus A_2}(m_1, m_2) = \nu_{A_1}(m_1) \vee \nu_{A_1}(m_2) \tag{4.12}$$

Now, from (4.9), (4.10), (4.11) and (4.12) we see that $\psi(B) \subseteq A_1 \oplus A_2$.
Thus the given short exact sequence $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ is weakly isomorphic (with identity map on A_1 and A_2) to the short exact sequence

$$0 \to A_1 \xrightarrow{i} A_1 \oplus A_2 \xrightarrow{\pi} A_2 \to 0.$$

In particular $B \cong A_1 \oplus A_2$. This proves the theorem. □

5 Semi-simple and split exact sequence of intuitionistic fuzzy G-modules

In this section we establish a relation between semi-simple intuitionistic fuzzy G-modules and split exact sequence of intuitionistic fuzzy G-modules.

Theorem 5.1. *Let M and N be two G-modules and let $A \in G(M), B \in G(N)$ where $A^* = M$. Then the sequence $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ is an exact sequence of intuitionistic fuzzy G-modules.*

Proof. We have to prove that $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ is exact.

Let $(m, n) \in M \oplus N$. Then

$$\mu_{i(A)}(m,n) = \vee\{\mu_A(t) : t \in M, i(t) = (m,n)\} = \begin{cases} \mu_A(m), & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases} \tag{5.1}$$

and

$$\nu_{i(A)}(m,n) = \wedge\{\nu_A(t) : t \in M, i(t) = (m,n)\} = \begin{cases} \nu_A(m), & \text{if } n = 0 \\ 0, & \text{if } n \neq 0. \end{cases} \tag{5.2}$$

Also,

$$\mu_{A+B}(m,n) = \mu_A(m) \wedge \mu_B(n) = \mu_A(m) if n = 0 \tag{5.3}$$

and

$$\nu_{A+B}(m,n) = \nu_A(m) \vee \nu_B(n) = \nu_A(m) if n = 0 \tag{5.4}$$

From (5.1), (5.2), (5.3) and (5.4) we get
$i(A)(m,n) \subseteq (A+B)(m,n)\ \forall (m,n) \in M \oplus N$.
Therefore,

$$i(A) \subseteq (A+B) \tag{5.5}$$

For $x \in N$ we have

$$\begin{aligned} \mu_{\pi(A+B)}(x) &= \vee\{\mu_{A+B}(m,n) : (m,n) \in M \oplus N; \pi(m,n) = x\} \\ &= \vee\{\mu_{A+B}(m,x) : x \in N\}[\text{ Since } \pi : M \oplus N \dashrightarrow N \text{ is a projection }] \\ &= \vee\{\mu_A(m) \wedge \mu_B(x) : x \in N\} \\ &= \mu_B(x).[\text{ Since } \mu_A(m) = 1 \text{ with } m = 0] \end{aligned}$$

Similarly we can get $\nu_{\pi(A+B)}(x) = \nu_B(x)$.
Thus, $\pi(A+B)(x) = B(x)\ \forall x \in N$.
Hence

$$\pi(A+B) = B. \tag{5.6}$$

Now, since $A^* = M$, it follows from (5.1) and (5.2) that
$\mu_{i(A)}(m,n) > 0$ and $\nu_{i(A)}(m,n) < 1$ if $n = 0$ i.e., if $(m,n) \in \ker \pi$ and
$\mu_{i(A)}(m,n) = 0$ and $\nu_{i(A)}(m,n) = 1$ if $n \notin 0$ i.e, if $(m,n) \notin \ker \pi$.
i.e.,

$$(i(A))^* = \ker \pi. \tag{5.7}$$

From (5.5), (5.6) and (5.7) we see that the sequence $0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$ is an exact sequence intuitionistic fuzzy G-modules. □

Definition 5.2. Let M and N be two left G-modules; let $C \in G(M), B \in G(N)$ and $A \in G(M \oplus N)$. Then a short exact sequence of intuitionistic fuzzy G-modules of the form

$$\begin{array}{c} 0 \longrightarrow M \xrightarrow{i} M \oplus N \xrightarrow{\pi} N \longrightarrow 0 \\ \text{and } 0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0 \end{array}$$

is said to be a short exact sequence if $A = C \oplus B$.

Definition 5.3. Let M, N and P be left G-modules and let $C \in G(M), A \in G(N), C \in G(P)$. Then a short exact sequnce of G-modules of the form

$$\begin{array}{c} 0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \\ \text{and } 0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0 \end{array}$$

is said to be a split exact sequence if $N = M \oplus P$ and $A = C \oplus B$ so that the given sequence is isomorphic to the short exact sequence

$$\begin{array}{c} 0 \longrightarrow M \xrightarrow{i} M \oplus P \xrightarrow{\pi} N \longrightarrow 0 \\ \text{and } 0 \longrightarrow C \longrightarrow C \oplus B \longrightarrow B \longrightarrow 0 \end{array}$$

Theorem 5.4. *All short exact sequences of G-modules are split if and only if G-modules are semi-simple.*

Proof. Assume that all short exact sequences of G-modules are split exact sequences. Let M be a semi-simple G-module and let $A \in G(M)$. Then we show that A is semi-simple. That is to show that if $C \in G(M)$ is given then $C \subseteq A, C \neq \Omega, C(x) = A(x)\ \forall x \in C^*, C^* \subseteq A^*$, then there exists a $B \in G(M)$ such that $B \subseteq A, B \neq \Omega, B(x) = A(x)\ \forall x \in B^*, B^* \subseteq A^*$ satisfying $A = C \oplus B$.
Since $C^* \subseteq A^*$ we have the short exact sequence of sequence of G-modules

$$0 \longrightarrow C^* \xrightarrow{i} A^* \xrightarrow{\pi} A^*/C^* \longrightarrow 0$$

We consider the intuitionistic fuzzy G-modules $C \in G(C^*), A \in G(A^*)$ and $A/C \in G(A^*/C^*)$. We claim that the sequence

$$\begin{array}{c} 0 \longrightarrow C^* \xrightarrow{i} A^* \xrightarrow{\pi} A^*/C^* \longrightarrow 0 \\ \text{and } 0 \longrightarrow C \longrightarrow A \longrightarrow A/C \longrightarrow 0 \end{array}$$

of intuitionistic fuzzy G-modules is exact. For:
(i)

$$\begin{aligned} \mu_{i(C)}(x) &= \vee\{\mu_C(t) : t \in C^*, i(t) = x\} \\ &= \begin{cases} \mu_C(x), & \text{if } x \in C^* \\ 0, & \text{if } x \notin C^*. \end{cases} \end{aligned}$$

Similarly,

$$\begin{aligned}\nu_{i(C)}(x) &= \wedge\{\nu_C(t) : t \in C^*, i(t) = x\} \\ &= \begin{cases}\nu_C(x), & \text{if } x \in C^* \\ 1, & \text{if } x \notin C^*.\end{cases}\end{aligned}$$

Since $C \subseteq A$ we get $i(C) \subseteq A$.
(ii)

$$\begin{aligned}\mu_{\pi(A)}(x + C^*) &= \vee\{\mu_A(t) : t \in A^*; \pi(0) = x + C^*\} \\ &= \vee\{\mu_A(t) : t \in A^*; t + C^* = x + C^*\} \\ &= \vee\{\mu_A(t) : t \in A^*; t \in x + C^*\} \\ &= \mu_{A/C}(x + C^*).\end{aligned}$$

Similarly, we can show that $\nu_{\pi(A)}(x + C^*) = \nu_{A/C}(x + C^*)$.
Thus $(\pi(A))(x + C^*) = A/C(x + C^*)$ and so $\pi(A) = A/C$.
and (iii) since

$$\mu_{i(C)}(x) = \begin{cases}\mu_c(x), & \text{if } x \in C^* \\ 0, & \text{if } x \in C^*\end{cases}; \quad \nu_{i(C)}(x) = \begin{cases}\nu_c(x), & \text{if } x \in C^* \\ 1, & \text{if } x \in C^*.\end{cases}$$

we see that
$\mu_{i(C)}(x) > 0$ and $\nu_{i(C)}(x) < 1$ if $x \in \ker \pi$. Also $\mu_{i(C)}(x) = 0$ and $\nu_{i(C)}(x) = 1$ if $x \notin \ker \pi$.
Since all exact sequence of intuitionistic fuzzy G-modules are split exact sequences we get $A^* = C^* \oplus A^*/C^*$ and $A = C \oplus A/C$ where $C \in G(C^*)$ and $A/C \in G(A^*/C^*)$.
Now A^*/C^* can be considered as a submodule N of A^* and hence of M and A/C can be considered as an intuitionistic fuzzy G-module of M.
Also we note that

$$\begin{aligned}(A/C)^* &= \{x + C^* \in A^*/C^* : \mu_{A/C}(x + C^*) > 0 \text{ and } \nu_{A/C}(x + C^*) < 1\} \\ &= \{x + C^* \in A^*/C^* : \vee\{\mu_A(t) : t \in x + C^*\} > 0 \text{ and } \wedge\{\nu_A(t) : t \in x + C^*\} < 1\} \\ &= \{x + C^* \in A^*/C^* : \exists t \in x + C^* \text{ with } \mu_A(t) > 0 \text{ and } \nu_A(t) < 1\} \\ &= A^*/C^*.\end{aligned}$$

Thus $A^* = C^* \oplus N$ where A^*, C^* and N are all submodules of M and since $A = C \oplus A/C$ it follows that $A/C(x) = A(x)\ \forall x \in N \Rightarrow A^*/C^* = (A/C)^*$.
Thus there exists a strictly proper intuitionistic fuzzy G-module A/C of A such that $A/C(x) = A(x)\ \forall x \in (A/C)^*$ and $A = C \oplus A/C$.
Therefore A is semi simple.
Conversely suppose that all G-modules are semi-simple.
Consider the short exact sequence of modules

$$\begin{array}{c} 0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \\ 0 \longrightarrow C^* \xrightarrow{f|_{C^*}} A^* \xrightarrow{g|_{A^*}} B^* \longrightarrow 0 \\ 0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0 \end{array}$$

To show that this sequence splits. For this it is enough to show that exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is isomorphic to $0 \to M \xrightarrow{i} M \oplus P \xrightarrow{g} P \to 0$.
So we consider the sequence

$$\begin{array}{c} 0 \longrightarrow M \xrightarrow{\mathrm{i}} M \oplus P \xrightarrow{g} P \longrightarrow 0 \\ 0 \longrightarrow C^* \xrightarrow{f|_{C^*}} A^* \xrightarrow{\pi|_{A^*}} B^* \longrightarrow 0 \\ 0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0 \end{array}$$

Obviously $C = i(C) \subseteq A$. Therefore since A is semi simple $A = C \oplus D$ for some strictly proper G-submodule D of A. Since $A^* = C^* \oplus D^*$. Also since a submodule of a semi simple G-module is semi-simple we see that A^* is semi simple and so we get $A^* = C^* \oplus B^*$. Hence $D^* \cong A^*/C^* \cong B^*$. So D can be considered as G-submodule of P and we can define D to be B so that $A = C \oplus B$. This completes the proof of the theorem. □

6 Conclusions

The main focus of this article is to introduce the concept of exact sequence of G-modules by intuitonistic fuzzification the concept in crisp theory. We established a relation between semi-simple G-modules and split exact sequences of G-modules. This is useful in the study of injective and projective intuitionistic fuzzy G-modules in terms of exact sequences.

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