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Intuitionistic fuzzy implication $\rightarrow_{187}$

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Abstract: In [4], some new intuitionistic fuzzy operations are defined and their properties are studied. On the basis of two of these new intuitionistic fuzzy operations, a new intuitionistic fuzzy implication is introduced here, numbered as $\rightarrow_{187}$ and some of its properties are examined.
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1 Introduction

In [4] five new intuitionistic fuzzy operations, containing multiplication were introduced. For the new intuitionistic fuzzy operations it was shown that three of the operations have conjunction properties and three – disjunction properties. Here, on the basis of the definitions of two of the new operations from [4], we introduce new operation implication and check some of its important properties.

In intuitionistic fuzzy logic (see [1, 2]), each proposition, variable or formula is evaluated with two degrees – "truth degree" or "degree of validity" and "falsity degree" or "degree of non-validity". Thus, to each one of these objects, e.g., p, two real numbers, $\mu(p)$ and $\nu(p)$, are assigned with the following constraint:

$$\mu(p), \nu(p) \in [0, 1] \text{ and } \mu(p) + \nu(p) \leq 1.$$

Let $\pi(p) = 1 - \mu(p) - \nu(p)$. This function determines the degree of uncertainty (indeterminacy). Let an evaluation function V be defined over a set of propositions $\mathcal{S}$, in such a way that for $p \in \mathcal{S}$:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

Hence the function $V : \mathcal{S} \to [0, 1] \times [0, 1]$ gives the truth and falsity degrees of all elements of $\mathcal{S}$. We assume that the evaluation function V assigns to the logical truth T, $V(T) = \langle 1, 0 \rangle$, and to the logical falsity F, $V(F) = \langle 0, 1 \rangle$.

As it was discussed in [2], the first (classical) intuitionistic fuzzy negation is

$$V(\neg_1 p) = \langle \nu(p), \mu(p) \rangle.$$

Below, for simplicity, we write $\neg$ instead of $\neg_1$.

Here, we define only the operations "disjunction" and "conjunction", originally introduced in [1], that have classical logic analogues, as follows:

$$V(p \vee q) = \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle,$$

$$V(p \wedge q) = \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle.$$

For the needs of the discussion below, we define the notions of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [1, 2]) and tautology.

Formula A is an IFT if and only if (iff) for every evaluation function V, if $V(A) = \langle a, b \rangle$, then, $a \geq b$, while it is a (classical) tautology if and only if for every evaluation function V, if $V(A) = \langle a, b \rangle$, then, $a = 1,\ b = 0$.

Below, when it is clear, we will omit notation "$V(A)$", using directly "A" instead of the intuitionistic fuzzy evaluation of A.

In [3], we called the object $\langle \mu(p), \nu(p) \rangle$ an Intuitionistic Fuzzy Pair (IFP).

For brevity, in a lot of places, instead of the IFP $\langle \mu(A), \nu(A) \rangle$ we will use the IFP $\langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$.

It is also suitable, if $\langle a, b \rangle$ and $\langle c, d \rangle$ are IFPs, to have

$$\langle a, b \rangle \leq \langle c, d \rangle \text{ iff } a \leq c \text{ and } b \geq d$$

and

$$\langle a, b \rangle \geq \langle c, d \rangle \text{ iff } a \geq c \text{ and } b \leq d.$$

If an IFP is an IFT, we call it Intuitionistic Fuzzy Tautological Pair (IFTP) and if it is a tautology – Tautological Intuitionistic Fuzzy Pair (TIFP).

2 Intuitionistic fuzzy operations $\times_1$ and $\times_5$

In [4], for two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$, we introduced the following novel operations from multiplicative type:

$$x \times_1 y = \langle \max(a, c), bd\rangle,$$
$$x \times_2 y = \langle \min(a, c), bd\rangle,$$
$$x \times_3 y = \langle ac, bd\rangle,$$
$$x \times_4 y = \langle ac, \min(b, d)\rangle,$$
$$x \times_5 y = \langle ac, \max(b, d)\rangle.$$

In the present paper, we discuss only first and fifth operations. For them, in [4] was proved the following.

First, both operations are defined correctly.

Second, let x and y have the above forms and let $z = \langle e, f\rangle$. Then, for $i = 1, 5$:

$$x \times_i y = y \times_i x,$$

$$(x \times_i y) \times_i z = x \times_i (y \times_i z).$$

Third, for each IFP x:

$$\langle 0, 1\rangle \times_1 x = x = x \times_1 \langle 0, 1\rangle,$$

$$\langle 1, 0\rangle \times_5 x = x = x \times_5 \langle 1, 0\rangle,$$

$$\langle 1, 0\rangle \times_1 x = \langle 1, 0\rangle = x \times_1 \langle 1, 0\rangle,$$

$$\langle 0, 1\rangle \times_5 x = \langle 0, 1\rangle = x \times_5 \langle 0, 1\rangle.$$

Let

$$\mathcal{L} = \{\langle a, b\rangle | a, b \in [0, 1] \ \&\ a + b \leq 1\}$$

be the set of all IFPs. The following assertion follows from above results.

Theorem 1. $\langle \mathcal{L}, \times_1, \langle 0, 1\rangle\rangle$ and $\langle \mathcal{L}, \times_5, \langle 1, 0\rangle\rangle$ are commutative monoids.

None of these two objects is a group.

Theorem 2. If x and y are IFTPs, then $x \times_1 y$, is an IFTP.

Theorem 3. If x and y are TPs, then $x \times_1 y$ and $x \times_5 y$ are TPs.

Fourth, in intuitionistic fuzzy propositional logic there are already definitions of 53 different intuitionistic fuzzy negations, only one from which is a classical one, as defined by

$$\neg\langle a, b\rangle = \langle b, a\rangle.$$

We see that for every two IFPs x and y:

$$\neg(\neg x \times_1 \neg y) = \neg(\neg\langle a, b\rangle \times_1 \neg\langle c, d\rangle)$$

$$\neg(\langle b, a\rangle \times_1 \langle d, c\rangle) = \neg\langle \max(b, d), ac\rangle$$

$$\langle ac, \max(b, d)\rangle = x \times_5 y$$

and

$$\neg(\neg x \times_5 \neg y) = \neg(\neg\langle a, b\rangle \times_5 \neg\langle c, d\rangle)$$

$$\neg(\langle b, a\rangle \times_5 \langle d, c\rangle) = \neg\langle bd, \max(a, c)\rangle$$

$$\langle \max(a, c), bd\rangle = x \times_1 y;$$

Therefore, operation $\times_1$ has the behaviour of operation disjunction, while operations $\times_5$ has the behaviour of operation conjunction.

3 Intuitionistic fuzzy implication $\rightarrow_{187}$ and its properties

Now, using the standard logical formula

$$x \rightarrow y = \neg x \vee y,$$

we obtain the new intuitionistic fuzzy implication

$$x \rightarrow_{187} y = \neg x \vee y = \langle \max(b, c), ad\rangle.$$

First, we see that

$$0 \le \max(b, c) + ad \le \max(b, c) + \min(a, d) \le \max(b, c) + \min(1 - b, 1 - c)$$

$$= \max(b, c) + 1 - \max(b, c) = 1,$$

i.e., implication $\rightarrow_{187}$ is defined correctly.

Second, we see that

$$\langle 0, 1\rangle \rightarrow_{187} \langle 0, 1\rangle = \langle 1, 0\rangle,$$

$$\langle 0, 1\rangle \rightarrow_{187} \langle 1, 0\rangle = \langle 1, 0\rangle,$$

$$\langle 1, 0\rangle \rightarrow_{187} \langle 0, 1\rangle = \langle 0, 1\rangle,$$

$$\langle 1, 0\rangle \rightarrow_{187} \langle 1, 0\rangle = \langle 1, 0\rangle,$$

i.e., this operation satisfy these basic properties of an implication.

Third, implication $\rightarrow_{187}$ generates the standard negation, because

$$\langle a, b\rangle \rightarrow_{187} \langle 0, 1\rangle = \langle b, a\rangle.$$

Fourth, we see that

$$(x \rightarrow_{187} y) \vee (y \rightarrow_{187} x)$$

$$= (\langle a, b\rangle \rightarrow_{187} \langle c, d\rangle) \vee (\langle c, d\rangle \rightarrow_{187} \langle a, b\rangle)$$

$$= \langle \max(b, c), ad\rangle \vee \langle \max(a, d), bc\rangle$$

$$= \langle \max(a, b, c, d), \min(ad, bc)\rangle.$$

Obviously, this IFP is not a TIFP, but it is an IFTP.

G.F. Rose's formula [9, 10] has the form:

$$((\neg\neg x \to_{187} x) \to_{187} (\neg\neg x \vee \neg x)) \to_{187} (\neg\neg x \vee \neg x).$$

Theorem 4. Rose's formula is an IFT.

Proof. Sequentially, we obtain:

$$((\neg\neg x \to_{187} x) \to_{187} (\neg\neg x \vee \neg x)) \to_{187} (\neg\neg x \vee \neg x)$$

$$= ((x \to_{187} x) \to_{187} (x \vee \neg x)) \to_{187} (x \vee \neg x)$$

$$= ((\langle a, b\rangle \to_{187} \langle a, b\rangle) \to_{187} (\langle a, b\rangle \vee \neg\langle a, b\rangle)) \to_{187} (\langle a, b\rangle \vee \neg\langle a, b\rangle)$$

$$= (\langle \max(a, b), ab\rangle \to_{187} (\langle a, b\rangle \vee \langle b, a\rangle)) \to_{187} (\langle a, b\rangle \vee \langle b, c\rangle)$$

$$= (\langle \max(a, b), ab\rangle \to_{187} \langle \max(a, b), \min(a, b)\rangle) \to_{187} \langle \max(a, b), \min(a, b)\rangle$$

$$= \langle \max(a, b, ab), \min(a, b, \max(a, b))\rangle \to_{187} \langle \max(a, b), \min(a, b)\rangle$$

$$= \langle \max(\min(a, b, \max(a, b)), \max(a, b)), \max(a, b, ab) \min(a, b)\rangle$$

(from

$$\max(\min(a, b, \max(a, b)), \max(a, b)) = \max(\min(a, b), \max(a, b)) = \max(a, b),$$

$$\max(a, b, ab) \min(a, b) = \max(a, b) \min(a, b) = ab,$$

we obtain)

$$= \langle \max(a, b), ab\rangle (= x \to_{187} x).$$ □

Obviously, this IFP is an IFTP, but not a TIFP.

Fifth, following [2], we discuss the well-known Contraposition Law

$$(x \to_{187} y) \to_{187} (\neg y \to_{187} \neg x).$$

Theorem 5. Contraposition Law is an IFT, but not a tautology.

Proof. Sequentially, we obtain:

$$(x \to_{187} y) \to_{187} (\neg y \to_{187} \neg x)$$

$$= (\langle a, b\rangle \to_{187} \langle c, d\rangle) \to_{187} (\neg\langle c, d\rangle \to_{187} \neg\langle a, b\rangle)$$

$$= (\langle a, b\rangle \to_{187} \langle c, d\rangle) \to_{187} (\langle d, c\rangle \to_{187} \langle b, a\rangle)$$

$$= \langle \max(b, c), ad\rangle \to_{187} \langle \max(b, c), ad\rangle$$

$$= \langle \max(b, c, ad), \max(b, c)ad\rangle.$$ □

Obviously, this IFP is an IFTP, but not a TIFP.

Sixth, some variants of fuzzy implications (marked by $I(x, y)$) are described in the book of Klir and Yuan [8] and the following nine axioms are discussed, where $I(x, y)$ denotes $x \to y$ for any of the possible forms of the operation implication, N is the operation negation related with operation $\to$, and for $a, b, c, d \in [0, 1], a + b \leq 1, c + d \leq 1$:

$$\langle a, b\rangle \leq \langle c, d\rangle \text{ iff } a \leq c \text{ and } b \geq d.$$

Axiom $A1$ $(\forall x, y)(x \leq y \to (\forall z)(I(x, z) \geq I(y, z)))$,
Axiom $A2$ $(\forall x, y)(x \leq y \to (\forall z)(I(z, x) \leq I(z, y)))$,
Axiom $A3$ $(\forall y)(I(0, y) = 1)$,
Axiom $A4$ $(\forall y)(I(1, y) = y)$,
Axiom $A5$ $(\forall x)(I(x, x) = 1)$,
Axiom $A6$ $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$,
Axiom $A7$ $(\forall x, y)(I(x, y) = 1 \text{ iff } x \leq y)$,
Axiom $A8$ $(\forall x, y)(I(x, y) = I(N(y), N(x)))$,
Axiom $A9$ I is a continuous function.

For our research, having in mind the specific forms of the intuitionistic fuzzy implication $\to_{187}$ and following [2], we modify two of these axioms, as follows.
Axiom $A5^*$ $(\forall x)(I(x, x)$ is an IFT).
Axiom $A7^*$ $(\forall x, y)($ if $x \leq y$, then, $I(x, y)$ is an IFT).

Theorem 5. Intuitionistic fuzzy implication $\to_{187}$ satisfies axioms $A1$ – $A4$, $A5^*$, $A6$, $A7^*$, $A8$ and $A9$.

Proof. Let $x = \langle a, b\rangle, y = \langle c, d\rangle, z = \langle e, f\rangle$. We obtain sequentially. Let $x \leq y$. Then for $A1$ is valid:

$$I(x, z) = \langle \max(b, e), af\rangle \geq \langle \max(d, e), cf\rangle = I(y, z).$$

The checks for $A2$ – $A4$ are similar. For $A5^*$ we have

$$I(x, x) = \langle a, b\rangle \to_{187} \langle a, b\rangle = \langle \max(a, b), ab\rangle.$$

Obviously, in the general case $\langle \max(a, b), ab\rangle \neq \langle 1, 0\rangle$, i.e. $A5$ is not valid, but $\langle \max(a, b), ab\rangle$ is an IFTP, i.e., $A5^*$ is valid. For $A6$ we have:

$$I(x, I(y, z)) = \langle a, b\rangle \to_{187} (\langle c, d\rangle \to_{187} \langle e, f\rangle)$$

$$\langle a, b\rangle \to_{187} \langle \max(d, e), cf\rangle$$

$$\langle \max(b, d, e), acf\rangle)$$

$$= \langle c, d\rangle \to_{187} \langle \max(b, e), af\rangle$$

$$= \langle c, d\rangle \to_{187} (\langle a, b\rangle \to_{187} \langle e, f\rangle) = I(y, I(x, z)).$$

From $x \leq y$ it follows for $I(x, y) = \langle \max(b, c), ad\rangle$ that $\max(b, c) \geq b \geq d \geq ad$, i.e. $A7^*$ is valid, but the opposite is not valid, because, e.g., for $x = \langle 0.2, 0.2\rangle, y = \langle 0.1, 0.9\rangle$ $I(x, y) = \langle 0.2, 0.18\rangle$ is an IFTP, but $x \geq y$. It is obvious that $A8$ and $A9$ are valid. $\square$

4 Conclusion

In next research other properties of the implication $\rightarrow_{187}$ will be introduced and studied. All the properties show that intuitionistc fuzzy sets and logics in the sense, described in [2] correspond to the ideas of Brouwer's intuitionism (see [5, 6, 7]).

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